

# Describing M5-branes using Loop Spaces

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Based on:

- CS, [arXiv:1007.3301](#), CMP ...
- C. Papageorgakis and CS, [arXiv:1103.6192](#), JHEP ...
- S. Palmer and CS, [arXiv:1105.3904](#)

# Motivation

Find an algorithm for the construction of self-dual string solutions

- **Effective description of M2-branes** proposed in 2007.
- This created lots of interest:  
**BLG-model**: >467 citations, **ABJM-model**: >605 citations
- Inspired by an idea by **Basu-Harvey**:  
Propose a lift of the **Nahm eqn.** describing D1-D3-system:  
**Basu-Harvey eqn.** describes M2-M5-brane system
- **Nahm** transform:  
go from Nahm eqn. to **Bogomolny monopole eqn.**  
switch perspective from D1-brane to D3-brane
- **Is there a lift for this Nahm transform?**  
go from BH eqn. to self-dual string eqn.  
switch perspective from M2-brane to M5-brane
- Such a transform would open up interesting possibilities:  
eff. description of **M5-branes**, new **integrable** structures, ...

(Alternative Approach: [Gustavsson, 0802.3456](#))

# Outline

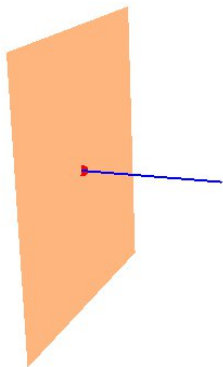
We will discuss the construction of monopoles and lift each ingredient to M-theory.

- Basu-Harvey lift of the Nahm equation and 3-Lie algebras
- Monopoles and self-dual strings
- Principal  $U(1)$ -bundles, abelian gerbes and loop space
- ADHMN construction and its lift
- Examples of self-dual string solutions
- Non-abelian tensor multiplet on loop space

# D1-D3-Branes and the Nahm Equation

D1-branes ending on D3-branes can be described by the Nahm equation.

dim	0	1	2	3	...	6
D1	×					×
D3	×	×	×	×		



$N$  D1-branes ending on D3-branes:

A **Monopole** appears.

$X^i \in \mathfrak{u}(N)$ : transverse fluctuations

**Nahm equation:** ( $s = x^6$ )

$$\frac{d}{ds} X^i + \varepsilon^{ijk} [X^j, X^k] = 0$$

Note  $SO(3)$ -invariance.

**Solution:**  $X^i = r(s)G^i$  with

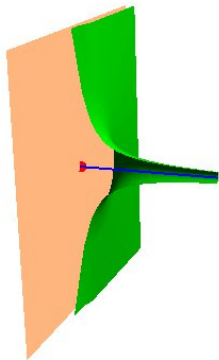
$$r(s) = \frac{1}{s}, \quad G^i = \varepsilon^{ijk} [G^j, G^k]$$

Nahm, Diaconescu, Tsimpis

# D1-D3-Branes and the Nahm Equation

The D1-branes end on the D3-branes by forming a fuzzy funnel.

dim	0	1	2	3	...	6
D1	×					×
D3	×	×	×	×		



**Solution:**  $X^i = r(s)G^i$

$$r(s) = \frac{1}{s}, \quad G^i = \varepsilon^{ijk} [G^j, G^k]$$

The D1-branes form a **fuzzy funnel**:

$G^i$  form irrep of  $\mathfrak{su}(2)$ :

coordinates on fuzzy sphere  $S_F^2$

D1-worldvolume polarizes:  $2d \rightarrow 4d$

Myers

# Lifting D1-D3-Branes to M2-M5-Branes

The lift to M-theory is performed by a T-duality and an M-theory lift

IIB	0	1	2	3	4	5	6
D1	×						×
D3	×	×	×	×			

T-dualize along  $x^5$ :

IIA	0	1	2	3	4	5	6
D2	×					×	×
D4	×	×	×	×		×	

Interpret  $x^4$  as M-theory direction:

M	0	1	2	3	4	5	6
M2	×					×	×
M5	×	×	×	×	×	×	

# The Basu-Harvey Lift of the Nahm Equation

M2-branes ending on M5-branes yield a Nahm equation with a cubic term.

M	0	1	2	3	4	5	6
M2	×					×	×
M5	×	×	×	×	×	×	

A **Self-Dual String** appears.

Substitute **SO(3)**-inv. **Nahm eqn.**

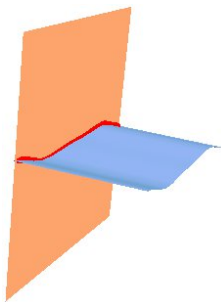
$$\frac{d}{ds} X^i + \varepsilon^{ijk} [X^j, X^k] = 0$$

by the **SO(4)**-invariant equation

$$\frac{d}{ds} X^\mu + \varepsilon^{\mu\nu\rho\sigma} [X^\nu, X^\rho, X^\sigma] = 0$$

**Solution:**  $X^\mu = r(s)G^\mu$  with

$$r(s) = \frac{1}{\sqrt{s}}, \quad G^\mu = \varepsilon^{\mu\nu\rho\sigma} [G^\nu, G^\rho, G^\sigma]$$



Basu, Harvey, hep-th/0412310

# The Basu-Harvey Lift of the Nahm Equation

M2-branes ending on M5-branes yield a Nahm equation with a cubic term.

M	0	1	2	3	4	5	6
M2	×					×	×
M5	×	×	×	×	×	×	

**Solution:**  $X^\mu = r(s)G^\mu$

$$r(s) = \frac{1}{\sqrt{s}}, \quad G^\mu = \varepsilon^{\mu\nu\rho\sigma}[G^\nu, G^\rho, G^\sigma]$$

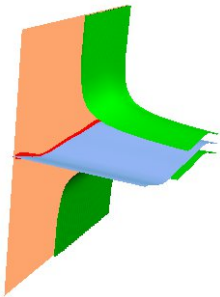
The M2-branes form a **fuzzy funnel**:

$G^\mu$  form a rep of  $\mathfrak{so}(4)$ :

coordinates on fuzzy sphere  $S_F^3$

M2-worldvolume polarizes:  $3d \rightarrow 6d$

**What is this triple bracket?**





# What is the algebra behind the triple bracket?

In analogy with Lie algebras, we can introduce 3-Lie algebras.

$$\frac{d}{ds} X^\mu + [A_s, X^\mu] + \varepsilon^{\mu\nu\rho\sigma} [X^\nu, X^\rho, X^\sigma] = 0, \quad X^\mu \in \mathcal{A}$$

Trivial:  $\mathcal{A}$  is a **vector space**,  $[\cdot, \cdot, \cdot]$  **trilinear+antisymmetric**.

▷ Gauge transformations from **inner derivations**:

Triple bracket forms a map  $D : \mathcal{A} \wedge \mathcal{A} \rightarrow \text{Der}(\mathcal{A}) =: \mathfrak{g}_{\mathcal{A}}$  via

$$D(A, B) \triangleright C := [A, B, C]$$

Demand a “3-Jacobi identity,” the **fundamental identity**:

$$\begin{aligned} D(A, B) \triangleright (D(C, D) \triangleright E) &:= [A, B, [C, D, E]] \\ &= [[A, B, C], D, E] + [C, [A, B, D], E] + [C, D, [A, B, E]] \end{aligned}$$

The inner derivations form indeed a **Lie algebra**:

$$[D(A, B), D(C, D)] \triangleright E := D(A, B) \triangleright (D(C, D) \triangleright E) - D(C, D) \triangleright (D(A, B) \triangleright E)$$

Bracket closes due to **fundamental identity**.

# Real and Hermitian 3-Algebras

There are two natural generalizations of 3-Lie algebras.

## Real 3-Algebras

Almost same as 3-Lie algebras: triple bracket only antisymmetric in first two slots.

S. Cherkis, CS, 0807.0808

## Hermitian 3-Algebras

Start from a complex vector space  $\mathcal{A}$ . Bracket  $[\cdot, \cdot; \cdot]$  satisfies

$$[A, B; C] = -[B, A; C], \quad [\alpha A, B; C] := \alpha[A, B; C], \quad [A, B; \alpha C] := \alpha^* [A, B; C]$$
$$[[C, D; E], A; B] - [[C, A; B], D; E] - [C, [D, A; B]; E] + [C, D; [E, B; A]] = 0$$

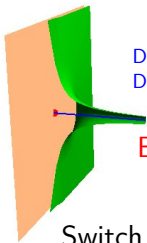
Aharony, Bergman, Jafferis, Maldacena, 0806.1218

Bagger, Lambert, 0807.0163

All our constructions generalize to these two types of 3-algebras.

# Monopoles and Self-Dual Strings

Lifting monopoles to M-theory yields self-dual strings.



	0	1	2	3	4	5	6
D1	×						×
D3	×	×	×	×			

**BPS configuration!**

Switch perspective:  $D1 \rightarrow D3$ :

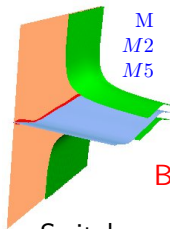
**Bogomolny monopole** eqn.:

$$F_{ij} = \varepsilon_{ijk} \nabla_k \Phi \Rightarrow \nabla^2 \Phi = 0$$

Single D3: **Dirac monopole**

$$\Phi = \frac{1}{r} \Rightarrow r(s) = \frac{1}{s}$$

$\Rightarrow$  **matching profile!**



	M	0	1	2	3	4	5	6
M2	×						×	×
M5	×	×	×	×	×	×	×	

**BPS configuration!**

Switch perspective:  $M2 \rightarrow M5$ :

**Self-dual string** eqn.:

$$H_{\mu\nu\rho} = \varepsilon_{\mu\nu\rho\sigma} \partial_\sigma \Phi \Rightarrow \partial^2 \Phi = 0$$

Only single M5 known:

$$\Phi = \frac{1}{r^2} \Rightarrow r(s) = \frac{1}{\sqrt{s}}$$

$\Rightarrow$  **matching profile!**

# Dirac Monopoles and Principal $U(1)$ -bundles

Dirac monopoles are described by principal  $U(1)$ -bundles over  $S^2$ .

Manifold  $M$  with cover  $(U_i)_i$ . **Principal  $U(1)$ -bundle** over  $M$ :

$$F \in \Omega^2(M, \mathfrak{u}(1)) ,$$

$$A_{(i)} \in \Omega^1(U_i, \mathfrak{u}(1)) \text{ with } F = dA_{(i)}$$

$$g_{ij} \in \Omega^0(U_i \cap U_j, U(1)) \text{ with } A_{(i)} - A_{(j)} = d \log g_{ij}$$

Consider monopole in  $\mathbb{R}^3$ , **but** describe it on  $S^2$  around monopole:

$S^2$  with patches  $U_+, U_-$ ,  $U_+ \cap U_- \sim S^1$ :  $g_{+-} = e^{-in\phi}$ ,  $n \in \mathbb{Z}$

$$c_1 = \frac{i}{2\pi} \int_{S^2} F = \frac{i}{2\pi} \int_{S^1} A^+ - A^- = \frac{1}{2\pi} \int_0^{2\pi} n d\phi = n$$

**Monopole charge:  $n$**

# Self-Dual Strings and Abelian Gerbes

Self-dual strings are described by abelian gerbes.

Manifold  $M$  with cover  $(U_i)_i$ . **Abelian (local) gerbe** over  $M$ :

$$H \in \Omega^3(M, \mathfrak{u}(1)) ,$$

$$B_{(i)} \in \Omega^2(U_i, \mathfrak{u}(1)) \text{ with } H = dB_{(i)}$$

$$A_{(ij)} \in \Omega^1(U_i \cap U_j, \mathfrak{u}(1)) \text{ with } B_{(i)} - B_{(j)} = dA_{ij}$$

$$h_{ijk} \in \Omega^0(U_i \cap U_j \cap U_k, \mathfrak{u}(1)) \text{ with } A_{(ij)} - A_{(ik)} + A_{(jk)} = dh_{ijk}$$

Note: Local gerbe: principal  $U(1)$ -bundles on intersections  $U_i \cap U_j$ .

Consider  $S^3$ , patches  $U_+, U_-, U_+ \cap U_- \sim S^2$ : **bundle over  $S^2$**

Reflected in:  $H^2(S^2, \mathbb{Z}) \cong H^3(S^3, \mathbb{Z}) \cong \mathbb{Z}$

$$\frac{i}{2\pi} \int_{S^3} H = \frac{i}{2\pi} \int_{S^2} B_+ - B_- = \dots = n$$

**Charge of self-dual string:  $n$**

# Abelian Gerbes and Loop Space

By going to loop space, one can reduce differential forms by one degree.

Consider the following **double fibration**:

$$\begin{array}{ccc} & \mathcal{L}M \times S^1 & \\ ev \swarrow & & \searrow pr \\ M & & \mathcal{L}M \end{array}$$

Identify  $T\mathcal{L}M = \mathcal{L}TM$ , then:  $x \in \mathcal{L}M \Rightarrow \dot{x}(\tau) \in T\mathcal{L}M$

## Transgression

$$\mathcal{T} : \Omega^{k+1}(M) \rightarrow \Omega^k(\mathcal{L}M), \quad v_i = \oint d\tau v_i^\mu(\tau) \frac{\delta}{\delta x^\mu(\tau)} \in T\mathcal{L}M$$

$$(\mathcal{T}\omega)_x(v_1(\tau), \dots, v_k(\tau)) := \oint_{S^1} d\tau \omega(v_1(\tau), \dots, v_k(\tau), \dot{x}(\tau))$$

An abelian local gerbe over  $M$  is a principal  $U(1)$ -bundle over  $\mathcal{L}M$ .

# Transgressed Self-Dual Strings

By going to loop space, one can reduce differential forms by one degree.

Recall the **self-dual string equation** on  $\mathbb{R}^4$ :

$$H_{\mu\nu\kappa} = \varepsilon_{\mu\nu\kappa\lambda} \frac{\partial}{\partial x^\lambda} \Phi$$

Its **transgressed form** is an equation for a **2-form**  $F$  on  $\mathcal{L}\mathbb{R}^4$ :

$$F(X, Y) = \oint d\tau \varepsilon_{\mu\nu\kappa\lambda} \dot{x}^\kappa(\tau) \frac{\partial}{\partial x^\lambda} \Phi(x) X^\mu(\tau) Y^\nu(\tau)$$

or

$$F_{(\mu\sigma)(\nu\rho)} = \delta(\sigma - \rho) \varepsilon_{\mu\nu\kappa\lambda} \dot{x}^\kappa(\tau) \frac{\partial}{\partial x^\lambda} \Phi(x)$$

Extend to **non-abelian** full loop space curvature:

$$F_{(\mu\sigma)(\nu\tau)}^\pm = (\varepsilon_{\mu\nu\kappa\lambda} \dot{x}^\kappa(\sigma) \nabla_{(\lambda\tau)} \Phi)_{(\sigma\tau)} \\ \mp (\dot{x}_\mu(\sigma) \nabla_{(\nu\tau)} \Phi + \dot{x}_\nu(\sigma) \nabla_{(\mu\tau)} \Phi - \delta_{\mu\nu} \dot{x}^\kappa(\sigma) \nabla_{(\kappa\tau)} \Phi)_{[\sigma\tau]}$$

where  $\nabla_{(\mu\sigma)} := \oint d\tau \delta x^\mu(\tau) \wedge \left( \frac{\delta}{\delta x^\mu(\tau)} + A_{(\mu\tau)} \right)$

**Goal:** Construct solutions to this equation.

# The ADHMN Construction

There is a map between and solutions to the Nahm equations.

**Nahm transform:** Instantons on  $T^4 \mapsto$  instantons on  $(T^4)^*$

Roughly here:

$$T^4: \begin{cases} 3 \text{ rad. } 0 \\ 1 \text{ rad. } \infty : \text{ D1 WV} \end{cases} \quad \text{and} \quad (T^4)^*: \begin{cases} 3 \text{ rad. } \infty : \text{ D3 WV} \\ 1 \text{ rad. } 0 \end{cases}$$

Introduce (twisted) “**Dirac operators**”:

$$\nabla_{s,x} = -\mathbb{1} \frac{d}{ds} + \sigma^i \otimes (iX^i + x^i \mathbb{1}_k), \quad \bar{\nabla}_{s,x} := \mathbb{1} \frac{d}{ds} + \sigma^i \otimes (iX^i + x^i \mathbb{1}_k)$$

Properties:

$$\Delta_{s,x} := \bar{\nabla}_{s,x} \nabla_{s,x} > 0, \quad [\Delta_{s,x}, \sigma^i] = 0 \Leftrightarrow X^i \text{ satisfy Nahm eqn.}$$

Normalized **zero modes**:  $\bar{\nabla}_{s,x} \psi_{s,x,\alpha} = 0$ ,  $\mathbb{1} = \int_{\mathcal{I}} ds \bar{\psi}_{s,x} \psi_{s,x}$  yield:

$$A_\mu := \int_{\mathcal{I}} ds \bar{\psi}_{s,x} \frac{\partial}{\partial x^\mu} \psi_{s,x} \quad \text{and} \quad \Phi := -i \int_{\mathcal{I}} ds \bar{\psi}_{s,x} s \psi_{s,x}$$

This is a solution to the Bogomolny monopole equations!



# Examples: Dirac monopoles

One can easily construct Dirac monopole solutions using the ADHMN construction.

**Charge 1:** Nahm eqn:  $\partial_s X^i = 0$ , so put  $X^i = 0$ . Zero mode:

$$\psi_+ = e^{-sR} \frac{\sqrt{R+x^3}}{x^1 - ix^2} \begin{pmatrix} x^1 - ix^2 \\ R - x^3 \end{pmatrix}$$

Monopole solution:

$$\Phi^+ = -\frac{i}{2R}, \quad A_i^+ = \frac{i}{2(x^1+x^2)^2} \left( x^2 \left( 1 - \frac{x^3}{R} \right), -x^1 \left( 1 - \frac{x^3}{R} \right), 0 \right)$$

**Charge 2:** Nahm eqn. nontrivial. Choose:

$$X^i = -\frac{1}{s} T^i \quad \text{with} \quad T^i = \frac{\sigma^i}{2i} = -\bar{T}^i$$

Resulting solution:

$$\Phi^+ = -\frac{i}{R}, \quad A_i^+ = \dots$$

# Lift of the “Dirac operator”

There is a natural lift of the Dirac operator to M-theory.

Type IIB (twisted):

$$\nabla_{s,x}^{\text{IIB}} = -\mathbb{1} \frac{d}{ds} + \sigma^i (iX^i + x^i \mathbb{1}_k)$$

<b>IIB</b>	0	1	2	3	4	5	6
<i>D1</i>	×						×
<i>D3</i>	×	×	×	×			

Type IIA (twisted):

$$\nabla_{s,x}^{\text{IIA}} = -\gamma_5 \mathbb{1}_k \frac{d}{ds} + \gamma^4 \gamma^i (X^i - ix^i)$$

<b>IIA</b>	0	1	2	3	4	5	6
<i>D2</i>	×					×	×
<i>D4</i>	×	×	×	×		×	

M-theory (untwisted):

$$\nabla_s^{\text{M}} = -\gamma_5 \frac{d}{ds} + \frac{1}{2} \gamma^{\mu\nu} D(X^\mu, X^\nu)$$

<b>M</b>	0	1	2	3	4	5	6
<i>M2</i>	×					×	×
<i>M5</i>	×	×	×	×	×	×	

M-theory (twisted):

$$\nabla_{s,x(\tau)}^{\text{M}} = -\gamma_5 \frac{d}{ds} + \frac{1}{2} \gamma^{\mu\nu} \left( D(X^\mu, X^\nu) - i \oint d\tau x^\mu(\tau) \dot{x}^\nu(\tau) \right)$$

# Lifted ADHMN Construction

The lifted ADHMN construction yields solutions to the loop space self-dual string eqns.

Recall:  $\Delta^{\text{IIB}} := \bar{\nabla}^{\text{IIB}} \nabla^{\text{IIB}}$ ,  $[\Delta^{\text{IIB}}, \sigma^i] = 0 \Leftrightarrow X^i$  satisfy Nahm eqn.

Here:  $\Delta^{\text{M}} := \bar{\nabla}^{\text{M}} \nabla^{\text{M}}$ ,  $[\Delta, \gamma^{\mu\nu}] = 0 \Leftarrow X^\mu$  satisfy BH eqn.

Recall **extended self-dual string equation** on loop space:

$$F_{(\mu\sigma)(\nu\tau)}^\pm = (\varepsilon_{\mu\nu\kappa\lambda} \dot{x}^\kappa(\sigma) \nabla_{(\lambda\tau)} \Phi)_{(\sigma\tau)} \\ \mp (\dot{x}_\mu(\sigma) \nabla_{(\nu\tau)} \Phi + \dot{x}_\nu(\sigma) \nabla_{(\mu\tau)} \Phi - \delta_{\mu\nu} \dot{x}^\kappa(\sigma) \nabla_{(\kappa\tau)} \Phi)_{[\sigma\tau]}$$

From normalized,  **$\mathcal{A}$ -valued** zero modes  $\psi_{s,x(\tau)}$  of  $\bar{\nabla}^{\text{M}}$  construct

$$A_{(\mu\tau)} = \int ds \bar{\psi}_{s,x} \frac{\delta}{\delta x^\mu(\tau)} \psi_{s,x}, \quad \Phi = -i \int ds \bar{\psi}_{s,x} s \psi_{s,x}$$

**These fields solve the loop space self-dual string equation.**

# Verification of the Construction

Verifying the construction is rather straightforward.

The proof is easy and follows that of the ADHMN construction:

$$\begin{aligned}
 F_{(\mu\sigma)(\nu\tau)}^{ab} &\stackrel{[1]}{=} 2 \int_{\mathcal{I}} ds (\delta_{(\mu\sigma)} \bar{\psi}_{s,x}^a, \delta_{(\nu\tau)} \psi_{s,x}^b) + 2 \int_{\mathcal{I}} ds \int_{\mathcal{I}} dt (\bar{\psi}_{s,x}^a, \delta_{(\mu\sigma)} \psi_{s,x}^c) (\bar{\psi}_{t,x}^c, \delta_{(\nu\tau)} \psi_{t,x}^b) \\
 &\stackrel{[2]}{=} -2 \int_{\mathcal{I}} ds \int_{\mathcal{I}} dt (\delta_{(\mu\sigma)} \bar{\psi}_{s,x}^a, (\nabla_{s,x} G_x(s,t) \bar{\kappa}_{t,x}^b) \delta_{(\nu\tau)} \psi_{t,x}^b) \\
 &\stackrel{[3]}{=} 2 \int_{\mathcal{I}} ds \int_{\mathcal{I}} dt (\bar{\psi}_{s,x}^a, (\gamma^{\mu\kappa} \dot{x}^\kappa(\sigma) G_x(s,t) \gamma^{\nu\lambda} \dot{x}^\lambda(\tau)) \psi_{t,x}^b) \\
 &\stackrel{[4]}{=} 2\varepsilon_{\mu\nu\kappa\lambda} \int_{\mathcal{I}} ds \int_{\mathcal{I}} dt (\bar{\psi}_{s,x}^a, G_x(s,t) \gamma^{\kappa\rho} \gamma_5 \dot{x}^\lambda(\sigma) \dot{x}^\rho(\tau) \psi_{t,x}^b) \\
 &\quad + \int_{\mathcal{I}} ds \int_{\mathcal{I}} dt (\bar{\psi}_{s,x}^a, G_x(s,t) (4\gamma^{\mu\lambda} \dot{x}^\nu(\sigma) \dot{x}^\lambda(\tau) - 2\delta^{\mu\nu} \gamma^{\kappa\lambda} \dot{x}^\kappa(\sigma) \dot{x}^\lambda(\tau)) \psi_{t,x}^b) \\
 &\stackrel{[5]}{=} i\varepsilon_{\mu\nu\kappa\lambda} \dot{x}^\kappa(\sigma) \int_{\mathcal{I}} ds ((\nabla_{(\lambda\tau)} \bar{\psi}_{s,x})^a, s \psi_{s,x}^b) + (\bar{\psi}_{s,x}^a, s (\nabla_{(\lambda\tau)} \psi_{s,x})^b) \\
 &\quad \mp 2i \dot{x}_\mu(\sigma) \int_{\mathcal{I}} ds ((\nabla_{(\nu\tau)} \bar{\psi}_{s,x})^a, s \psi_{s,x}^b) + (\bar{\psi}_{s,x}^a, s (\nabla_{(\nu\tau)} \psi_{s,x})^b) \\
 &\quad \mp 2i \dot{x}_\nu(\sigma) \int_{\mathcal{I}} ds ((\nabla_{(\mu\tau)} \bar{\psi}_{s,x})^a, s \psi_{s,x}^b) + (\bar{\psi}_{s,x}^a, s (\nabla_{(\mu\tau)} \psi_{s,x})^b) \\
 &\quad \pm i \delta_{\mu\nu} \dot{x}^\kappa(\sigma) \int_{\mathcal{I}} ds ((\nabla_{(\kappa\tau)} \bar{\psi}_{s,x})^a, s \psi_{s,x}^b) + (\bar{\psi}_{s,x}^a, s (\nabla_{(\kappa\tau)} \psi_{s,x})^b) \\
 &\stackrel{[6]}{=} (\varepsilon_{\mu\nu\kappa\lambda} \dot{x}^\kappa(\sigma) \nabla_{(\lambda\tau)} \Phi \mp \dot{x}_\mu(\sigma) \nabla_{(\nu\tau)} \Phi \mp \dot{x}_\nu(\sigma) \nabla_{(\mu\tau)} \Phi \pm \delta_{\mu\nu} \dot{x}^\kappa(\sigma) \nabla_{(\kappa\tau)} \Phi)^{ab} .
 \end{aligned}$$

# Reduction to the ADHMN Construction

The lift reduces in the expected way to the ADHMN construction.

On  $\mathcal{L}S^3 \subset \mathcal{L}\mathbb{R}^4$ :  $x^\mu x^\mu = \dot{x}^\mu \dot{x}^\mu = R^2$ ,  $x^\mu \dot{x}^\mu = 0$ .

Reduction (cf. [Mukhi/Papageorgakis, 0803.3218](#)):

$$\langle X^4 \rangle = \frac{r}{\ell_p^{3/2}} e_4 = g_{\text{YM}} e_4, \quad \dot{x}^4(\tau) = R \Rightarrow \dot{x}^i(\tau) = x^4(\tau) = 0$$

$$\nabla^{\text{M}} = -\gamma_5 \frac{d}{ds} + \frac{1}{2} \gamma^{\mu\nu} \left( D(X^\mu, X^\nu) - i \oint d\tau x^\mu(\tau) \dot{x}^\nu(\tau) \right)$$

$$\rightarrow -\gamma_5 \frac{d}{ds} + \frac{1}{2} \gamma^{\mu\nu} (D(X^\mu, X^\nu) - 2\pi i R x_0^\mu \delta_4^\nu)$$

$$= -\gamma_5 \frac{d}{ds} + R \gamma^{4i} (X^{i\alpha} D(e_\alpha, e_4) - i x_0^i) + \dots = \nabla^{\text{IIA}} + \dots$$

$$\frac{d}{ds} X^\mu = \frac{1}{3!} \varepsilon^{\mu\nu\rho\sigma} [X^\nu, X^\rho, X^\sigma] \rightarrow \frac{d}{ds} X^i = \frac{1}{2} \varepsilon^{ijk} R [X^j, X^k] + \dots$$

$$F_{\mu\nu} = \varepsilon_{\mu\nu\rho\sigma} \dot{x}^\rho \frac{\partial}{\partial x^\sigma} \Phi_{\text{SDS}} \rightarrow F_{ij} = \varepsilon_{ijk} \frac{\partial}{\partial x^k} R \Phi_{\text{SDS}} + \dots$$

# Examples

Our examples reproduce the expected solutions.

**Charge 1:** Choose again **trivial Nahm data**. Introduce

$$y^{\mu\nu} := \oint d\tau x^{[\mu}(\tau)\dot{x}^{\nu]}(\tau), \quad r_{\pm}^2 := \frac{1}{2}\sqrt{(y^{\mu\nu} \pm \frac{1}{2}\varepsilon_{\mu\nu\kappa\lambda}y^{\kappa\lambda})^2}$$

The **zero modes** of the adjoint of the Dirac operators are:

$$\psi_{s,x} \sim e^{-r_-^2 s} \begin{pmatrix} i(r_-^2 + y^{12} - y^{34}) \\ y^{13} + y^{24} + i(y^{23} - y^{14}) \\ 0 \\ 0 \end{pmatrix}$$

The solution then reads as  $\Phi = \frac{i}{2r_-^2}$  and

$$A(\sigma) = \frac{i}{2r_-^2(r_-^2 + (y^{12} - y^{34}))} \begin{pmatrix} \dot{x}^3(\sigma)(y^{23} - y^{14}) + \dot{x}^4(\sigma)(y^{13} + y^{24}) \\ \dot{x}^4(\sigma)(y^{23} - y^{14}) - \dot{x}^3(\sigma)(y^{13} + y^{24}) \\ \dot{x}^1(\sigma)(y^{14} - y^{23}) + \dot{x}^2(\sigma)(y^{13} + y^{24}) \\ \dot{x}^2(\sigma)(y^{14} - y^{23}) - \dot{x}^1(\sigma)(y^{13} + y^{24}) \end{pmatrix}$$

**This is indeed a solution.**

# Examples

Our examples reproduce the expected solutions.

Charge 2:

Nahm data:

$$X^\mu = \frac{e_\mu}{\sqrt{2s}}, \quad e_\mu \text{ generate } \mathcal{A}$$

Solution:

$$\Phi(x) = \frac{i}{r_-^2}$$

As expected: twice the charge of the case  $k = 1$ .

# Remarks

Our lift of the ADHMN construction is very natural and rather straightforward.

- The **lift of the Dirac operator** was natural considering the corresponding brane configurations.
- It is natural to go to **loop space** to describe self-dual strings.
- It can be trivially rendered **nonabelian**:
- The construction nicely involves the **Basu-Harvey equation**.
- It **reduces naturally** to the ADHMN construction.
- The construction does produce **transgressed self-dual strings**.
- It extends to **real** and **hermitian 3-algebras** ( $\rightarrow$  ABJM model)



# The non-abelian tensor multiplet

A recently proposed 3-Lie algebra valued tensor-multiplet implies a transgression.

Recall the **transgression map**:

$$(\mathcal{T}\omega)_x(v_1(\tau), \dots, v_k(\tau)) := \int_{S^1} d\tau \omega(v_1(\tau), \dots, v_k(\tau), \dot{x}(\tau))$$

Equations found by **Lambert, Papageorgakis, 1007.2982**:

$$\nabla^2 X^I - \frac{i}{2}[\bar{\Psi}, \Gamma_\nu \Gamma^I \Psi, C^\nu] - [X^J, C^\nu, [X^J, C_\nu, X^I]] = 0$$

$$\Gamma^\mu \nabla_\mu \Psi - [X^I, C^\nu, \Gamma_\nu \Gamma^I \Psi] = 0$$

$$\nabla_{[\mu} H_{\nu\lambda\rho]} + \frac{1}{4}\varepsilon_{\mu\nu\lambda\rho\sigma\tau}[X^I, \nabla^\tau X^I, C^\sigma] + \frac{i}{8}\varepsilon_{\mu\nu\lambda\rho\sigma\tau}[\bar{\Psi}, \Gamma^\tau \Psi, C^\sigma] = 0$$

$$F_{\mu\nu} - D(C^\lambda, H_{\mu\nu\lambda}) = 0$$

$$\nabla_\mu C^\nu = D(C^\mu, C^\nu) = 0$$

$$D(C^\rho, \nabla_\rho X^I) = D(C^\rho, \nabla_\rho \Psi) = D(C^\rho, \nabla_\rho H_{\mu\nu\lambda}) = 0$$

**Factorization** of  $C^\rho = C\dot{x}^\rho$ . Here, **3-Lie algebra transgression**:

$$(\mathcal{T}\omega)_x(v_1(\tau), \dots, v_k(\tau)) := \int_{S^1} d\tau D(\omega(v_1(\tau), \dots, v_k(\tau), \dot{x}(\tau)), C)$$

# The non-abelian tensor multiplet on loop space

The corresponding equations can all be rewritten on loop space.

Transgression of matter fields (not in [Huang, Huang, 1008.3834](#))

$$\dot{X}^I(x(\tau)) := R D(C, X^I(x(\tau))), \quad \dot{\Psi}(x(\tau)) := \Gamma^\rho \dot{x}_\rho D(C, \Psi(x(\tau)))$$

**Equations of motion** (SYM-like):

$$\begin{aligned} \nabla^2 \dot{X}^I + \frac{i}{2} \frac{1}{R} \dot{x}^\nu [\bar{\dot{\Psi}}, \Gamma_\nu \Gamma^I \dot{\Psi}] + [\dot{X}^J, [\dot{X}^J, \dot{X}^I]] &= 0, \\ \frac{1}{R} \Gamma^{\mu\nu} \dot{x}_\nu \nabla_\mu \dot{\Psi} - \Gamma^I [\dot{X}^I, \dot{\Psi}] &= 0, \\ \nabla_\mu \dot{F}^{\mu\nu} + [\dot{X}^I, \nabla^\nu \dot{X}^I] + \frac{i}{2} \left( [\bar{\dot{\Psi}}, \Gamma^\nu \dot{\Psi}] - \frac{2}{R^2} \dot{x}^\sigma \dot{x}^\nu [\bar{\dot{\Psi}}, \Gamma_\sigma \dot{\Psi}] \right) &= 0, \end{aligned}$$

**Supersymmetry transformations** (SYM-like):

$$\begin{aligned} \delta \dot{X}^I &= \frac{1}{R} i \bar{\varepsilon} \Gamma^I \dot{x}^\rho \Gamma_\rho \dot{\Psi}, \\ \delta \dot{A}_\mu &= \frac{1}{R^2} i \bar{\varepsilon} \Gamma_{\mu\lambda} \Gamma_\rho \dot{x}^\lambda \dot{x}^\rho \dot{\Psi}, \\ \delta \dot{\Psi} &= \frac{1}{R} \Gamma^{\nu\mu} \dot{x}_\nu \Gamma^I \nabla_\mu \dot{X}^I \varepsilon + \frac{1}{2} \Gamma_{\mu\nu} \dot{F}^{\mu\nu} \varepsilon - \frac{1}{2} \Gamma^{IJ} [\dot{X}^I, \dot{X}^J] \varepsilon, \\ \delta C^\mu &= 0 \end{aligned}$$

# Remarks

The loop space tensor multiplet fits well into the picture.

- Get **SYM theory on loop space** from the tensor multiplet
- **BPS equation** is essentially the self-dual string equation on loop space:

$$F_{\mu\nu} = \varepsilon_{\mu\nu\rho\sigma} \dot{x}^\rho \nabla^\sigma X^6$$

- We used a reduced loop space derivative:  $\partial_\mu := \oint d\tau \frac{\delta}{\delta x^\mu(\tau)}$
- **Right direction**, more work in progress to get rid of  $\dot{x}$  etc.

# Conclusions

## Summary and Outlook.

### Summary:

- ✓ Reformulation of self-dual string equation on **loop space**
- ✓ **Generalized ADHMN construction** for self-dual string
- ✓ **Explicit constructions** in many cases
- ✓ Reformulate **non-abelian tensor multiplet** eqns. on loop space
- ✓ **Partially** generalized ADHMN construction

### Future directions:

- ▷ Find **(2,0)-theory** using loop space
- ▷ Study **classical integrability** in more detail (diff. dims., ...)
- ▷ Loop space **quantization** of  $S^3$ , cf. Dirac operator
- ▷ Quantization of  $S^3$  via **gerbes** and **2-groupoids**

# Describing M5-branes using Loop Spaces

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