

Quantization of 2-plectic Manifolds

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Based on:

- CS and Richard Szabo, work in progress

2-plectic Manifolds

Multisymplectic manifolds are a natural generalization of symplectic manifolds.

Symplectic manifolds

Manifold M with closed 2-form ω such that $\iota_v \omega = 0 \Leftrightarrow v = 0$.

- Poisson structure \rightarrow Phase spaces in **Hamiltonian dynamics**.
- Starting point for **quantization**.

p -plectic manifolds

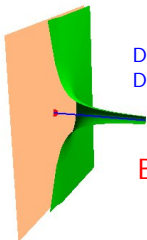
Manifold M with closed $p + 1$ -form ω such that $\iota_v \omega = 0 \Leftrightarrow v = 0$.

- 1-plectic: symplectic, 2-plectic: 3-form ω
- (Often) Nambu-Poisson structure \rightarrow multiphase spaces in **Nambu mechanics**.
- Starting point for **higher quantization** (?)

- **Why** should we be interested in such manifolds?
- **Why** should we quantize them?

Monopoles and Self-Dual Strings

Lifting monopoles to M-theory yields self-dual strings.



	0	1	2	3	4	5	6
D1	×						×
D3	×	×	×	×			

BPS configuration!

Nahm equation:

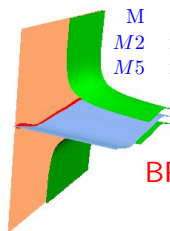
$$\frac{d}{dx^6} X^i + \varepsilon^{ijk} [X^j, X^k] = 0$$

Sol. $X^i = \frac{1}{x^6} G^i,$

$$G^i = \varepsilon^{ijk} [G^j, G^k]$$

⇒ Fuzzy Funnel

Nahm, Diaconescu, Tsimpis



	M	0	1	2	3	4	5	6
M2		×					×	×
M5		×	×	×	×	×	×	

BPS configuration!

Basu-Harvey equation:

$$\frac{d}{dx^6} X^\mu + \varepsilon^{\mu\nu\rho\sigma} [X^\nu, X^\rho, X^\sigma] = 0$$

Sol. $X^\mu = \frac{1}{\sqrt{x^6}} G^\mu,$

$$G^\mu = \varepsilon^{\mu\nu\rho\sigma} [G^\nu, G^\rho, G^\sigma]$$

⇒ Fuzzy Funnel

Basu, Harvey

Further Motivation

There are more appearances of 2-plectic manifolds in string-/M-theory.

- **M5-brane perspective:** Turning on 3-form background,

$$C = \theta dx^0 \wedge dx^1 \wedge dx^2 + \theta' dx^0 \wedge dx^1 \wedge dx^2 ,$$

one gets **interesting noncommutative deformations:**

- Noncommutative loop space
 - Kawamoto, Sasakura and Bergshoeff et al. (2000)
 - $[x^0, x^1, x^2] = \theta$ and $[x^3, x^4, x^5] = \theta'$ Chu, Smith (2009)
- Non-associative structures from strings in H-field backgrounds
 - Blumenhagen, Deser, Lüst, Plauschinn, Rennecke (2010/11)
- Baez et al.: Phase space of **bosonic string** is 2-plectic

- Berezin quantization of $\mathbb{C}P^1$
- Quantization of duals of Lie algebras
- From Lie algebras to Lie algebroids
- Hawkins approach to quantization
- Loop spaces and gerbes
- Example: Quantization of \mathbb{R}^3

Berezin Quantization of $\mathbb{C}P^1 \simeq S^2$

The fuzzy sphere is the Berezin quantization of $\mathbb{C}P^1$.

Hilbert space

\mathcal{H} : space of global **polarized** sections of **prequantum line bundle**.
Line bundle $L_k = \mathcal{O}(k)$, Kähler polarization: $\mathcal{H}_k = H^0(\mathbb{C}P^1, L_k)$.

$$\mathcal{H}_k \cong \text{span}(z_{\alpha_1} \dots z_{\alpha_k}) \cong \text{span}(\hat{a}_{\alpha_1}^\dagger \dots \hat{a}_{\alpha_k}^\dagger |0\rangle)$$

Coherent states

For any $z \in \mathcal{M}$: coherent st. $|z\rangle \in \mathcal{H}$. Here: $|z\rangle = \frac{1}{k!} (\bar{z}_\alpha \hat{a}_\alpha^\dagger)^k |0\rangle$.

Quantization

Quantization is the **inverse map** on the **image** $\Sigma = \sigma(\mathcal{C}^\infty(\mathcal{M}))$ of

$$f(z) = \sigma(\hat{f}) = \text{tr} \left(\frac{|z\rangle\langle z|}{\langle z|z\rangle} \hat{f} \right), \quad \text{Bridge: } \mathcal{P} := \frac{|z\rangle\langle z|}{\langle z|z\rangle}$$

Quantization of the Dual of a Lie Algebra

The quantization of the dual of a Lie algebra is straightforward and well known.

Kirillov-Kostant-Souriau Poisson structure on $M = \mathfrak{g}^*$

Linear functions on M : elements of \mathfrak{g} . Define

$$\{g_1, g_2\}(x) = \langle x, [g_1, g_2] \rangle, \quad g_1, g_2 \in \mathfrak{g}.$$

Extend to polynomial functions using the Leibniz identity.

Quantization: $C_0(\mathfrak{g}^*) \rightarrow$ convolution algebra $C^*(G)$

$$\begin{array}{ccc} C_0(\mathfrak{g}^*) & \xrightarrow{\text{Fourier-transform}} & C_0(\mathfrak{g}) \\ & & \downarrow \text{identify via exp} \\ C^*(\mathfrak{g}^*) & \xleftarrow{\text{transform back}} & C^*(G) \leftarrow \text{convolution product} \end{array}$$

For nilpotent Lie algebras: equiv. to Kontsevich/univ. env. algebra

Example: κ -Minkowski Space

κ -Minkowski space can be obtained from a quantized dual of a Lie algebra.

κ -Minkowski space: $[g^0, g^i] = i\hbar g^i$, $[g^i, g^j] = 0$, $i > 0$

Group: $G_\hbar \cong \mathbb{R} \rtimes_\alpha \mathbb{R}$ generated by $W(k_0, \vec{k}) = V_{\vec{k}} U_{k_0}$ with $U_{k_0} := \exp(i k_0 g^0)$, $V_{\vec{k}} = \exp(-i \sum_i k_i g^i)$ and

$$W(k_0, \vec{k}) W(k'_0, \vec{k}') = W(k_0 + k'_0, \vec{k} + e^{-\hbar k_0} \vec{k}')$$

Convolution algebra generated by

$$W(\tilde{f}) := \int_{\mathbb{R}^d} dk_0 d\vec{k} e^{\hbar k_0} \tilde{f}(k_0, \vec{k}) W(k_0, \vec{k})$$

Convolution product:

$$(\tilde{f} \circledast_{\hbar} \tilde{g})(k_0, \vec{k}) = \int_{\mathbb{R}^d} dk'_0 d\vec{k}' e^{\hbar k'_0} \tilde{f}(k'_0, \vec{k}') \tilde{g}(k_0 - k'_0, e^{\hbar k'_0} (\vec{k} - \vec{k}'))$$

Lie Groupoids and Lie Algebroids

Lie groupoids and Lie algebroids are generalizations of Lie groups and Lie algebras.

Group:

Category with one object $\mathbb{1}$, every morphism (group elt.) invertible.

Groupoid:

Category with every morphism invertible.

Lie group(oid): objects/morphisms manifolds, maps differentiable.

Equivalently: Symmetric digraph (vertices B and arrows \mathcal{G}) with:

- source- and target maps $s, t : \mathcal{G} \rightarrow B: s(a) \xrightarrow{a} t(a)$
- embedding $i : B \hookrightarrow \mathcal{G}$ with $t \circ i = s \circ i = \text{id}_B$
- partial associative product $m : \mathcal{G} \times \mathcal{G} \rightarrow \mathcal{G}$ with $m(a, iv) = a$

Examples:

- Group G : one object pt , arrows: group elements, $i : pt \mapsto \mathbb{1}_G$
- Pair groupoid $M \times M: x \xrightarrow{(x,y)} y \xrightarrow{(y,z)} z, i(x) = (x, x).$

Lie Groupoids and Lie Algebroids

Lie groupoids and Lie algebroids are generalizations of Lie algebras and Lie groups.

Lie algebra: tangent space of Lie group at identity.

Lie algebroids: $\text{Lie}(\mathcal{G}) = \cup_{x \in B} T_{i(x)}(t^{-1}(x)) \subset T\mathcal{G}$:

$$\begin{array}{ccc} \text{Lie}(\mathcal{G}) & \xrightarrow{\#} & TB \\ & \searrow & \downarrow \\ & & B \end{array}$$

with **Lie bracket structure** on sections of $\text{Lie}(\mathcal{G})$, $\#$ compatible.

Examples:

- Group G : $\text{Lie}(\mathcal{G}) = \mathfrak{g} \rightarrow pt$, $\#$: trivial.
- Pair groupoid: $\text{Lie}(M \times M) = \cup_{x \in M} x \times T_x M = TM$.

Note: not every Lie algebroid can be integrated to a Lie groupoid!

Quantizing Duals of Lie Algebroids

The elements of Lie algebra quantization can be lifted to Lie algebroids and Lie groupoids.

Goal:

Extend quant. of duals of Lie algebras to duals of Lie algebroids.

Before: $C_0(\mathfrak{g}^*) \rightarrow C^*(G)$. Now:

$$C_0(\text{Lie}^*(\mathcal{G})) \rightarrow C^*(\mathcal{G})$$

This is interesting, as T^*M of Poisson manifold M is Lie algebroid.

Eli Hawkins (2006), Weinstein, Renault, ...

Yesterday's talk: Podles sphere

Bonechi, Ciccoli, Staffolani, Tarlini

Hawkins' Groupoid Approach to Quantization

In Hawkins' approach, the elements of geometric quantization are lifted to groupoids.

Groupoid \mathcal{G} , composable arrows: \mathcal{G}_2 . Maps: $\text{pr}_1, \text{pr}_2, m: \mathcal{G}_2 \rightarrow \mathcal{G}$

Key idea in Hawkins: Reduce polarization issues as follows

T^*M , M Poisson manifold, integrates to Lie groupoid \Leftrightarrow

$\exists s, t: (\Sigma, \omega) \rightrightarrows M$ with t Poisson, $\partial^*\omega = 0$, $\partial^* = \text{pr}_1^* + \text{pr}_2^* - m^*$

Crainic, Fernandes (2002)

Hawkins' quantization algorithm:

- 1 Integrating groupoid $s, t: \Sigma \rightrightarrows M$, ω , $\partial^*\omega = 0$, t Poisson
- 2 Construct a **prequantization** of Σ with data (L, ∇)
- 3 Endow Σ with a groupoid **polarization**
- 4 Construct a **twist element**
- 5 Obtain **twisted polarized convolution algebra** of Σ .

Note: Many questions concerning existence and uniqueness remain.

Approach **avoids Hilbert spaces**, this might help in 2-plectic case.

Example: Groupoid Quantization of \mathbb{R}^2

The Moyal plane is conveniently reproduced in groupoid language.

Starting point: $M = \mathbb{R}^2$, Poisson structure θ^{ij} , $i, j = 1, 2$.

- ① **Lie groupoid**: $\Sigma = M \times M^*$, coords. (x^i, y_i) , $\omega = dx^i \wedge dy_i$

$$x^i + \frac{1}{2}\theta^{ij}y_j \xrightarrow{(x^i, y_i)} x^i - \frac{1}{2}\theta^{ij}y_j$$

Note: t is indeed a **Poisson map**: $\{t^*f, t^*g\}_\omega = t^*\{f, g\}_\pi$

$$x^i + \theta^{ij}(y_j + y'_j) \longrightarrow x^i + \theta^{ij}(y_j - y'_j) \longrightarrow x^i - \theta^{ij}(y_j + y'_j)$$

From this: pr_1, pr_2 and m . $\partial^*\omega = pr_1^*\omega - m^*\omega + pr_2^*\omega = 0$

- ② **Prequantization**: L trivial line bundle over Σ , $F = -i2\pi\omega$
- ③ **Polarization**: Induced by symplectic prepotential $\vartheta = -x^i dy_i$
- ④ **Twist element**: $\partial^*\vartheta = \sigma_0^{-1}d\sigma_0 = d(-\frac{1}{2}\theta^{ij}y_i y'_j)$
- ⑤ **Twisted polarized convolution algebra**: Moyal product on M

Hawkins

Other Examples

Hawkins' method reproduces all canonical examples.

Analogously:

- Quantization of κ -Minkowski space with $\Sigma = T^*G$, $\sigma_0 = 0$
- Berezin Quantization of Kähler manifolds M : $\Sigma = M \times M$.

2-plectic Manifolds and Gerbes

Certain 2-plectic manifolds naturally come with a prequantum gerbe.

Symplectic manifold (M, ω) with $\omega \in H^2(M, \mathbb{Z})$:

\Rightarrow **Prequantum line bundle** with connection ∇ , $F_\nabla = 2\pi i\omega$.

2-plectic manifold (M, ϖ) with $\varpi \in H^3(M, \mathbb{Z})$:

\Rightarrow **Prequant. abelian gerbe** with connect. struct. incl. $H = 2\pi i\varpi$.

Idea: **Categorify Hawkins' approach** (2-groupoids, etc.)

(work in progress, cf. **Freed, Baez, Rogers ...**)

Line Bundles over Loop Spaces from Gerbes

A prequantum gerbe over a manifold yields a prequantum line bundle over its loop space.

Alternative approach:

Map the 2-plectic form to a symplectic form over **loop space**.

Consider the following **double fibration**:

$$\begin{array}{ccc} & \mathcal{L}M \times S^1 & \\ ev \swarrow & & \searrow pr \\ M & & \mathcal{L}M \end{array}$$

Transgression

$$\mathcal{T} : H^{k+1}(M) \rightarrow H^k(\mathcal{L}M), \quad \mathcal{T} = pr! \circ ev^*$$

$$(\mathcal{T}\omega)_x(v_1(\tau), \dots, v_k(\tau)) := \int_{S^1} d\tau \omega(v_1(\tau), \dots, v_k(\tau), \dot{x}(\tau))$$

Previously successfully applied: Lift ADHMN constr. to M-theory
CS, Papageorgakis&CS, Palmer&CS

Towards a Groupoid Quantization of \mathbb{R}^3

The manifold $\mathcal{L}\mathbb{R}^3$ comes with a natural symplectic structure.

Explicitly, this works as follows:

We start from \mathbb{R}^3 with **2-plectic form** $\varpi = \varepsilon_{ijk} dx^i \wedge dx^j \wedge dx^k$.

Transgression yields a symplectic form on loop space $\mathcal{L}\mathbb{R}^3$:

$$\omega = \oint d\tau \oint d\sigma \varepsilon_{ijk} \dot{x}^k(\tau) \delta(\tau - \sigma) \delta x^i(\tau) \wedge \delta x^j(\sigma)$$

Kernel of ω :

$$\iota_X(\mathcal{T}\varpi) = 0 \quad \text{for} \quad X = \oint d\rho \dot{x}^i(\rho) \frac{\delta}{\delta x^i(\rho)}$$

This vector field generates reparameterizations of the loops in $\mathcal{L}\mathbb{R}^3$.

We can therefore invert ω and obtain the **Poisson bracket**

$$\{f, g\} := \oint d\tau \oint d\rho \delta(\tau - \rho) \theta^{ijk} \frac{\dot{x}_k(\rho)}{|\dot{x}(\rho)|^2} \left(\frac{\delta}{\delta x^i(\tau)} f \right) \left(\frac{\delta}{\delta x^j(\rho)} g \right)$$

The Integrating Symplectic Groupoid of $\mathcal{L}\mathbb{R}^3$

To lowest order in θ , the integrating groupoid is trivially found.

Start: $M = \mathbb{R}^3$, Poisson structure $\theta^{i\sigma, j\rho} := \delta(\sigma - \rho)\theta^{ijk}\dot{x}_k(\sigma)$

① **Int. groupoid.**: $\Sigma = \mathcal{L}(T^*M) \cong T^*\mathcal{L}M$, coords. $(x^i(\sigma), y_i(\sigma))$

Groupoid structure on $T^*\mathcal{L}M$:

$$x^i + \frac{1}{2}\theta^{ijk}y_j\dot{x}_k \xrightarrow{(x^i(\sigma), y_i(\sigma))} x^i - \frac{1}{2}\theta^{ijk}y_j\dot{x}_k$$

2-nerve $\Sigma_2 = \mathcal{L}(\mathbb{R}^3 \times \mathbb{R}^3 \times \mathbb{R}^3)$, coords. $(x^i(\sigma), y_i(\sigma), y'_i(\sigma))$:

$$x^i + \theta^{ijk}(y_j + y'_j)\dot{x}_k \rightarrow x^i + \theta^{ijk}(y_j - y'_j)\dot{x}_k \rightarrow x^i - \theta^{ijk}(y_j + y'_j)\dot{x}_k$$

Symplectic structure:

$$\omega = \oint d\rho \oint d\sigma \delta(\rho - \sigma)\delta x^i(\rho) \wedge \delta y_i(\sigma)$$

t is Poisson map and $\partial^*\omega = 0$ only to lowest order in θ !

Groupoid Quantization of \mathbb{R}^3

The groupoid quantization can be extended to $\mathcal{L}\mathbb{R}^3$, reproducing M-theory results.

- ② **Prequantization:** L trivial line bundle over Σ
Connection ∇ with $F = 2\pi i\omega = 2\pi i\mathcal{T}\varpi$
- ③ **Polarization:** Induced by symplectic prepotential:

$$\vartheta = \oint d\rho x^i(\rho)\delta y_i(\rho)$$

- ④ **Twist element:** $\partial^*\vartheta = \sigma_0^{-1}d\sigma_0$ with

$$\sigma_0(y_i(\rho), y'_j(\sigma)) = -\frac{1}{2}\theta^{ijk} y_i(\rho)y'_j(\sigma)\dot{x}_k(\sigma)\delta(\rho - \sigma)$$

- ⑤ **Twisted polarized convolution algebra:** difficult, but we have

$$[x^i(\tau), x^j(\sigma)] = \theta^{ijk}\dot{x}_k(\tau)\delta(\tau - \sigma) + \mathcal{O}(\theta)$$

Agrees with one-form quantization of [Baez et al.](#)

Agrees with [Kawamoto](#), [Sasakura](#) and [Bergshoeff et al. \(2000\)](#)

Remains: Compute $\mathcal{O}(\theta)$ -corrections and compare to [KS](#)

Conclusions

Summary and Outlook.

Summary:

- ✓ **Groupoids** offer a nice approach to quantization
- ✓ Can be extended to **loop spaces/2-plectic manifolds**

Future directions:

- ▷ **Complete** the loop space picture, **other spaces** (S^3 , T^3).
- ▷ Extend quantization of \mathbb{R}^3 to **2-groupoid**.
- ▷ **Unify picture**: Higher Poisson structures? Courant algebroids?
- ▷ **Rewrite M2-brane models** using the new function algebras.

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