Marginal Deformations and 3-Algebra Structures

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Based on:

- S. A. Cherkis, CS, PRD 78 (2008) 066019
- S. A. Cherkis, V. Dotsenko, CS, PRD 79 (2009) 086002
- N. Akerblom, CS and M. Wolf, Nucl. Phys. B (2009) ...

Outline

- Introductory part
  - The Nahm equation or D1-D3 branes
  - The Basu-Harvey equation or M2-M5 branes
  - 3-Lie algebras
  - Stacks of flat M2-branes: The BLG model
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The Nahm Equation or D1-D3-Branes

In type IIB string theory, monopoles can be seen as D1-branes ending on D3-branes.

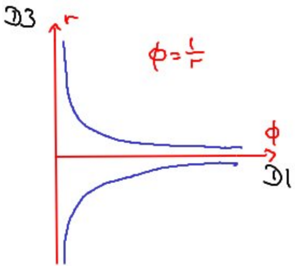
Consider a D3-brane in directions 0234.

A BPS solution to the SYM equations is the magnetic monopole with Higgs field $\phi \sim \frac{1}{r}$: A D1-brane appears.

As they are BPS, one trivially forms a stack of N D1-branes.

From the perspective of the D1-brane, the effective dynamics is described by the Nahm equations:

$$\frac{\mathrm{d}}{\mathrm{d}\phi} X^i + \varepsilon^{ijk}[X^j, X^k] = 0 \ .$$



dim	0	1	2	3	4
$D1$	$\times$	$\times$			
$D3$	$\times$		$\times$	$\times$	$\times$

These equations have the following solution ("fuzzy funnel")

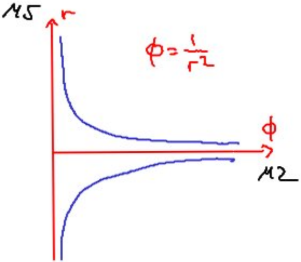
$$X^i = r(\phi) G^i \ , \quad r(\phi) = \frac{1}{\phi} \ , \quad G^i = \varepsilon^{ijk}[G^j, G^k]$$

The Basu-Harvey Equation or M2-M5-Branes

M2 branes ending on M5 branes should be described by Nahm-type equations.

M5-brane in directions 013456:

$$G^{mn}\nabla_m\nabla_n X^{a'} = 0$$
$$G^{mn}\nabla_m H_{npq} = 0$$



Ansatz for a soliton:

$$X^{5'} = \phi$$

$$H_{01m} = v_m \quad H_{mnp} = \varepsilon_{mnpq} v^q$$

Solution:

$$H_{01m} \sim \partial_m \phi \quad \phi \sim \frac{1}{r^2}$$

dim	0	1	2	3	4	5	6
$M2$	×	×				×	
$M5$	×		×	×	×	×	×

Perspective of M2: postulate four scalar fields X^i, satisfying

$$\frac{\mathrm{d}}{\mathrm{d}\phi} X^i + \varepsilon^{ijkl}[X^j, X^k, X^l] = 0$$

Basu, Harvey, hep-th/0412310

The Basu-Harvey Equation or M2-M5-Branes

M2 branes ending on M5 branes should be described by Nahm-type equations.

Basu-Harvey equation:

$$\frac{\mathrm{d}}{\mathrm{d}\phi}X^i + \varepsilon^{ijkl}[X^j, X^k, X^l] = 0$$

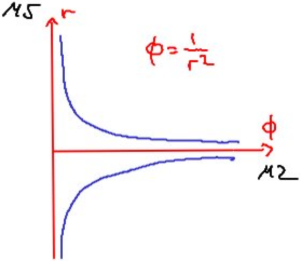
Solution (similar to D1-D3 case):

$$X^i = r(\phi)G^i \quad r(\phi) = \frac{1}{\sqrt{\phi}}$$

$$G^i = \varepsilon^{ijkl}[G^j, G^k, G^l]$$

Interprete this again as a fuzzy funnel, this time with a fuzzy S^3 at every point ϕ (not quite...).

$R \sim \sqrt{N}$ dofs $\sim R^3 \sim N^{3/2}$ ✓



dim	0	1	2	3	4	5	6
$M2$	×	×				×	
$M5$	×		×	×	×	×	×

What is the algebra behind the triple bracket?

In analogy with Lie algebras, we can introduce 3-Lie algebras.

Basu-Harvey equation:

$$\frac{\mathrm{d}}{\mathrm{d}\phi}X^i + \varepsilon^{ijkl}[X^j, X^k, X^l] = 0~, \quad X^i(\phi) \in \mathcal{A}$$

▷ $\mathcal{A}$ forms a vector space.

▷ $[\cdot,\cdot,\cdot]$ is a totally antisymmetric, linear map $\mathcal{A}\wedge\mathcal{A}\wedge\mathcal{A}\to\mathcal{A}$.

What is the algebra behind the triple bracket?
In analogy with Lie algebras, we can introduce 3-Lie algebras.

Basu-Harvey equation:

$$\frac{\mathrm{d}}{\mathrm{d}\phi}X^i + [A_\phi, X^i] + \varepsilon^{ijkl}[X^j, X^k, X^l] = 0 \ , \quad X^i \in \mathcal{A}$$

▷ Gauge transformations from inner derivations:

The triple bracket forms a map $\delta : \mathcal{A} \wedge \mathcal{A} \to \mathrm{Der}(\mathcal{A}) =: \mathfrak{g}_\mathcal{A}$ via

$$\delta_{A\wedge B}(C) := [A, B, C]$$

Demand a "3-Leibniz rule":

$$\begin{aligned}\delta_{A\wedge B}(\delta_{C\wedge D}(E)) &:= [A, B, [C, D, E]] \\ &= [[A, B, C], D, E] + [C, [A, B, D], E] + [C, D, [A, B, E]]\end{aligned}$$

The inner derivations form indeed a Lie algebra:

$$[\delta_{A\wedge B}, \delta_{C\wedge D}](E) := \delta_{A\wedge B}(\delta_{C\wedge D}(E)) - \delta_{C\wedge D}(\delta_{A\wedge B}(E))$$

Bracket closes due to "3-Leibniz rule".

What is the algebra behind the triple bracket?

In analogy with Lie algebras, we can introduce 3-Lie algebras.

To write down an action, i.e. gauge invariant terms, we need an invariant pairing on $\mathcal{A}$:

$$(\cdot,\cdot) : \mathcal{A} \otimes \mathcal{A} \to \mathbb{C}$$

Invariance under gauge transformations:

$$([A,B,C],D) + (C,[A,B,D]) = 0$$

On $\mathrm{Der}(\mathcal{A})$, there are now two pairings $(\!(\cdot,\cdot)\!)$:

1. The usual Killing form
2. A pairing induced from the pairing on $\mathcal{A}$:

$$(\!(\delta_{A\wedge B}, \delta_{C\wedge D})\!) = (D,[A,B,C])$$

Key to constructing a maximally SUSY model later: use the latter.

Short Remark: L_∞-algebras

This structures form a simple strong homotopy Lie algebra.

Note: we could combine $\mathcal{A}$ and $\mathrm{Der}(A)$ into one space $\mathcal{V}$ with two brackets $[\cdot,\cdot]$ and $[\cdot,\cdot,\cdot]$.

Jacobi-Identity and 3-Leibniz rule $\leftrightarrow$ Homotopy Jacobi identities.

$\mathcal{V}$ forms therfore an L_∞- or strong homotopy Lie algebra.

The Basu-Harvey equation is then precisely the homotopy Maurer-Cartan equation for the L_∞ algebra $\mathcal{V}\otimes\Omega^\bullet(\mathbb{R})$.

L_∞ algebras are behind most things coming up. Usefulness unclear.

C. I. Lazaroiu, D. McNamee, CS and A. Zejak, 0901.3905

Example: The Metric 3-Lie Algebra A_4

The 3-Lie algebra A_4 is the most important 3-Lie algebra in the context of BLG.

Consider the vector space $\mathbb{R}^4$ with basis $\tau_1, \ldots, \tau_4$.
Then define the bracket $[\cdot,\cdot,\cdot]$ as the linear extension of

$$[\tau_a, \tau_b, \tau_c] = \sum_d \varepsilon_{abcd}\tau_d \quad .$$

Additional structure:
The bilinear symmetric map $(\cdot,\cdot)$ is given as the lin. extension of

$$(\tau_a, \tau_b) = \delta_{ab} \quad .$$

The associated Lie algebra is $\mathfrak{g}_{A_4} = \mathfrak{so}(4) \cong \mathfrak{su}(2) \times \mathfrak{su}(2)$;
its bilinear pairing $(\!(\cdot,\cdot)\!)$ has split signature:

$$(\!(\delta_{\tau_a \wedge \tau_b}, \delta_{\tau_c \wedge \tau_d})\!) = \varepsilon_{abcd}$$

Approaching the Effective Description of M2-Branes

Spacetime symmetries and BPS equations give helpful constraints on the description.

A stack of flat M2-branes in $\mathbb{R}^{1,10}$ should be effectively described by a conformal field theory with the following constraints:

Spacetime symmetries: $\mathrm{SO}(1,10) \rightarrow \mathrm{SO}(1,2) \times \mathrm{SO}(8)$
extended by $\mathcal{N}=8$ SUSY.

Field content: $X = \Gamma_I X^I$, $I = 1, \ldots, 8$, and superpartners Ψ_α

Assumption

Take BPS/SUSY transformations from Basu-Harvey equation and therefore the matter fields take values in a metric 3-Lie algebra.

$$\delta X = \mathrm{i}\Gamma_I \bar{\varepsilon} \Gamma^I \Psi \qquad \delta\Psi = \partial_\mu X \Gamma^\mu \varepsilon - \tfrac{1}{6}[X, X, X]\varepsilon$$

Recipe: Try to close SUSY algebra. Constraints yield equations of motion for matter fields.

The Bagger-Lambert-Gustavsson Model
This model is an unconventional supersymmetric Chern-Simons matter theory.

BLG found that for SUSY, we need to introduce gauge symmetry.
⇒ Field content: $X \in \mathcal{A}$, $\Psi \in \mathcal{A}$ and gauge potential $A_\mu \in \mathfrak{g}_\mathcal{A}$.

The Bagger-Lambert-Gustavsson model

$$\begin{aligned}\mathcal{L}_{\rm BLG} \;=\; &+\tfrac{k}{4\pi}\varepsilon^{\mu\nu\kappa}\left(\!(\!(A_\mu,\partial_\nu A_\kappa)\!)+\tfrac{1}{3}(\!(A_\mu,[A_\nu,A_\kappa])\!)\right)\\ &-\tfrac{1}{2}(\nabla_\mu X,\nabla^\mu X)_{Cl}+\tfrac{\mathrm{i}}{2}(\bar\Psi,\Gamma^\mu\nabla_\mu\Psi)\\ &+\tfrac{\mathrm{i}}{4}(\bar\Psi,[X,X,\Psi])-\tfrac{1}{12}([X,X,X],[X,X,X])_{Cl}\end{aligned}$$

This model is invariant under the supersymmetry transformations:

$$\delta X = \mathrm{i}\Gamma_I\bar\varepsilon\Gamma^I\Psi\ ,\qquad \delta\Psi = \nabla_\mu X\Gamma^\mu\varepsilon-\tfrac{1}{6}[X,X,X]\varepsilon\ ,$$
$$\delta A_\mu = \mathrm{i}\bar\varepsilon\Gamma_\mu(\delta_{X\wedge\Psi})$$

Consistency checks

The BLG model passes a number of consistency checks.

$$\begin{aligned}\mathcal{L}_{\mathrm{BLG}} \;=\; &+\tfrac{k}{4\pi}\varepsilon^{\mu\nu\kappa}\left((\!(A_\mu,\partial_\nu A_\kappa)\!)+\tfrac{1}{3}(\!(A_\mu,[A_\nu,A_\kappa])\!)\right)\\ &-\tfrac{1}{2}(\nabla_\mu X,\nabla^\mu X)_{Cl}+\tfrac{\mathrm{i}}{2}(\bar\Psi,\Gamma^\mu\nabla_\mu\Psi)\\ &+\tfrac{\mathrm{i}}{4}(\bar\Psi,[X,X,\Psi])-\tfrac{1}{12}([X,X,X],[X,X,X])_{Cl}\end{aligned}$$

Further results:

- The model is classically conformal and seems rather unique.
- If $\mathcal{N}=8$ SUSY not anomalous $\Rightarrow$ vanishing β-function
- The model is parity invariant.
- Under some assumptions: reduction mechanism M2→D2.

(Mukhi, Papageorgakis,0803.3218)

- $k=2$: moduli space matches 2 M2-branes at tip of $\mathbb{R}^8/\mathbb{Z}_2$.
- Recast into the ABJM version, it yields integrable spin chain.

Manifestly $\mathcal{N}=2$ SUSY Formulation

There is a manifestly $\mathcal{N}=2$ SUSY formulation, allowing for various deformations.

Approach: Take $\mathcal{N}=1$, 4d superspace $\mathbb{R}^{1,3|4}$ and reduce to 3d.

Field content of the theory:

- The matter fields X^I, Ψ are encoded in four chiral multiplets:

$$\Phi^i(y)=\phi^i(y)+\sqrt{2}\theta\psi^i(y)+\theta^2F^i(y)\ ,$$

- The gauge potential A_μ is contained in a vector superfield:

$$\begin{aligned}V(x)\ &=\ -\theta^\alpha\bar{\theta}^{\dot{\alpha}}(\sigma^\mu_{\alpha\dot{\alpha}}A_\mu(x)+\mathrm{i}\varepsilon_{\alpha\dot{\alpha}}\sigma(x))\\&\quad+\mathrm{i}\theta^2(\bar{\theta}\bar{\lambda}(x))-\mathrm{i}\bar{\theta}^2(\theta\lambda(x))+\tfrac{1}{2}\theta^2\bar{\theta}^2D(x)\ ,\end{aligned}$$

$\mathcal{N}=2$ superspace formulation of BLG (Cherkis, CS, 0807.0808)

$$\begin{aligned}\mathcal{L}=\int \mathrm{d}^4\theta\ \kappa\left(\mathrm{i}(\!(V,(\bar{D}_\alpha D^\alpha V)\,)\!)+\tfrac{2}{3}(\!(V,\{(\bar{D}^\alpha V),(D_\alpha V)\})\!)\right)\\+(\bar{\Phi}_i,\mathrm{e}^{2iV}\cdot\Phi^i)+\alpha\left(\int \mathrm{d}^2\theta\ \varepsilon_{ijkl}([\Phi^i,\Phi^j,\Phi^k],\Phi^l)+c.c.\right)\end{aligned}$$

Manifestly $\mathcal{N}=4$ Supersymmetric Formulation

In projective superspace, one can make $\mathcal{N}=4$ SUSY in the BLG model manifest.

Field content of the BLG model in projective superspace:

- Matter X^I, Ψ: 4 $\mathcal{N}=2$ chiral mltps. $\Rightarrow$ 2 $\mathcal{N}=4$ hypermltps.
- Gauge A_μ: $\mathcal{N}=2$ vector multiplet $\Rightarrow$ $\mathcal{N}=4$ tropical multiplet

Supersymmetric action: (Cherkis, Dotsenko, CS, 0812.3127)

$$\int \mu\ \kappa\Big(\mathrm{i}(\!(\mathcal{V},(\bar{\mathcal{D}}_\alpha\mathcal{D}^\alpha\mathcal{V}))\!)+\tfrac{2}{3}(\!(\mathcal{V},\{(\bar{\mathcal{D}}^\alpha\mathcal{V}),(\mathcal{D}_\alpha\mathcal{V})\})\!)\Big)+\big(\bar{\eta}_k,\mathrm{e}^{2\mathrm{i}\mathcal{V}}\cdot\eta_k\big)$$

Observations:

- Chern-Simons term completely reduces to $\mathcal{N}=1$ form.
- The complex linear superfield Σ in the hypermultiplet
  $$\eta_k \;=\; \bar{\Phi}\frac{1}{\zeta^2}+\bar{\Sigma}\frac{1}{\zeta}+X-\zeta\Sigma+\zeta^2\Phi$$
  can be dualized to a chiral multiplet.
- To compute the interaction terms, one would have to solve a Riemann-Hilbert problem. However, its symmetries tell us that this is the BLG model.

Motivation: Marginal deformations of $\mathcal{N}=4$ SYM

The BLG model shares features with $\mathcal{N}=4$ SYM. What about marginal deformations?

Observation: BLG/ABJM seems similar to $\mathcal{N}=4$ SYM
($\rightarrow$ integrable spin chains).

$\mathcal{N}=4$ SYM admits (exactly) marginal deformations:

$$\mathcal{W} = \varepsilon_{ijk}\,\mathrm{tr}\,([\Phi^i,\Phi^j]_\beta\Phi^k)$$
$$[\Phi^i,\Phi^j]_\beta := \mathrm{e}^{\mathrm{i}\beta}\Phi^i\Phi^j - \mathrm{e}^{-\mathrm{i}\beta}\Phi^j\Phi^i$$

R. G. Leigh and M.J. Strassler, Nucl. Phys. B 447 (1995).

Conformality for β-deformed SYM to all orders in perturbation theory: S. Ananth, S. Kovacs, H. Shimada, JHEP 01 (2007) 046.

Such deformations correspond to deformations of $AdS_5 \times S^5$.

Similar deformations for $AdS_4 \times S^7$ in the literature.
What about BLG/ABJM and their deformations on quantum level?

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metric 3-Lie algebras (Filippov, 1985)

$\mathcal{A}$ a real vector space with a bracket $[\cdot,\cdot,\cdot]:\Lambda^3\mathcal{A}\to\mathcal{A}$ satisfying

$$[A,B,[C,D,E]] = [[A,B,C],D,E]+[C,[A,B,D],E]+[C,D,[A,B,E]] \quad \text{(FI)}$$

and a bilinear symmetric map $(\cdot,\cdot):\mathcal{A}\otimes\mathcal{A}\to\mathbb{R}$ satisfying

$$([A,B,C],D)+(C,[A,B,D]) = 0 \quad \text{(Cmp)}$$

There is a map from $\mathcal{A}\wedge\mathcal{A}$ to $\mathrm{Der}(\mathcal{A})$ given by linearly extending

$$D_{A\wedge B}(C) := [A,B,C]\ , \quad A,B,C\in\mathcal{A}$$

The inner derivations $\mathfrak{g}_\mathcal{A} := \mathrm{im}(D_{\mathcal{A}\wedge\mathcal{A}})$ form a Lie algebra.
Two invariant pairings on $\mathfrak{g}_\mathcal{A}$: $(\!(\delta_{A\wedge B},\delta_{C\wedge D})\!) := ([A,B,C],D)$
and induced Killing form.

Extending The Structure of a 3-Lie Algebra

The notion of a 3-Lie algebra is too restrictive and one has to find a generalized notion.

Problem: Given a three-algebra $\mathcal{A}$, if its bilinear form $(\cdot,\cdot)$ is positive definite, then $\mathcal{A}$ is A_4 or a direct sum thereof.

A_4 supposedly describes a stack of 2 M2-branes, not enough.

Mukhi, Papageorgakis, 0803.3218

Possible extensions:

(1) Assume, 3-Lie algebras appear accidentally $\Rightarrow$ ABJM model
(2) Give up positive definiteness of $(\cdot,\cdot)$ $\Rightarrow$ ghosts
(3) Relax conditions on 3-Lie algebras (+monopole operators)

Guideline: Demand gauge invariance of the $\mathcal{N}=2$ Lagrangian

$$\mathcal{L}=\int \mathrm{d}^4\theta\ \kappa\left(\mathrm{i}(\!(V,(\bar{D}_\alpha D^\alpha V))\!)+\tfrac{2}{3}(\!(V,\{(\bar{D}^\alpha V),(D_\alpha V)\})\!)\right)$$

$$+(\bar{\Phi}_i,\mathrm{e}^{2iV}\cdot\Phi^i)+\alpha\left(\int \mathrm{d}^2\theta\ \varepsilon_{ijkl}([\Phi^i,\Phi^j,\Phi^k],\Phi^l)+c.c.\right)$$

Demanding gauge invariance in above theory yields the condition:

$$
\begin{aligned}
([A,B,C],D) &= -([B,A,C],D) \\
&= -([A,B,D],C) = ([C,D,A],B)
\end{aligned}
$$

Cherkis, CS, 0807.0808

Generalized metric 3-Lie algebras or real 3-algebras

Same as a 3-Lie algebra, but relax $([A,B,C],D)$ from totally antisymmetric to the above symmetry properties.

Alternatively to our way of extending 3-Lie algebras:

Reduce supersymmetry to $\mathcal{N}=6$, i.e. assume the following:

$$\begin{aligned}\delta\phi^i &= \sqrt{2}\bar{\varepsilon}^{ij}\bar{\psi}_j\ ,\\ \delta\bar{\psi}_i &= -\mathrm{i}\sqrt{2}\sigma^\mu\varepsilon_{ij}\nabla_\mu\phi^j+[\phi^j,\phi^k;\bar{\phi}_j]\varepsilon_{ik}+[\phi^j,\phi^k;\bar{\phi}_i]\varepsilon_{jk}\ ,\\ \delta A_\mu &= -\mathrm{i}\varepsilon_{ij}\sigma_\mu\phi^i\wedge\psi^j+\mathrm{i}\bar{\varepsilon}^{ij}\sigma_\mu\bar{\phi}_i\wedge\bar{\psi}_j\ .\end{aligned}$$

where ε^{ij} is in the $\mathbf{6}$ of $SU(4)$. Closure of this algebra implies:

$$[A,B;C]=-[B,A;C]\qquad ([A,B;C],D)=(B,[C,D;A])\ .$$

$$[[C,D;E],A;B]-[[C,A;B],D;E]-[C,[D,A;B];E]+[C,D;[E,B;A]]\ =\ 0\ .$$

An associated Lie algebra $\mathfrak{g}_{\mathcal{A}}:=\mathrm{im}(D_{\mathcal{A}\otimes\mathcal{A}})$ is induced by

$$D_{A\wedge B}(C):=[C,A;B]\ ,\quad A,B,C\in\mathcal{A}$$

This leads to the ABJM model, a Chern-Simons-matter theory.

Aharony, Bergman, Jafferis, Maldacena, 0806.1218

Bagger, Lambert, 0807.0163

Current Situation:
It is not clear, if 3-Lie algebras are necessary at all.

Observations:

- 3-Lie algebras too restrictive, only one example: A_4.
- Generalizations lead to less than $\mathcal{N} = 8$ supersymmetry.
- All models can be rewritten as gauge theories.

$\Rightarrow$ We need more input from physics.

Particularly important here: AdS/CFT correspondence

We need some kind of $N \to \infty$ limit, so let's look at representations of (generalized) 3-Lie algebras in terms of matrix algebras.

Classifications of $*$-Algebra Representations of 3-Algebras

Representations on matrix algebras, which are useful for $N \to \infty$, can be constructed.

Representation of metric 3-algebras on $*$-algebras:

Take a $*$- or matrix algebra equipped with a trace form. Construct a 3-bracket on this algebra from matrix products and the involution and use the Hilbert-Schmidt scalar product $(A, B) = \text{tr}\,(A^\dagger B)$.

Classification of all such representations in the real and hermitian case using MuPad done in Cherkis, Dotsenko, CS, 0812.3127

Representations on matrix algebras, which are useful for $N \to \infty$, can be constructed.

The Real case. $[A, B, C] :=$

$$\begin{aligned}
\mathrm{I}:&\ \alpha([[A^\dagger, B], C] + [[A, B^\dagger], C] + [[A, B], C^\dagger] - [[A^\dagger, B^\dagger], C^\dagger])\\
\mathrm{II}:&\ \alpha([[A, B^\dagger], C] + [[A^\dagger, B], C])\\
\mathrm{III}:&\ \alpha(AB^\dagger - BA^\dagger)C + \beta C(A^\dagger B - B^\dagger A)\\
\mathrm{IV}:&\ \alpha([[A, B], C] + [[A^\dagger, B^\dagger], C] + [[A^\dagger, B], C^\dagger] + [[A, B^\dagger], C^\dagger])\\
&\ + \beta([[A, B], C^\dagger] + [[A^\dagger, B], C] + [[A, B^\dagger], C] + [[A^\dagger, B^\dagger], C^\dagger])\ .
\end{aligned}$$

The class of examples $\mathcal{C}_{2d}$,

$$[\gamma_a, \gamma_b, \gamma_c] := [[\gamma_a, \gamma_b]\gamma_{ch}, \gamma_c]\ ,$$

is contained in III, with $\alpha = \beta = -1$ and the ∗-algebra is the algebra of $d \times d$ matrices.

The Hermitian case. $[A, B, C] :=$

$$\mathrm{I}_\alpha : A, B, C \mapsto \alpha(AC^\dagger B - BC^\dagger A) \ .$$

This is precisely the Hermitian 3-Lie algebra used in Bagger, Lambert, 0807.0163 to obtain the ABJM model in 3-algebra form.

Associated 3-products

Analogously to matrix products, one can introduce 3-products.

Recall:

In gauge theories: gauge fields live in Lie algebra, matter fields live in representations of this Lie algebra.

Representations can carry (additional) products, e.g. the adjoint:

$$A \star_{a,b} B = aAB + bBA$$

which still transform covariantly under gauge transformations:

$$\delta_\Lambda(A \star_{a,b} B) = \delta_\Lambda(A) \star_{a,b} B + A \star_{a,b} \delta_\Lambda(B)$$

These are used in the superpotential of $\mathcal{N} = 1^*$ SYM theory.

Associated 3-products

Analogously to matrix products, one can introduce 3-products.

Assume now: Matter fields of BLG in representation $\mathcal{R}$ of $\mathcal{A}$.

A generalized or associated 3-product is a map

$$\langle A, B, C\rangle : \mathcal{R} \times \mathcal{R} \times \mathcal{R} \to \mathcal{R}$$

and has to satisfy (real 3-Lie algebras):

$$[A, B, \langle C, D, E\rangle] = \langle [A, B, C], D, E\rangle + \langle C, [A, B, D], E\rangle + \langle C, D, [A, B, E]\rangle$$

Use these products: more general terms in BLG action:

$$\mathcal{L} = \int \mathrm{d}^4\theta\ \kappa \left(\mathrm{i}(\!(V, (\bar{D}_\alpha D^\alpha V))\!) + \tfrac{2}{3}(\!(V, \{(\bar{D}^\alpha V), (D_\alpha V)\})\!)\right) + (\bar{\Phi}_i, \mathrm{e}^{2iV} \cdot \Phi^i) + \alpha\left(\int \mathrm{d}^2\theta\ \varepsilon_{ijkl}([\Phi^i, \Phi^j, \Phi^k], \Phi^l) + c.c.\right)$$

Which associated 3-products exist in the representations?

The admissible classes of associated 3-products are limited.

Real 3-algebras:

Class III: $[A, B, C] := \alpha(AB^\dagger - BA^\dagger)C + \beta C(A^\dagger B - B^\dagger A)$

Most general associated 3-product:

$$\begin{aligned}\langle A, B, C\rangle \ &= \alpha_1 AB^T C + \alpha_2 CB^T A + \beta_1 BC^T A + \beta_2 AC^T B + \\ &\gamma_1 CA^T B + \gamma_2 BA^T C\end{aligned}$$

Hermitian 3-algebras:

Class I: $[A, B, C] := \alpha(AC^\dagger B - BC^\dagger A)$

Most general associated 3-product:

$$\langle A, B; C\rangle \ = \ \alpha_1 AC^\dagger B - \alpha_2 BC^\dagger A$$

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Action in terms of $\mathcal{N} = 2$ superfields

We generalize the superpotential, using associated 3-products and multitraces.

Real case:

$$S_0^R = \mathrm{i}\sqrt{\kappa} \int \mathrm{d}^{3|4} z \int_0^1 \mathrm{d}t \, (\!(V, \bar{D}^\alpha (\mathrm{e}^{-\frac{2\mathrm{i}}{\sqrt{\kappa}} tV} D_\alpha \mathrm{e}^{\frac{2\mathrm{i}}{\sqrt{\kappa}} tV}))\!) + \int \mathrm{d}^{3|4} z \, (\bar{\Phi}_i, \mathrm{e}^{-\frac{2\mathrm{i}}{\sqrt{\kappa}} V} \Phi^i)$$

Superpotential:

$$S_1^R = \int \mathrm{d}^{3|2} z \left[R^{(1)}_{ijkl} (\Phi^l, [\Phi^i, \Phi^j, \Phi^k]) + R^{(2)}_{ijkl} (\Phi^i, \Phi^j)(\Phi^k, \Phi^l) \right] + \int \mathrm{d}^{3|2} \bar{z} \left[R^{ijkl}_{(1)} (\bar{\Phi}_l, [\bar{\Phi}_i, \bar{\Phi}_j, \bar{\Phi}_k]) + R^{ijkl}_{(2)} (\bar{\Phi}_i, \bar{\Phi}_j)(\bar{\Phi}_k, \bar{\Phi}_l) \right]$$

Here, we restrict ourselves to multitraces, as the ordinary 3-bracket already allows for marginal deformations.

Action in terms of $\mathcal{N} = 2$ superfields

We generalize the superpotential, using associated 3-products and multitraces.

Hermitian case:

Technical issue: $\mathrm{SU}(4)$ R-symmetry does not respect chirality:
$\mathrm{SU}(4)$ multiplet: $(\Phi^m, \bar{\Phi}_{\dot{m}})$, $m, \dot{m} = 1, 2$ and thus:

$$S_0^H = \mathrm{i}\sqrt{\kappa} \int \mathrm{d}^{3|4}z \int_0^1 \mathrm{d}t \, (\!(V, \bar{D}^\alpha (\mathrm{e}^{-\frac{2\mathrm{i}}{\sqrt{\kappa}}tV} D_\alpha \mathrm{e}^{\frac{2\mathrm{i}}{\sqrt{\kappa}}tV}))\!) + \int \mathrm{d}^{3|4}z \left[(\Phi^m, \mathrm{e}^{-\frac{2\mathrm{i}}{\sqrt{\kappa}}V} \Phi^m) + (\bar{\Phi}_{\dot{m}}, \mathrm{e}^{\frac{2\mathrm{i}}{\sqrt{\kappa}}V} \bar{\Phi}_{\dot{m}})\right]$$

Superpotential:

$$S_1^H = \int \mathrm{d}^{3|2}z \left[H^{(1)}_{mn\dot{m}\dot{n}} (\bar{\Phi}_{\dot{n}}, [\Phi^m, \Phi^n; \bar{\Phi}_{\dot{m}}]_\beta) + H^{(2)}_{mn\dot{m}\dot{n}} (\bar{\Phi}_{\dot{m}}, \Phi^m)(\bar{\Phi}_{\dot{n}}, \Phi^n) \right] + c.c.$$

where we defined $[\tau_a, \tau_b; \tau_c]_\beta := \mathrm{e}^{\mathrm{i}\beta} \tau_a \tau_c^\dagger \tau_b - \mathrm{e}^{-\mathrm{i}\beta} \tau_b \tau_c^\dagger \tau_a$.

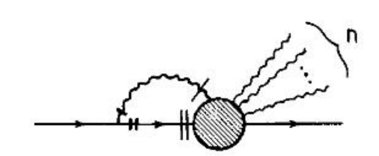
$\mathcal{N} = 2$ Superfield Feynman Rules
Supergraph rules can be derived in a straightforward manner.

Super Feynman rules for Chern-Simons theory with matter are easily derived, see e.g. S.J. Gates, H. Nishino, PLB281 (1992) 72

Gluon propagator (Landau gauge): $\frac{\mathrm{i}\bar{D}^\alpha D_\alpha(p,\theta^1)}{p^2}\delta^4(\theta^1 - \theta^2)$
Matter propagator: $\frac{1}{p^2}\delta^4(\theta^1 - \theta^2)$
Vertices: from action, insert $-\frac{1}{4}\bar{D}^2/-\frac{1}{4}D^2$ for $\Phi/\bar{\Phi}$
add the usual loop integrals, symmetry factors, ...

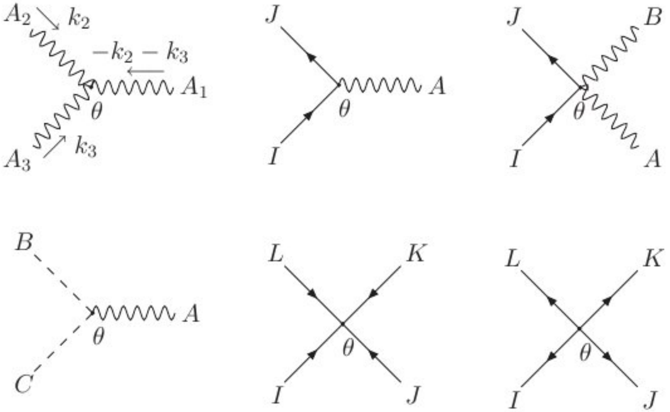
Vanishing lemma

The following diagrams essentially vanish due to $D^3 = \bar{D}^3 = 0$:



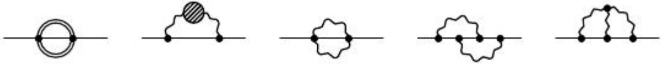
Types of vertices for 2-loop computations

Of the infinite vertices, only finitely many contribute at 2-loop level.



Example: Field strength renormalization.

In components: Gaiotto, Yin, JHEP 08 (2007) 056

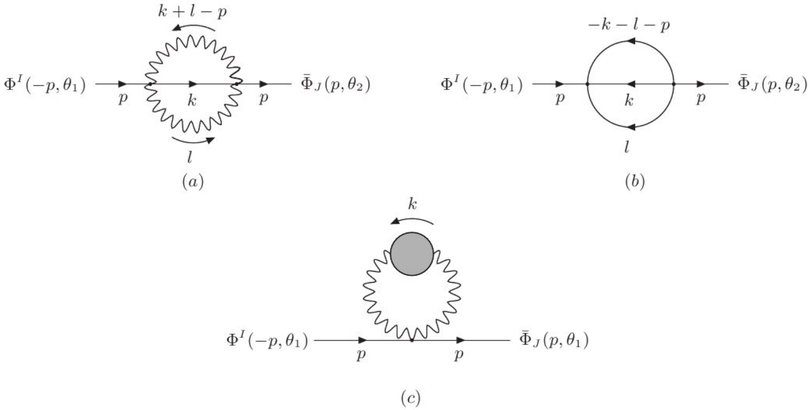


With supergraphs, only the 1st and 3rd diagrams survives (lemma).

Contributing diagrams

At 2 loop level, only three classes of diagrams contribute.

Contributing diagrams (only 2-pt contributions are divergent):



Potential flow of the couplings due to anomalous dimensions.

2nd reason for matrix representations: Need to compute Casimirs.

Introduce $f_{abcd} = (\tau_d, [\tau_a, \tau_b, \tau_c])$ and $h_{ab} = (\tau_a, \tau_b)$. Examples:

$$f_{ac}{}^{cb} = c_1 \delta_a{}^b \quad f_{acde} f^{edcb} = -c_2 \delta_a{}^b \quad f_{acde} f^{bcde} = -c_3 \delta_a{}^b$$

and similar objects on the associated Lie algebras $\mathfrak{g}_{\mathcal{A}}$.

These can easily be computed in $\mathrm{u}(N)$ and $\mathrm{so}(N)$, e.g.

$$\mathrm{tr}\,(\tau_a^T \tau_b) = \delta_{ab} =: h_{ab} \quad \text{and} \quad (\tau_a)_{ij}(\tau_a)_{kl} = \delta_{ik}\delta_{jl}$$

for $\mathrm{so}(N)$.

Results: The β-function in the real case

The BLG model is conformally invariant at two loops.

Recall: All flow from anomalous dimensions at two loops.

Total anomalous dimension:

$$\gamma_i{}^j = \frac{1}{8\pi^2\kappa^2}\Big\{\big[k_2 + k_1^2 + \tfrac{1}{12}(2k_2 + N_f k_3)\big]\delta_i{}^j + 8\kappa^2\Big[R^{(1)}_{iklm}\big(-c_3 R^{jklm}_{(1)} + 2c_2 R^{jmlk}_{(1)} + 2c_1 R^{jmlk}_{(2)}\big) + R^{(2)}_{iklm}\big(d\, R^{jklm}_{(2)} + 2R^{jmlk}_{(2)} + 2c_1 R^{jmlk}_{(1)}\big)\Big]\Big\}$$

Quick test: BLG. $R^{(2)}_{ijkl} = 0$, $\mathcal{A} = A_4$, therefore $R^{(1)}_{ijkl} = \lambda\varepsilon_{ijkl}$ and

$$d = 4 \quad k_1 = 0 \quad k_2 = -3 \quad k_3 = 6 \quad c_1 = 0 \quad c_2 = c_3 = -6$$

The β-function reads as (the phase does not flow)

$$\beta^{(1)}_{ijkl} = -\tfrac{3}{4\pi^2\kappa^2}\big[1 - (4!\kappa)^2|\lambda|^2\big]R^{(1)}_{ijkl} \quad \text{so} \quad |\lambda| = \frac{1}{4!\kappa}$$

At point where $\beta^{(1)}_{ijkl} = 0$, regularization scheme irrelevant.

Results: The β-function in the hermitian case

The ABJM model is conformally invariant at two loops.

$$
\begin{aligned}
\gamma_m{}^n \;=\; & \frac{1}{8\pi^2\kappa^2}\Big\{\big[k_2+k_1^2+\tfrac{1}{12}(2k_2+N_f k_3)\big]\delta_m{}^n \\
& + \kappa^2\Big[\big(H^{(1)}_{mk\dot m\dot n}H_{(1)}^{\dot m\dot nkn}-H^{(1)}_{mk\dot m\dot n}H_{(1)}^{\dot n\dot mkn}\big)c_2\cos^2\beta \\
& \quad + \big(H^{(1)}_{mk\dot m\dot n}H_{(1)}^{\dot m\dot nkn}+H^{(1)}_{mk\dot m\dot n}H_{(1)}^{\dot n\dot mkn}\big)c_2'\sin^2\beta \\
& \quad + \big(H^{(1)}_{mk\dot m\dot n}H_{(2)}^{\dot m\dot nkn}+H^{(2)}_{mk\dot m\dot n}H_{(1)}^{\dot m\dot nkn}\big)\big(c_1\cos\beta+\mathrm{i}c_1'\sin\beta\big) \\
& \quad - \big(H^{(1)}_{mk\dot m\dot n}H_{(2)}^{\dot n\dot mkn}+H^{(2)}_{mk\dot m\dot n}H_{(1)}^{\dot n\dot mkn}\big)\big(c_1\cos\beta-\mathrm{i}c_1'\sin\beta\big) \\
& \quad + \big(H^{(2)}_{mk\dot m\dot n}H_{(2)}^{\dot m\dot nkn}+d\,H^{(2)}_{mk\dot m\dot n}H_{(2)}^{\dot n\dot mkn}\big)\Big]\Big\}
\end{aligned}
$$

Quick test: ABJM. $\beta=0$, $N_f=4$ $H^{(1)}_{mn\dot m\dot n}=\lambda\varepsilon_{mn}\varepsilon_{\dot m\dot n}$, $H^{(2)}_{ijkl}=0$

The β-function reads as

$$
\gamma_m{}^n \;=\; \frac{1}{16\pi^2\kappa^2}(1-N^2)\big[1-(4\kappa)^2|\lambda|^2\big]\delta_m{}^n \quad \text{so} \quad |\lambda|=\frac{1}{4\kappa}
$$

The running of the coupling is exactly as expected.

Real case:

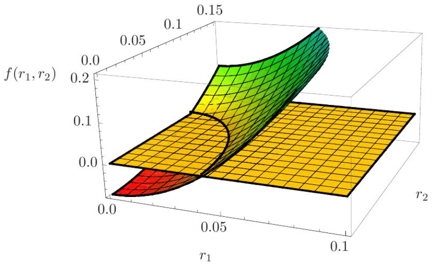
For simplicity, we take $\mathcal{A} = A_4$ and the superpotential

$$R^{(1)}_{ijkl} = \frac{\lambda_1}{\kappa}\varepsilon_{ijkl} \quad \text{and} \quad R^{(2)}_{ijkl} = \frac{\lambda_2}{\kappa}\delta_{ij}\delta_{kl}\ , \quad \lambda_i = r_i \mathrm{e}^{\varphi_i}$$

The β-function at two loops reads as

$$\beta^{(\ell)}_{ijkl} = \frac{f(r_1,r_2)}{\kappa^2}R^{(\ell)}_{ijkl} \qquad f(r_1,r_2) := -\frac{3}{4\pi^2}\left[1-96\left(6r_1^2+r_2^2\right)\right]$$

(again, the phases do not flow)



BLG: $r_1 = \frac{1}{24}, r_2 = 0$

points on ellipse:
IR fixed points

similar for hermitian case

Done:

- Identification of extended 3-algebraic structures
- Verified conformal invariance up to 2 loops for BLG/ABJM
- Found classes of marginal deformations

Future directions:

- Other representations?
- Finiteness to all orders in perturbation theory?
- Integrability for subsectors?
- Identify dual geometries
- h deformations?

Marginal Deformations and 3-Algebra Structures

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10th Anniversary of the EMPG, Sep 30th 2009