Symmetries from Derived Brackets

Abstract

Generalized Complex Geometry brings together complex and symplectic geometry. The key underlying mathematical structure is the standard Courant algebroid, which is most conveniently understood as a symplectic Lie 2-algebroid. Each symplectic Lie n-algebroid comes with an associated Lie n-algebra of symmetries which is readily constructed from derived brackets. These categorified symmetry algebras make symplectic Lie n-algebroids particular interesting for various applications in differential geometry and theoretical physics. I will review the description of Lie n-algebroids as symplectic Q-manifolds as well as the relevant derived bracket construction. I will then discuss some interesting special cases and show how they relate to symmetries of categorified principal U(1)-bundles. I will close with a brief discussion with an example relevant in Double Field Theory.

Literature:

- arXiv:1611.02772 (with A. Deser)
- arXiv:1803.01659 (with A. Deser)

1. Motivation

Old example:

- Symplectomorphisms $\leftrightarrow$ Canonical Transformations
- (M,ω) $\quad f\in\mathcal{C}^\infty(M)$ $\;\rightarrow\;$ X_f via $\iota_{X_f}\omega = \mathrm{d}f$.
- X_f generates symplectomorphism: $\mathcal{L}_{X_f}\omega = \iota_{X_f}\mathrm{d}\omega + \mathrm{d}\iota_{X_f}\mathrm{d}\omega = \mathrm{d}^2 f = 0$
- $X_f(g)=\{f,g\}$ or:
  $\pi=\frac{1}{2}\pi^{\mu\nu}\frac{\partial}{\partial x^\mu}\wedge\frac{\partial}{\partial x^\nu}$: Poisson tensor
  Schouten bracket on polyvectors:
  $[X_1\wedge\cdots\wedge X_m, Y_1\wedge\cdots\wedge Y_n]_{\mathrm{Sch}} := \pm\sum[X_i,Y_j]X_1\wedge\ldots\hat{X}_i\ldots\wedge X_m\wedge Y_1\ldots\hat{Y}_j\ldots\wedge Y_n$.
  Then: $X_f(g)=[[\pi,f],g]_{\mathrm{Sch}}$. This is a derived bracket

Modern reformulation:

- $T^*[1]M$, coordinates $\overset{0}{x^\mu}$, $\overset{1}{\xi_\mu}$, $\mathcal{C}^\infty(T^*[1]M)\cong$ polyvector fields

- $\omega = \mathrm{d}x^\mu \wedge \mathrm{d}\xi_\mu$ induces Poisson bracket $\{-,-\}$ = Schouten bracket
- $\mathcal{Q} = \frac{1}{2}\pi^{\mu\nu}\xi_\mu\xi_\nu$, $\{\mathcal{Q},\mathcal{Q}\} = 0 \Leftrightarrow \pi$ Poisson tensor on M
- $\{f,g\} = \{\{\mathcal{Q},f\},g\}$ for $f,g \in \mathcal{C}^\infty_0(T^*[1]M)$

Idea:

- Many structures used in physics, in particular symmetries obtained from derived brackets on symplectic graded manifolds with homological vecor field.

Reason:

- Symmetry $\rightarrow$ Group action $\alpha : G \times X \rightarrow X$.
- Action Lie groupoid: $G \times X \rightrightarrows X$, $x \xrightarrow{(g,x)} \alpha(g,x)$
- infinitesimally: action Lie algebroid, action L_∞-algebroid
- Action respecting a structure (metric, symplectic form, ...) $\rightarrow$ symplectic L_∞-algebroid
- Symplectic L_∞-algebroids come with an L_∞-algebra from antisymmetrized derived brackets

2. Symplectic L_∞-algebroids

Graded manifolds (mostly $\mathbb{N}$, extendable to $\mathbb{Z}$)

- Actually: locally ringed spaces, simpler picture sufficient:
- Algebra of functions $\mathcal{C}^\infty(M) = \mathcal{C}^\infty_0(M) \oplus \mathcal{C}^\infty_1(M) \oplus \dots$
- Generated by coordinates of degree $0, 1, 2, \dots$; polynomial in degrees > 0
- Example: $T[1]M$, coordinates $\overset{0}{x^\mu}$, $\overset{1}{\xi^\mu}$
  $\mathcal{C}^\infty_n(M) = \{f = \sum f_{\mu_1\dots\mu_n}(x)\xi^{\mu_1}\dots\xi^{\mu_n}\}$, $\mathcal{C}^\infty(M) = \Omega^n(M)$

Q-manifolds

- graded manifold M with vector field Q of degree 1, $Q^2 = 0$
- Note $(\mathcal{C}^\infty(M), Q)$: differential graded algebra (dga)
- Examples:
  $T[1]M$, $Q = \xi^\mu \frac{\partial}{\partial x^\mu}$
  $\mathfrak{g}[1]$, coords $\overset{1}{\xi^\alpha}$, $Q = -\frac{1}{2}f^\alpha_{\beta\gamma}\xi^\beta\xi^\gamma\frac{\partial}{\partial \xi^\alpha}$, $Q^2 = 0 \Leftrightarrow$ Jacobi identity

- Q-mfds. *concentrated* in degrees $0,\dots,n$: n-term L_∞-algebroids/Lie n-algebroids
- Q-mfds. *concentrated* in degrees $1,\dots,n$: n-term L_∞-algebras/Lie n-algebras

Symplectic Q-manifolds

- (M,Q), ω symplectic form on M compatible with Q: $\mathcal{L}_Q\omega=0$
- $\mathbb{N}$-/$\mathbb{Z}$-degree of ω: *degree of Q-manifold*
- For degree $\neq -1$: Q is Hamiltonian: $Q=\{\mathcal{Q},-\}$
- Example: $T^*[1]M$, coordinates $\overset{0}{x}{}^\mu$, $\overset{1}{\xi}_\mu$, $\mathcal{C}^\infty(M)=\mathfrak{X}^\bullet(M)$ (polyvector fields), $\omega=\mathrm{d}x^\mu\wedge\mathrm{d}\xi_\mu$, $Q=\pi^{\mu\nu}\xi_\mu\frac{\partial}{\partial x^\mu}$, $\mathcal{Q}=x^\mu\xi_\mu$

Important hierarchies

- Ševera: symplectic Lie n-algebroid of degree ...
  0: symplectic manifold ($Q=0$)
  1: Poisson manifold (above example, π Poisson due to $Q^2=0$)
  2: Courant algebroids...
- Vinogradov n-algebroids: $\mathcal{V}_n(M)=T^*[n]T[1]M$, coords $\overset{0}{x}{}^\mu,\overset{1}{\xi}{}^\mu,\overset{n-1}{\zeta_\mu},\overset{n}{p}_\mu$
  $\omega=\mathrm{d}x^\mu\wedge\mathrm{d}p_\mu+\mathrm{d}\xi^\mu\wedge\mathrm{d}\zeta_\mu$ $\mathcal{Q}=\xi^\mu p_\mu$, $Q=\xi^\mu\frac{\partial}{\partial x^\mu}+p_\mu\frac{\partial}{\partial\zeta_\mu}$
  More general: twist by closed n-form ϖ: $\mathcal{Q}=\xi^\mu p_\mu+\frac{1}{n!}\varpi_{\mu_1\dots\mu_n}\xi^{\mu_1}\dots\xi^{\mu_n}$
  n-gerbe $\leftrightarrow\mathcal{V}_{n+1}(M)$
  $\mathcal{V}_2(M)$: contains/descibes exact Courant algebroid $E=TM\oplus T^*M$ of GG

Additional structure

- Every symplectic L_∞-algebroid comes with associated Lie n-algebra:
  $$\begin{aligned}\mathsf{L}(\mathcal{M})\;:=&\quad\mathcal{C}^\infty_0(\mathcal{M})\to\quad\mathcal{C}^\infty_1(\mathcal{M})\to\;\dots\to\;\mathcal{C}^\infty_{n-2}(\mathcal{M})\to\;\mathcal{C}^\infty_{n-1}(\mathcal{M})\\ =&\quad\mathsf{L}_{n-1}(\mathcal{M})\to\;\mathsf{L}_{n-2}(\mathcal{M})\to\;\dots\to\;\mathsf{L}_1(\mathcal{M})\to\;\mathsf{L}_0(\mathcal{M})\quad,\end{aligned}$$
  with brackets
  $$\begin{aligned}\mu_1(\ell)&=Q\ell-\delta\ell\ ,\\ \mu_2(\ell_1,\ell_2)&=\tfrac{1}{2}\big(\{\delta\ell_1,\ell_2\}\pm\{\delta\ell_2,\ell_1\}\big)\ ,\\ \mu_3(\ell_1,\ell_2,\ell_3)&=-\tfrac{1}{12}\big(\{\{\delta\ell_1,\ell_2\},\ell_3\}\pm\dots\big)\ ,\end{aligned}$$
  where we abbreviated
  $$\delta(\ell)=\begin{cases}Q\ell & \ell\in\mathcal{C}^\infty_0(\mathcal{M})\ ,\\ 0 & \text{else}\ .\end{cases}$$
- Note: On $\mathcal{V}_n(M)$, $\mathsf{L}_0=\mathfrak{X}(M)\oplus\wedge^{n-1}(M)$
  L describes symmetries (diffeos+gauge) of gravity + n-form potential

3. Applications

Lie bracket as derived bracket:

- Take any $\mathcal{V}_n(M)$, then $\mathsf{L}_0 \supset \mathfrak{X}(M)$
- $[X,Y] = \mu_2(X,Y) = \frac{1}{2}(\{QX,Y\} - \{QY,X\}) = \{QX,Y\}$

Dorfman and Courant brackets as derived brackets:

- On Courant algebroid $TM \oplus T^*M$:
  Dorfman bracket: $\nu_2(X+\alpha, Y+\beta) := [X,Y] + \mathcal{L}_X\beta - \iota_Y \mathrm{d}\alpha$
  Courant bracket: $\mu_2(X+\alpha, Y+\beta) := [X,Y] + \mathcal{L}_X\beta - \mathcal{L}_Y\alpha - \frac{1}{2}\mathrm{d}(\iota_X\beta - \iota_Y\alpha)$
- Work on $\mathcal{V}_2(M)$
  Dorfman: $\nu_2(X+\alpha, Y+\beta) := \{\{\mathcal{Q}, X+\alpha\}, Y+\beta\}$
  Courant: $\mu_2(X+\alpha, Y+\beta) := \frac{1}{2}\big(\{\{\mathcal{Q}, X+\alpha\}, Y+\beta\} - (X+\alpha) \leftrightarrow (Y+\beta)\big)$

More generally: Supergravity

- theory of gravity coupled to various p-form curvatures/$p-2$-gerbes
- Einstein-Maxwell: gravity + 2-form curvature
  Symmetries: $\mathsf{L}(\mathcal{V}_1(M)) = \mathsf{L}_0 = \underbrace{\mathfrak{X}(M) \oplus \mathcal{C}^\infty(M)}_{\text{diffeo+gauge}}$
  What do actually higher powers of ζ do?
- GG (NS-sector of type II): gravity + 3-form curvature
  Symmetries: $\mathsf{L}(\mathcal{V}_2(M)) = \mathsf{L}_1 \to \mathsf{L}_0 = \underbrace{\mathcal{C}^\infty(M)}_{\text{gauge of gauge}} \to \underbrace{\mathfrak{X}(M) \oplus \Omega^1(M)}_{\text{diffeo+gauge}}$
  Application: Characterize T-duality transformations
- 11d SUGRA: "$TM \oplus \wedge^2 T^*M \oplus \wedge^5 T^*M$":
  Symmetries: $\mathsf{L}(\mathcal{V}_3(M) \oplus \mathbb{R}[3])$, $\mathsf{L}_0 = \mathfrak{X}(M) \oplus \Omega^2(M) \oplus \Omega^5(M)$
- etc

Also: BV-quantization $\omega = \mathrm{d}\phi^i \wedge \mathrm{d}\phi_i^\dagger$, $\mathcal{Q} = S_{\mathrm{BV}}$
What is associated L_∞-algebra?

4. Relation to multisymplectic geometry

Multisymplectic manifold

- (M,ϖ), $\varpi \in \Omega^n(M)$ such that $\mathrm{d}\varpi = 0$, $\iota_X\varpi = 0 \Leftrightarrow X = 0$

"Poisson L_∞-algebra" $\Pi(M,\varpi)$ of observables on (M,ϖ)

- Vector field X correspond to Hamiltonian $n-1$ form α: $\iota_X\varpi = \mathrm{d}\alpha$
- Lie $(n-1)$-algebra from $\mu_2(\alpha,\beta) \sim \iota_{X_\alpha}\iota_{X_\beta}\varpi$, $\mu_3(\alpha,\beta,\gamma) \sim \iota_{X_\alpha}\iota_{X_\beta}\iota_{X_\gamma}\varpi$, ...

Theorem (for $n=2$: Rogers, 2010):

- $\Pi(M,\varpi)$ categorically $\cong$ to $\mathsf{L}(\mathcal{V}_{n-1}(M,\varpi))$ after restrict. L_0 to $(X_\alpha, \alpha = \iota_{X_\alpha}\varpi)$

5. Relation to higher gauge theory

2-plectic structure on a compact Lie group G (Baez, Rogers, 2009)

- $k(-,[-,-])$ on $\mathfrak{g} = \mathsf{Lie}(\mathsf{G})$ yields left-invariant 2-plectic form ϖ
- all left-invariant forms are Hamiltonian: $\Omega^1_{\mathrm{Ham}}(M) \supset \mathfrak{g}^* \cong \mathfrak{g}$
- $\Pi(\mathsf{G},\varpi)|_{\mathbb{R}\to\mathfrak{g}^*}$ is (isomorphic to) the *String Lie 2-algebra* of $\mathfrak{g}$
- By above reasoning: also $\mathcal{V}_2(\mathsf{G},\varpi)$, restricted to (X_α,α).
- String Lie 2-algebra: particularly suitable for higher gauge theory

6. Generalizations

More general brackets

- Heterotic supergravity: $\mathcal{V}_2(M)$, but deformed Poisson:

$$\{f,g\}_{\alpha'} = \{f,g\} + \alpha' \left(f \frac{\overleftarrow{\partial}}{\partial x^\mu} \frac{\overleftarrow{\partial}}{\partial \zeta_\nu} \frac{\overrightarrow{\partial}}{\partial x^\nu} \frac{\overrightarrow{\partial}}{\partial \zeta_\mu} g \right) .$$

- This yields

$$\begin{aligned} \langle X,Y\rangle_{\alpha'} &= \langle X,Y\rangle_0 - \alpha'\partial_\mu X^\nu \partial_\nu Y^\mu = \{X,Y\}_{\alpha'} \\ \nu_2(X,Y)_{\alpha'} &= \nu_2(X,Y)_0 - \alpha'\partial_\mu Y^\nu \mathrm{d}\partial_\nu X^\mu = \{QX,Y\}_{\alpha'} \\ \mu_2(X,Y)_{\alpha'} &= \mu_2(X,Y)_0 - \tfrac{1}{2}\alpha'(\partial_\mu Y^\nu \mathrm{d}\partial_\nu X^\mu - \partial_\mu X^\nu \mathrm{d}\partial_\nu Y^\mu) \\ &= \tfrac{1}{2}\big(\{QX,Y\}_{\alpha'} - \{QY,X\}_{\alpha'}\big) \end{aligned}$$

- Q: What are admissible brackets?

More general notion of algebroid

- Double Field Theory: $\mathcal{V}_2(T^*M)|_{\text{polarized}}$

- $Q^2 \neq 0$ only on subspace (solution to "section condition")
- similar to BRST complex $Q^2 = 0$ only on-shell
- Application: T-duality cov. description, T-dualities: canonical transformation