# Integrability and Geometric Quantization with Loop Spaces

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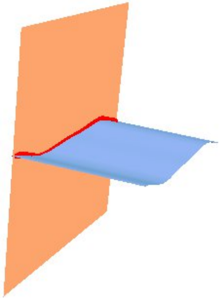
Based on the following papers:

- 1007.3301, 1206.0432
- with S Palmer: 1105.3904
- with R Szabo: 1211.0395

# (My) Motivation and Outline

Understanding a configuration of M-branes leads to questions invoking loop spaces.

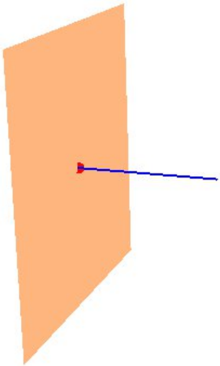


- An M2-brane can end on an M5-brane with 1-dimensional boundary, constant in time.
- Spatial boundary: self-dual string.
- M-theory makes this configuration smooth: M2-brane gains 3 quantized spatial dim.
- Describing this requires quantizing a 3-sphere.
- Approach: Quantize the loop space of $S^3$.
- Duality yielding integrability between
  - Description from the M2-brane perspective: Fields are operators on $\mathcal{H}_{S^3}$.
  - Description from the M5-brane perspective: Theory of self-dual strings.

Note: My loops are actually Brylinski's singular oriented knots.

# What is Known: The Reduced D-brane Picture

A D-brane configuration that should lift to the M-brane configuration is well understood.



- $k$ D1-branes ending on $N$ D3-brane.

| dim | 0 | 1 | 2 | 3 | ... | 6 |
|---|---|---|---|---|---|---|
| $D1$ | $\times$ | | | | | $\times$ |
| $D3$ | $\times$ | $\times$ | $\times$ | $\times$ | | |

- D3-brane perspective: Magnetic Monopole
  U($N$)-bundle with connection $\nabla$ over $\mathbb{R}^3$,
  Scalar fields $\Phi$ in adjoint of $\mathfrak{u}(N)$,

$$F = \star\nabla\Phi\ .$$

- D1-brane perspective: Nahm equation
  U($k$)-bundle with connection $\nabla$ over $(0,\infty)$,
  Scalar fields $X^{1,2,3}$ in adjoint of $\mathfrak{u}(k)$,

$$\nabla X^i + \varepsilon^{ijk}[X^j, X^k] = 0\ .$$

# Quantized Sphere from D1-D3-branes

A fuzzy funnel of quantized 2-spheres appears in the geometry.

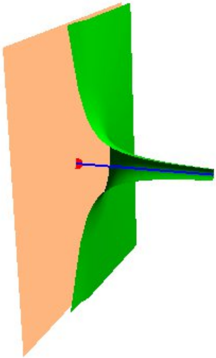
- D3-brane perspective: $F = \star\nabla\Phi$

  Dirac monopole with $\Phi \sim \frac{1}{r}$ on $\mathbb{R}^3$

- D1-brane perspective: $\nabla X^i + \varepsilon^{ijk}[X^j, X^k] = 0$
  - Gauge potential can be gauged away
  - Product ansatz: $X^i(s) = r(s)G^i$
  - Solution:

$$r(s) = \frac{1}{s}\ , \quad G^i = \varepsilon^{ijk}[G^j, G^k]$$

  - +irrep: Fuzzy/Geometrically Quantized Sphere



- "Smooth" transition between D1 and D3 via noncommutative geometry

# Lifting D1-D3-Branes to M2-M5-Branes

The lift to M-theory is performed by a T-duality and an M-theory lift

| IIB | 0 | 1 | 2 | 3 | 4 | 5 | 6 |
|---|---|---|---|---|---|---|---|
| $D1$ | × | | | | | | × |
| $D3$ | × | × | × | × | | | |

T-dualize along $x^5$:

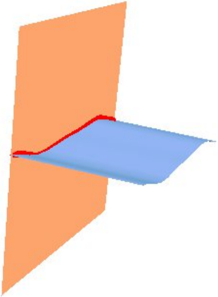
| IIA | 0 | 1 | 2 | 3 | 4 | 5 | 6 |
|---|---|---|---|---|---|---|---|
| $D2$ | × | | | | | × | × |
| $D4$ | × | × | × | × | | × | |

Interpret $x^4$ as M-theory direction:

| M | 0 | 1 | 2 | 3 | 4 | 5 | 6 |
|---|---|---|---|---|---|---|---|
| $M2$ | × | | | | | × | × |
| $M5$ | × | × | × | × | × | × | |

# What is Known: The M-brane Configuration

M2-branes ending on M5-branes yield a Nahm-type equation with a cubic term.



- 2 M2-branes ending on 1 M5-brane.

| dim | 0 | 1 | 2 | 3 | 4 | 5 | 6 |
|---|---|---|---|---|---|---|---|
| $M2$ | $\times$ | | | | | $\times$ | $\times$ |
| $M5$ | $\times$ | $\times$ | $\times$ | $\times$ | $\times$ | $\times$ | |

- M5-brane perspective: Self-Dual String
  2-form potential $B$ over $\mathbb{R}^4$ (abelian gerbe),
  Scalar field $\Phi$ with values in $\mathfrak{u}(1)$,

$$H := \mathrm{d}B = \star \mathrm{d}\Phi \ .$$

- M2-brane perspective: "Basu-Harvey" equation
  G-bundle with connection $\nabla$ over $(0, \infty)$,
  Scalar fields $X^{1,2,3,4}$ in some rep of G,

$$\nabla X^\mu + \varepsilon^{\mu\nu\kappa\lambda}[X^\nu, X^\kappa, X^\lambda] = 0 \ .$$

# Quantized 3-Sphere in M2-M5-Branes



In M-theory, the quantized 2-sphere becomes a quantized 3-sphere.

- M5-brane perspective: $H = \star \mathrm{d}\Phi$

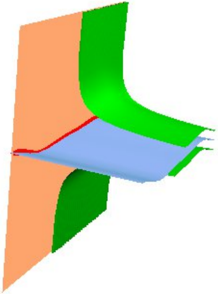
  Higher Dirac monopole with $\Phi \sim \frac{1}{r^2}$ on $\mathbb{R}^3$

- M2-brane perspective:
  $\nabla X^\mu + \varepsilon^{\mu\nu\kappa\lambda}[X^\nu, X^\kappa, X^\lambda] = 0$

  - Gauge potential can be gauged away
  - Product ansatz: $X^\mu(s) = r(s)G^\mu$
  - Solution:

$$r(s) = \frac{1}{\sqrt{s}}\ , \quad G^\mu = \varepsilon^{\mu\nu\kappa\lambda}[G^\nu, G^\kappa, G^\lambda]$$

  - Fuzzy/Geometrically Quantized $S^3$?



- "Smooth" transition between M2 and M5 via nonassociative geometry

### 3-Lie algebra (do not confuse with Lie 3-algebras)

$\mathcal{A}$ is a vector space, $[\cdot,\cdot,\cdot]$ trilinear+antisymmetric.

Satisfies a "3-Jacobi identity," the fundamental identity:

$$[A,B,[C,D,E]] = [[A,B,C],D,E] + [C,[A,B,D],E] + [C,D,[A,B,E]]$$

Filippov (1985)

Algebra of inner derivations closes due to fundamental identity

$$D : \mathcal{A}\wedge\mathcal{A} \to \mathsf{Der}(\mathcal{A}) =: \mathfrak{g}_\mathcal{A} \qquad D(A,B) \rhd C := [A,B,C]$$

- 3-algebras $\overset{1:1}{\longleftrightarrow}$ metric Lie algebras $\mathfrak{g} \cong \mathsf{Der}(\mathcal{A})$ faithful orthog. representations $V \cong \mathcal{A}$
  J Figueroa-O'Farrill et al., 0809.1086
- They form strict Lie 2-algebras. S Palmer & CS, 1203.5757
- M2-brane models are higher gauge theories.

- Duality in D1-D3-brane configuration tightly connected to integrability
- Can be described in 3 related ways:
  - Twistor description: Solutions from holomorphic cocycles
  - Nahm transform: Duality is Fourier-Mukai transform
  - Spectral curves
- What is known/conjectured for M2-M5-brane system

| | Twistors | Nahm transform | Spectral curves |
|---|---|---|---|
| NA-Gerbes | Yes[1] | WIP | ? |
| Loop space | WIP | Yes[2] | ? |

[1]: with Jurco/Wolf: 1205.3108, 1305.4870, 1403.7185
[2]: with Palmer: 1007.3301, 1105.3904

- Focus on Nahm transform with loop spaces.

# Goal: A Nahm Transform on Loop Space

Each ingredient of the Nahm transform should have a nice reduction.

- Assume M-theory description involves loop space.
- Reduction to string theory: Loop along M-theory direction.
- Lift the "gerby" self-dual string equation to loop space.
- Find a Nahm transform yielding solutions to this equation.
- Constraint: Reduction produces the known string formulas.
- see also: Gustavsson 0802.3456

# Abelian Gerbes and Loop Space

By going to loop space, one can reduce differential forms by one degree.



Consider the following double fibration:

$$
\begin{array}{ccc}
 & \mathcal{L}M \times S^1 & \\
 {\scriptstyle ev} \swarrow & & \searrow {\scriptstyle pr} \\
M & & \mathcal{L}M
\end{array}
$$

Identify $T\mathcal{L}M = \mathcal{L}TM$, then: $x \in \mathcal{L}M \Rightarrow \dot{x}(\tau) \in T\mathcal{L}M$

### Transgression

$$\mathcal{T} : \Omega^{k+1}(M) \to \Omega^k(\mathcal{L}M) \ , \quad v_i = \oint \mathrm{d}\tau \, v_i^\mu(\tau) \frac{\delta}{\delta x^\mu(\tau)} \in T\mathcal{L}M$$

$$(\mathcal{T}\omega)_x(v_1(\tau), \ldots, v_k(\tau)) := \oint_{S^1} \mathrm{d}\tau \, \omega(x(\tau))(v_1(\tau), \ldots, v_k(\tau), \dot{x}(\tau))$$

Nice properties: reparameterization invariant, chain map, ...

An abelian local gerbe over $M$ is a principal $\mathsf{U}(1)$-bundle over $\mathcal{L}M$.

# Transgressed Self-Dual Strings

By going to loop space, one can reduce differential forms by one degree.

Recall the self-dual string equation on $\mathbb{R}^4$: $H_{\mu\nu\kappa} = \varepsilon_{\mu\nu\kappa\lambda}\frac{\partial}{\partial x^\lambda}\Phi$

Its transgressed form is an equation for a 2-form $F$ on $\mathcal{L}\mathbb{R}^4$:

$$F_{(\mu\sigma)(\nu\rho)} = \delta(\sigma-\rho)\varepsilon_{\mu\nu\kappa\lambda}\dot{x}^\kappa(\tau)\,\frac{\partial}{\partial y^\lambda}\Phi(y)\Big|_{y=x(\tau)}$$

Extend to full non-abelian loop space curvature:

$$\begin{aligned} F^{\pm}_{(\mu\sigma)(\nu\tau)} =& \big(\varepsilon_{\mu\nu\kappa\lambda}\dot{x}^\kappa(\sigma)\nabla_{(\lambda\tau)}\Phi\big)_{(\sigma\tau)} \\ &\mp\big(\dot{x}_\mu(\sigma)\nabla_{(\nu\tau)}\Phi + \dot{x}_\nu(\sigma)\nabla_{(\mu\tau)}\Phi - \delta_{\mu\nu}\dot{x}^\kappa(\sigma)\nabla_{(\kappa\tau)}\Phi\big)_{[\sigma\tau]} \end{aligned}$$

where $\nabla_{(\mu\sigma)} := \oint \mathrm{d}\tau\,\delta x^\mu(\tau)\wedge\left(\frac{\delta}{\delta x^\mu(\tau)} + A_{(\mu\tau)}\right)$

- Check: Fix loops $x^\mu(\tau) = x_0^\mu + \kappa\delta^{\mu 4}\tau$ yields monopole eqn.
- Goal: Construct solutions to this equation.

Nahm transform: Instantons on $T^4 \mapsto$ instantons on $(T^4)^*$

Roughly here:

$T^4$: $\begin{cases} \text{3 rad. } 0 \\ \text{1 rad. } \infty \;:\; \text{D1 WV} \end{cases}$ and $(T^4)^*$: $\begin{cases} \text{3 rad. } \infty \;:\; \text{D3 WV} \\ \text{1 rad. } 0 \end{cases}$

Introduce (twisted) "Dirac operators":

$$\nabla\!\!\!/_{s,x} = -\mathbb{1}\tfrac{\mathrm{d}}{\mathrm{d}s} + \sigma^i \otimes (\mathrm{i}X^i + x^i \mathbb{1}_k)\ , \quad \bar{\nabla\!\!\!/}_{s,x} := \mathbb{1}\tfrac{\mathrm{d}}{\mathrm{d}s} + \sigma^i \otimes (\mathrm{i}X^i + x^i \mathbb{1}_k)$$

Properties:

$\Delta_{s,x} := \bar{\nabla\!\!\!/}_{s,x}\nabla\!\!\!/_{s,x} > 0$, $[\Delta_{s,x}, \sigma^i] = 0 \Leftrightarrow X^i$ satisfy Nahm eqn.

Normalized zero modes: $\bar{\nabla\!\!\!/}_{s,x}\psi_{s,x,\alpha} = 0$, $\mathbb{1} = \int_{\mathcal{I}} \mathrm{d}s\, \bar{\psi}_{s,x}\psi_{s,x}$ yield:

$$A_\mu := \int_{\mathcal{I}} \mathrm{d}s\, \bar{\psi}_{s,x} \frac{\partial}{\partial x^\mu} \psi_{s,x} \quad \text{and} \quad \Phi := -\mathrm{i}\int_{\mathcal{I}} \mathrm{d}s\, \bar{\psi}_{s,x}\, s\, \psi_{s,x}$$

This is a solution to the Bogomolny monopole equations!

# Examples: Dirac Monopoles

One can easily construct Dirac monopole solutions using the ADHMN construction.

Charge 1: Nahm eqn: $\partial_s X^i = 0$, so put $X^i = 0$. Zero mode:

$$\psi_+ = \mathrm{e}^{-sR} \frac{\sqrt{R+x^3}}{x^1 - \mathrm{i}x^2} \begin{pmatrix} x^1 - \mathrm{i}x^2 \\ R - x^3 \end{pmatrix}$$

Monopole solution:

$$\Phi^+ = -\frac{\mathrm{i}}{2R}\,,\ A_i^+ = \frac{\mathrm{i}}{2(x^1+x^2)^2}\left(x^2\left(1-\frac{x^3}{R}\right), -x^1\left(1-\frac{x^3}{R}\right), 0\right)$$

Charge 2: Nahm eqn. nontrivial. Choose:

$$X^i = -\frac{1}{s}T^i \quad \text{with} \quad T^i = \frac{\sigma^i}{2\mathrm{i}} = -\bar{T}^i$$

Resulting solution:

$$\Phi^+ = -\frac{\mathrm{i}}{R}\,,\ A_i^+ = \ldots$$

# Lift of the "Dirac Operator"

There is a natural lift of the Dirac operator to M-theory.

Type IIB (twisted):

$$\nabla^{\rm IIB}_{s,x} = -\mathbb{1}\frac{\rm d}{{\rm d}s} + \sigma^i({\rm i}X^i + x^i\mathbb{1}_k)$$

| IIB | 0 | 1 | 2 | 3 | 4 | 5 | 6 |
|---|---|---|---|---|---|---|---|
| $D1$ | × | | | | | | × |
| $D3$ | × | × | × | × | | | |

Type IIA (twisted):

$$\nabla^{\rm IIA}_{s,x} = -\gamma_5\mathbb{1}_k\frac{\rm d}{{\rm d}s} + \gamma^4\gamma^i(X^i - {\rm i}x^i)$$

| IIA | 0 | 1 | 2 | 3 | 4 | 5 | 6 |
|---|---|---|---|---|---|---|---|
| $D2$ | × | | | | | × | × |
| $D4$ | × | × | × | × | | × | |

M-theory (untwisted):

$$\nabla^{\rm M}_{s} = -\gamma_5\frac{\rm d}{{\rm d}s} + \tfrac{1}{2}\gamma^{\mu\nu}D(X^\mu, X^\nu)$$

| M | 0 | 1 | 2 | 3 | 4 | 5 | 6 |
|---|---|---|---|---|---|---|---|
| $M2$ | × | | | | | × | × |
| $M5$ | × | × | × | × | × | × | |

M-theory (twisted):

$$\nabla^{\rm M}_{s,x(\tau)} = -\gamma_5\frac{\rm d}{{\rm d}s} + \tfrac{1}{2}\gamma^{\mu\nu}\left(D(X^\mu, X^\nu) - {\rm i}\oint {\rm d}\tau\, x^\mu(\tau)\dot{x}^\nu(\tau)\right)$$

# Lifted ADHMN Construction

The lifted ADHMN construction yields solutions to the loop space self-dual string eqns.

Recall: $\Delta^{\rm IIB} := \bar{\nabla\!\!\!/}^{\rm IIB}\nabla\!\!\!/^{\rm IIB}$, $[\Delta^{\rm IIB},\sigma^i]=0 \Leftrightarrow X^i$ satisfy Nahm eqn.

Here: $\Delta^{\rm M} := \bar{\nabla\!\!\!/}^{\rm M}\nabla\!\!\!/^{\rm M}$, $[\Delta,\gamma^{\mu\nu}]=0 \Leftarrow X^\mu$ satisfy BH eqn.

Recall extended self-dual string equation on loop space:

$$F^\pm_{(\mu\sigma)(\nu\tau)} = \Big(\varepsilon_{\mu\nu\kappa\lambda}\dot{x}^\kappa(\sigma)\nabla_{(\lambda\tau)}\Phi\Big)_{(\sigma\tau)} \mp \Big(\dot{x}_\mu(\sigma)\nabla_{(\nu\tau)}\Phi + \dot{x}_\nu(\sigma)\nabla_{(\mu\tau)}\Phi - \delta_{\mu\nu}\dot{x}^\kappa(\sigma)\nabla_{(\kappa\tau)}\Phi\Big)_{[\sigma\tau]}$$

From normalized, $\mathcal{A}$-valued zero modes $\psi_{s,x(\tau)}$ of $\bar{\nabla\!\!\!/}^{\rm M}$ construct

$$A_{(\mu\tau)} = \int \mathrm{d}s\,\bar{\psi}_{s,x}\frac{\delta}{\delta x^\mu(\tau)}\psi_{s,x}~,\quad \Phi = -\mathrm{i}\int \mathrm{d}s\,\bar{\psi}_{s,x}\,s\,\psi_{s,x}$$

These fields solve the loop space self-dual string equation.

# Verification of the Construction

## Verifying the construction is rather straightforward.

The proof is easy and follows that of the ADHMN construction:

$$
\begin{aligned}
F^{ab}_{(\mu\sigma)(\nu\tau)} &\overset{[\cdot]}{=} 2\int_{\mathcal{I}} \mathrm{d}s\,(\delta_{(\mu\sigma)}\bar\psi^a_{s,x},\delta_{(\nu\tau)}\psi^b_{s,x}) + 2\int_{\mathcal{I}}\mathrm{d}s\int_{\mathcal{I}}\mathrm{d}t\,(\bar\psi^a_{s,x},\delta_{(\mu\sigma)}\psi^c_{s,x})(\bar\psi^c_{t,x},\delta_{(\nu\tau)}\psi^b_{t,x})\\
&\overset{[\cdot]}{=} -2\int_{\mathcal{I}}\mathrm{d}s\int_{\mathcal{I}}\mathrm{d}t\left(\delta_{(\mu\sigma)}\bar\psi^a_{s,x}\,,\,\left(\nabla\!\!\!\!/_{s,x}G_x(s,t)\bar{\nabla\!\!\!\!/}_{t,x}\right)\delta_{(\nu\tau)}\psi^b_{t,x}\right)\\
&\overset{[\cdot]}{=} 2\int_{\mathcal{I}}\mathrm{d}s\int_{\mathcal{I}}\mathrm{d}t\left(\bar\psi^a_{s,x},\left(\gamma^{\mu\kappa}\dot x^\kappa(\sigma)G_x(s,t)\gamma^{\nu\lambda}\dot x^\lambda(\tau)\right)\psi^b_{t,x}\right)\\
&\overset{[\cdot]}{=} 2\varepsilon_{\mu\nu\kappa\lambda}\int_{\mathcal{I}}\mathrm{d}s\int_{\mathcal{I}}\mathrm{d}t\left(\bar\psi^a_{s,x},\,G_x(s,t)\gamma^{\kappa\rho}\gamma_5\dot x^\lambda(\sigma)\dot x^\rho(\tau)\psi^b_{t,x}\right)\\
&\qquad + \int_{\mathcal{I}}\mathrm{d}s\int_{\mathcal{I}}\mathrm{d}t\left(\bar\psi^a_{s,x},\,G_x(s,t)\left(4\gamma^{\mu\lambda}\dot x^\nu(\sigma)\dot x^\lambda(\tau)-2\delta^{\mu\nu}\gamma^{\kappa\lambda}\dot x^\kappa(\sigma)\dot x^\lambda(\tau)\right)\psi^b_{t,x}\right)\\
&\overset{[\cdot]}{=} \mathrm{i}\varepsilon_{\mu\nu\kappa\lambda}\dot x^\kappa(\sigma)\int_{\mathcal{I}}\mathrm{d}s\left((\nabla_{(\lambda\tau)}\bar\psi_{s,x})^a,\,s\,\psi^b_{s,x}\right)+\left(\bar\psi^a_{s,x},\,s\,(\nabla_{(\lambda\tau)}\psi_{s,x})^b\right)\\
&\qquad \mp 2\mathrm{i}\dot x_\mu(\sigma)\int_{\mathcal{I}}\mathrm{d}s\left((\nabla_{(\nu\tau)}\bar\psi_{s,x})^a,\,s\,\psi^b_{s,x}\right)+\left(\bar\psi^a_{s,x},\,s\,(\nabla_{(\nu\tau)}\psi_{s,x})^b\right)\\
&\qquad \mp 2\mathrm{i}\dot x_\nu(\sigma)\int_{\mathcal{I}}\mathrm{d}s\left((\nabla_{(\mu\tau)}\bar\psi_{s,x})^a,\,s\,\psi^b_{s,x}\right)+\left(\bar\psi^a_{s,x},\,s\,(\nabla_{(\mu\tau)}\psi_{s,x})^b\right)\\
&\qquad \pm \mathrm{i}\delta_{\mu\nu}\dot x^\kappa(\sigma)\int_{\mathcal{I}}\mathrm{d}s\left((\nabla_{(\kappa\tau)}\bar\psi_{s,x})^a,\,s\,\psi^b_{s,x}\right)+\left(\bar\psi^a_{s,x},\,s\,(\nabla_{(\kappa\tau)}\psi_{s,x})^b\right)\\
&\overset{[\cdot]}{=} \left(\varepsilon_{\mu\nu\kappa\lambda}\dot x^\kappa(\sigma)\nabla_{(\lambda\tau)}\Phi\mp\dot x_\mu(\sigma)\nabla_{(\nu\tau)}\Phi\mp\dot x_\nu(\sigma)\nabla_{(\mu\tau)}\Phi\pm\delta_{\mu\nu}\dot x^\kappa(\sigma)\nabla_{(\kappa\tau)}\Phi\right)^{ab}.
\end{aligned}
$$

Reduction (cf. Mukhi/Papageorgakis, 0803.3218):

$$\langle X^4 \rangle = \frac{r}{\ell_p^{3/2}} e_4 = g_{\mathrm{YM}} e_4 \;, \quad \dot{x}^\mu(\tau) = \delta^{\mu 4} R$$

$$\begin{aligned} \nabla\!\!\!/^{\mathrm{M}} &= -\gamma_5 \frac{\mathrm{d}}{\mathrm{d}s} + \tfrac{1}{2}\gamma^{\mu\nu}\left( D(X^\mu, X^\nu) - \mathrm{i} \oint \mathrm{d}\tau x^\mu(\tau)\dot{x}^\nu(\tau) \right) \\ &\rightarrow \; -\gamma_5 \frac{\mathrm{d}}{\mathrm{d}s} + \tfrac{1}{2}\gamma^{\mu\nu}\left( D(X^\mu, X^\nu) - 2\pi\mathrm{i} R x_0^\mu \delta_4^\nu \right) \\ &= -\gamma_5 \frac{\mathrm{d}}{\mathrm{d}s} + R\gamma^{4i}\left( X^{i\alpha} D(e_\alpha, e_4) - \mathrm{i} x_0^i \right) + \ldots = \nabla\!\!\!/^{\mathrm{IIA}} + \ldots \end{aligned}$$

$$\frac{\mathrm{d}}{\mathrm{d}s} X^\mu = \tfrac{1}{3!}\varepsilon^{\mu\nu\kappa\lambda}[X^\nu, X^\kappa, X^\lambda] \quad \rightarrow \quad \frac{\mathrm{d}}{\mathrm{d}s} X^i = \tfrac{1}{2}\varepsilon^{ijk} R [X^j, X^k] + \ldots$$

$$F_{\mu\sigma,\nu\tau} = \varepsilon_{\mu\nu\kappa\lambda}\dot{x}^{\kappa\sigma}\delta_{\lambda\tau}\Phi_{\mathrm{SDS}} + \ldots \quad \rightarrow \quad F_{ij} = \varepsilon_{ijk}\frac{\partial}{\partial x^k} R\Phi_{\mathrm{SDS}} + \ldots$$

Charge 1: Choose again trivial Nahm data. Introduce

$$y^{\mu\nu} := \oint \mathrm{d}\tau\, x^{[\mu}(\tau)\dot{x}^{\nu]}(\tau)\ , \quad r_\pm^2 := \tfrac{1}{2}\sqrt{(y^{\mu\nu} \pm \tfrac{1}{2}\varepsilon_{\mu\nu\kappa\lambda}y^{\kappa\lambda})^2}$$

The zero modes of the adjoint of the Dirac operators are:

$$\psi_{s,x} \sim \mathrm{e}^{-r_-^2 s}\begin{pmatrix} \mathrm{i}\left(r_-^2 + y^{12} - y^{34}\right) \\ y^{13} + y^{24} + \mathrm{i}(y^{23} - y^{14}) \\ 0 \\ 0 \end{pmatrix}$$

The solution then reads as $\Phi = \frac{\mathrm{i}}{2r_-^2}$ and

$$A(\sigma) = \frac{\mathrm{i}}{2r_-^2(r_-^2 + (y^{12} - y^{34}))}\begin{pmatrix} \dot{x}^3(\sigma)(y^{23} - y^{14}) + \dot{x}^4(\sigma)(y^{13} + y^{24}) \\ \dot{x}^4(\sigma)(y^{23} - y^{14}) - \dot{x}^3(\sigma)(y^{13} + y^{24}) \\ \dot{x}^1(\sigma)(y^{14} - y^{23}) + \dot{x}^2(\sigma)(y^{13} + y^{24}) \\ \dot{x}^2(\sigma)(y^{14} - y^{23}) - \dot{x}^1(\sigma)(y^{13} + y^{24}) \end{pmatrix}$$

This is indeed a solution.

# Examples

## Our examples reproduce the expected solutions.

Charge 2:

Nahm data:

$$X^\mu = \frac{e_\mu}{\sqrt{2s}}\ , \quad e_\mu \text{ generate } \mathcal{A}$$

Solution:

$$\Phi(x) = \frac{\mathrm{i}}{r_-^2}$$

As expected: twice the charge of the case $k = 1$.

# Remarks on the Construction



The construction is very natural and behaves as expected.

- Can easily make the discussion non-abelian.
- Nahm eqn. and Basu-Harvey eqn. play analogous roles.
- Construction extends to general. Basu-Harvey eqn. (ABJM).
- One can construct many examples explicitly.
- It reduces nicely to ADHMN via the M2-Higgs mechanism.

CS, 1007.3301, S Palmer & CS, 1105.3904

Let's now come to the other problem:

Quantizing $S^3$.

The ingredients of the quantization of the 2-sphere need to be lifted.

Geometric quantization prescription: (e.g. fuzzy sphere)

Special symplectic manifold $(M,\omega)$ $\rightarrow$ line bundle $L$ with $(h,\nabla)$ over $M$ $\rightarrow$ Hilbert space $\mathscr{H}$: global holomorphic sections of $L$

Quantization map: $[\hat{f},\hat{g}]=\mathrm{i}\hbar\widehat{\{f,g\}}+\mathcal{O}(\hbar^2)$

Explicitly for the fuzzy sphere:

- Symplectic form: $\omega=k\mathrm{vol}$, $k\in\mathbb{N}$
- Line bundle: $L_k=\mathcal{O}(k)$
- Hilbert space: $\mathscr{H}_k=H^0(\mathbb{C}P^1,L_k)$

$$\mathscr{H}_k\cong\mathrm{span}(z_{\alpha_1}\dots z_{\alpha_k})\cong\mathrm{span}(\hat{a}^\dagger_{\alpha_1}\dots\hat{a}^\dagger_{\alpha_k}|0\rangle)$$

- Prequantization of $S^3$
  - $S^3$ as 2-plectic manifold (closed, non-degenerate 3-form)
  - $\varpi = k\mathrm{vol}$ defines prequantum gerbes
- Quantization map: need analogue of Poisson algebra
  - Use approach of Baez, Hoffnung, Rogers, 0808.0246
  - Hamiltonian one-forms: $\iota_{X_\alpha}\omega = \mathrm{d}\alpha$
  - $L = \mathcal{C}^\infty(M) \oplus \Omega^1_{\mathrm{Ham}}(M)$
  - $\pi_1 = \mathrm{d}$, $\pi_2(\alpha,\beta) = \iota_{X_\alpha}\iota_{X_\beta}\varpi$, $\pi_3(\alpha,\beta,\gamma) = \iota_{X_\alpha}\iota_{X_\beta}\iota_{X_\gamma}\varpi$.
  - Example: Heisenberg Lie 2-algebra of $\mathbb{R}^3$:
    $\xi_i = \frac{1}{2}\varepsilon_{ijk}x^j\mathrm{d}x^k$, $\pi_2(\xi_i,\xi_j) = \varepsilon_{ijk}\mathrm{d}x^k$, $\pi_3(\xi_i,\xi_j,\xi_k) = -\varepsilon_{ijk}$.
- 2-plectic manifold $\rightarrow$ symplectic loop space (Transgression).
- Compatibility: $\mathcal{T}\{\alpha,\beta\}_M = \{\mathcal{T}\alpha,\mathcal{T}\beta\}_{\mathcal{L}M}$
- Try to quantize loop space instead:
  - Prequantum gerbe becomes prequantum line bundle $L$
  - There is a natural Kähler structure on $\mathcal{L}S^3$
  - Hilbert space: global holomorphic sections of $L$

# Groupoid Quantization of $\mathcal{L}\mathbb{R}^3$

The loop space of $\mathbb{R}^3$ carries a Kähler structure.



- Quantizing dual of Lie(G) $\rightarrow$ convolution algebra of G
- $T^*M$ of a Poisson manifold is a Lie algebroid
- Lift quantization to Lie algebroids/groupoids

Hawkins (math.SG/0612363)

- Use this to quantize $\mathcal{L}\mathbb{R}^3$:
  - 2-plectic form
    $\varpi = \theta_{ijk}\mathrm{d}x^i \wedge \mathrm{d}x^j \wedge \mathrm{d}x^k := \theta^{-1}\varepsilon_{ijk}\mathrm{d}x^i \wedge \mathrm{d}x^j \wedge \mathrm{d}x^k$
  - Symplectic form on loop space from transgression:
    $$\omega = \oint \mathrm{d}\tau \oint \mathrm{d}\sigma\ \theta_{ijk}\dot{x}^k(\tau)\delta(\tau-\sigma)\ \delta x^i(\tau) \wedge \delta x^j(\sigma)$$
  - Kernel of $\omega$: Vector fields generating reparameterizations
    $$\iota_X(\mathcal{T}\varpi) = 0 \quad \text{for} \quad X = \oint \mathrm{d}\rho\ \dot{x}^i(\rho)\,\frac{\delta}{\delta x^i(\rho)}$$
  - Poisson bracket on knot space of $\mathbb{R}^3$:
    $$\{f,g\} := \oint \mathrm{d}\tau \oint \mathrm{d}\rho\,\delta(\tau-\rho)\,\theta^{ijk}\,\frac{\dot{x}_k(\rho)}{|\dot{x}(\rho)|^2}\left(\frac{\delta}{\delta x^i(\tau)}f\right)\left(\frac{\delta}{\delta x^j(\rho)}g\right)$$

- Going through the machinery, we end up with

$$[\hat{x}^i(\tau), \hat{x}^j(\sigma)] = \theta^{ijk} \frac{\hat{\dot{x}}_k(\tau)}{|\hat{\dot{x}}(\tau)|^2} \delta(\tau - \sigma) + \mathcal{O}(\theta^2)$$

- Satisfies compatibility with 2-plectic approach
- Agrees with Kawamoto, Sasakura and Bergshoeff et al. (2000)

# Details of Quantization of $\mathcal{L}S^3$

The definition of holomorphic sections is straightforward.



- 2-plectic form $\varpi_{S^3} = \frac{1}{2\pi^2}\, \sin^2\theta^1\, \sin\theta^2\, \mathrm{d}\theta^1 \wedge \mathrm{d}\theta^2 \wedge \mathrm{d}\theta^3$.
- Prequantum gerbe with Dixmier-Douady class $k\varpi_{S^3}$.
- Construct corresponding connective structure.
- Cover loop space $\mathcal{L}S^3$ by $U_a = \mathcal{L}(S^3\{a\})$, $a \in S^3$.
- Connection on $L \to \mathcal{L}S^3$: transgress connective structure, e.g.

$$\mathcal{A}_+ := \mathcal{T}B_+ = -\frac{\mathrm{i}\,k}{2} \oint \mathrm{d}\tau \left(\theta^{1\tau} - \frac{1}{2}\sin(2\theta^{1\tau})\right) \sin\theta^{2\tau} \left(\dot\theta^{2\tau}\,\delta\theta^{3\tau} - \dot\theta^{3\tau}\,\delta\theta^{2\tau}\right)$$

- Complex structure:

$$\mathcal{J}_{S^3} \frac{\delta}{\delta\theta^{i\tau}} = \tilde\varepsilon_{ijk} \frac{\dot\theta^{k\tau}}{|\dot\theta^\tau|} g^{jl} \frac{\delta}{\delta\theta^{l\tau}}$$

- Antiholomorphic part of the connection:

$$\nabla^{0,1}_+ = \frac{1}{2} \oint \mathrm{d}\tau\ \delta\theta^{i\tau} \left( \frac{\delta}{\delta\theta^{i\tau}} + \mathrm{i}\,\tilde\varepsilon_{ijk} \frac{\dot\theta^{k\tau}}{|\dot\theta^\tau|} g^{jl} \frac{\delta}{\delta\theta^{l\tau}} \right) + \mathcal{A}^{0,1}_+$$

# Details of Quantization of $\mathcal{L}S^3$

## Constructing all holomorphic sections is difficult.

- Can show: hol. sections of $L$ $\Leftrightarrow$ hol. functions on $\mathcal{L}\mathbb{R}^3$.
- Constructing hol. functions non-trivial.
- $\Rightarrow$ Method by Lempert, also Drinfeld/LeBrun using twistors

Consider the space $\mathbb{C}P^1 \times \mathbb{R}^3$ with coordinates

$$\lambda_\alpha\ , \quad \alpha = 1,2\ , \quad \lambda^{\alpha\tau} := \varepsilon^{\alpha\beta}\,\lambda_{\beta\tau} \quad \text{and} \quad \hat{\lambda}^{\alpha\tau} := \varepsilon^{\alpha\beta}\,\overline{\lambda}^{\,\beta\tau}$$

$$\left(x^{\alpha\beta}\right) = \left(x^{\beta\alpha}\right) = \begin{pmatrix} x^1 + \mathrm{i}\,x^2 & -x^3 \\ -x^3 & -x^1 + \mathrm{i}\,x^2 \end{pmatrix}$$

- $\dot{x}^{\alpha\beta\tau}$ is of the form $\dot{x}^{\alpha\beta\tau} = \lambda^{\alpha\tau}\,\hat{\lambda}^{\beta\tau} + \hat{\lambda}^{\alpha\tau}\,\lambda^{\beta\tau}$, $\lambda \in \mathbb{C}P^1$
- $\tilde{x} = (x(\tau), \lambda(\tau)) : S^1 \to \mathbb{C}P^1 \times \mathbb{R}^3$, $\tilde{x} \in \mathcal{K}_T(\mathbb{C}P^1 \times \mathbb{R}^3)$.
- One can show that $\tilde{x}^* f$ is hol. if $f$ hol. on $\mathcal{K}_T(\mathbb{C}P^1 \times \mathbb{R}^3)$
- Constructing holomorphic functions on $\mathcal{K}_T(\mathbb{C}P^1 \times \mathbb{R}^3)$:
  - Take $(1,0)$-form $\alpha$ on $\mathcal{O}_{\mathbb{C}P^1}(1) \oplus \mathcal{O}_{\mathbb{C}P^1}(1) \supset \mathbb{C}P^1 \times \mathbb{R}^3$.
  - Then the following map is holomorphic for $y \in \mathcal{K}_T(\mathbb{C}P^1 \times \mathbb{R}^3)$:

$$y \;\longmapsto\; f(y) = (\mathcal{T}\alpha)_y = \oint \mathrm{d}\tau\ \iota_{\dot{y}}(ev^*\alpha)\ .$$

# Details of Quantization of $\mathcal{L}S^3$

One can find explicit examples for holomorphic sections.

Example:

- Consider $\alpha = \frac{x^{1\alpha}\,\lambda_\alpha\,\mathrm{d}\left(x^{2\beta}\,\lambda_\beta\right)}{(\lambda_1)^2}$
- This yields the function ($\dot{x}^{\alpha\beta\tau} = \lambda^{\alpha\tau}\,\hat{\lambda}^{\beta\tau}$)

$$f(x) = \oint \mathrm{d}\tau\ \frac{x^{1\alpha\tau}\,\lambda_{\alpha\tau}\,\left(\dot{x}^{2\beta\tau}\,\lambda_{\beta\tau} + x^{2\beta\tau}\,\dot{\lambda}_{\beta\tau}\right)}{(\lambda_{1\tau})^2}\ .$$

- Reduction to loops $\dot{x}^{1\tau} = \dot{x}^{2\tau} = 0 \neq \dot{x}^{3\tau}$, $\lambda^1 = \hat{\lambda}^2 = 0 = \dot{\lambda}^\alpha$:

$$f(x) = \left(x^1 + \mathrm{i}\,x^2\right)\ \oint \mathrm{d}\tau\ \hat{\lambda}^{1\tau}\,\lambda^{2\tau} = \left(x^1 + \mathrm{i}\,x^2\right)V\ ,$$

  where $V$ is some constant volume factor.
- This is a rough reduction to section of $\mathcal{O}(1)$.

Remaining Problem: Construct all holomorphic functions on $\mathcal{L}\mathbb{R}^3$.

# Wishlist

A number of questions that would be nice to have an answer to...



- What is an intrinsic definition for the transgression of a self-dual gerbe over $\mathbb{R}^{1,5}$?
- What are interesting representations of $\{\gamma^\mu(\sigma), \gamma^\nu(\tau)\} = 2\eta^{\mu\nu}\delta(\sigma - \tau)$?
- What is the Dirac operator on $\mathcal{L}\mathbb{R}^{1,5}$ and/or $\mathcal{L}(T^1 \times T^5)$?

Summary:

- ✓ Loop space description of self-dual strings
- ✓ Generalized construction of loop space SDS solutions
- ✓ Beginnings of quantization of $\mathcal{L}S^3$

Future directions:

- ▷ Twistor spaces of loop spaces
- ▷ Full duality with loop spaces
- ▷ Quantization with higher Hilbert spaces

# Integrability and Geometric Quantization with Loop Spaces

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