Extended Riemannian Geometry I: Local Double Field Theory

Andreas Deser and Christian Sämann

arXiv:1611.02772

1. Motivation

Double Field Theory (DFT):

- target space formulations useful (SUSY gauge theories, supergravity)
- T-duality very interesting
- ⇒ combine both
- useful ✓ (get supergravity actions that were not derivable from string theory.)
- sensible? strange truncation of string theory modes? seems to work ...

Idea:

- Construct a big space that carries representation of T-duality group $\mathsf{O}(D, D, \mathbb{Z})$
- Have a procedure for choosing a T-duality frame
- Should reproduce generalized geometry, massless string modes g, B, ϕ.

Our research:

- Understand DFT properly
- Underlying local geometry (first step towards global formulation and Riemannian geometry)
- correct section conditions $\partial^\mu \alpha \partial_\mu \beta = 0$ for tensors α, β
- correct symmetries
- global symmetries (second step towards global formulation)

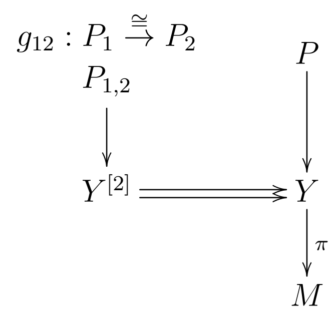
2. Generalized geometry

Describe massless string modes g, B, ϕ

- B belongs to connective struct. of a gerbe (Gawedzki 1987, Freed Witten 1999)
- Recall: principal $\mathsf{U}(1)$-bundle:

$$\mathsf{U}(1) \to P \to M$$

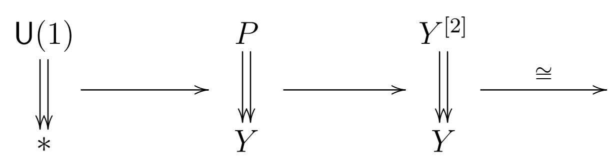
Also: with $Y := \cup_i U_i$, $Y^{[2]} := \cup_{i,j} U_i \cap U_j$, ...



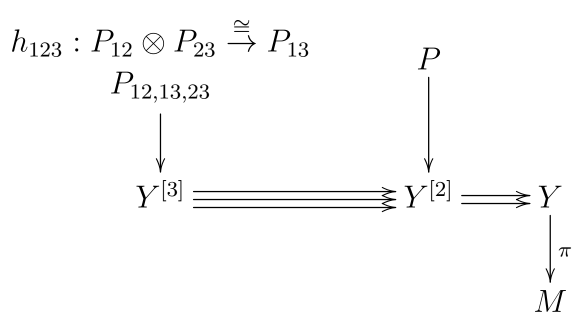
Altogether: $g_{ij} \in \mathcal{C}^\infty(U_i \cap U_j, \mathsf{U}(1))$, $A_i \in \Omega^1(U_i, \mathfrak{u}(1))$ with

$$g_{ij} g_{jk} = g_{ik} \ , \quad A_i = g_{ij}^{-1}(A_j + \mathrm{d}) g_{ij} \ .$$

- Gerbe works similarly:



Also: with $Y := \cup_i U_i$, $Y^{[2]} := \cup_{i,j} U_i \cap U_j$, $Y^{[3]} := \cup_{i,j,k} U_i \cap U_j \cap U_k$

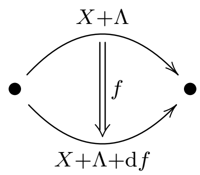


Altogether: $h_{ijk} \in \mathcal{C}^\infty(U_i \cap U_j \cap U_k, \mathsf{U}(1))$, $A_{ij} \in \Omega^1(U_i \cap U_j, \mathfrak{u}(1))$, $B_i \in \Omega^1(U_i, \mathfrak{u}(1))$ with

$$h_{ijk} h_{ik\ell} = h_{ij\ell} h_{jk\ell} \ , \quad A_{ij} - A_{ik} + A_{jk} = \mathrm{d}\log(h_{ijk}) \ , \quad B_i - B_j = \mathrm{d}A_{ij} \ .$$

Symmetries

- form Lie 2-algebra (diffeos $X \in \mathfrak{X}(M)$ and gauge symmetries $\Lambda \in \Omega^1(M)$, $f \in \Omega^0(M)$):



  Fully fixed by preserving $\mathsf{O}(D, D, \mathbb{R})$-structure on $\mathfrak{X}(M) \oplus \Omega^1(M)$:
  $(X + \alpha, Y + \beta) = \iota_X \alpha + \iota_Y \beta$

- Description in terms of derived brackets on higher algebroids:

$$\mathcal{V}_2 = T^*[2]T[1]M \ , \quad \text{coordinates}(x^\mu, \xi^\mu, \zeta_\mu, p_\mu)$$

  Functions of degree 1: sections of $TM \oplus T^*M$, of degree 0: functions $\mathcal{C}^\infty(M)$
  Homological vector field: Q, symplectic form ω:

$$Q = \xi^\mu \frac{\partial}{\partial x^\mu} + p_\mu \frac{\partial}{\partial \zeta_\mu} \ , \quad \omega = \mathrm{d}x^\mu \wedge \mathrm{d}p_\mu + \mathrm{d}\xi^\mu \wedge \mathrm{d}\zeta_\mu$$

  Lie 2-algebra: $\mathcal{C}^\infty(M) \to \Gamma(TM \oplus T^*M)$

$$\begin{aligned} \mu_1(f) &= Qf \ , \\ \mu_2(f, X) &= -\tfrac{1}{2}\{QX, f\} \ , \quad \mu_2(X, Y) = \tfrac{1}{2}(\{QX, Y\} - \{QY, X\}) \\ \mu_3(X, Y, Z) &= -\tfrac{1}{12}(\{\{QX, Y\}, Z\} \pm \text{permuations}) \ . \end{aligned}$$

  μ_2 is called Courant bracket, also: action on tensor fields

- Note: Any L_∞-algebroid comes with derived bracket L_∞-algebra
  (Roytenberg, Fiorenza, Getzler)

3. Double Field Theory

- Observation: ordinary manifolds not good enough $\Rightarrow$ categorified spaces

- double space, i.e. replace M by T^*M

- Algebroid: $\mathcal{V}_2(T^*M) = T^*[2]T[1](T^*M)$
  coordinates: $(x^M, p_M, \xi^M, \zeta_M) = (x^\mu, x_\mu, \dots, \zeta_\mu, \zeta^\mu)$
  symplectic form and Q as before

- Have: $\mathsf{O}(D, D, \mathbb{R})$-structure $\eta_{MN} = \eta^{MN} = \begin{pmatrix} 0 & \mathbb{1} \\ \mathbb{1} & 0 \end{pmatrix}$.

- New coordinate: $\theta^M = \frac{1}{\sqrt{2}}(\xi^M + \eta^{MN}\zeta_N)$ and $\beta^M = \frac{1}{\sqrt{2}}(\xi^M - \eta^{MN}\zeta_N)$
- Restrict to $\beta^M = 0$:

$$\omega = \mathrm{d}x^M \wedge \mathrm{d}p_M + \tfrac{1}{2}\eta_{MN}\mathrm{d}\theta^M \wedge \mathrm{d}\theta^N \ , \quad Q = \theta^M \frac{\partial}{\partial x^M} + p_M \eta^{MN}\frac{\partial}{\partial \theta^N}$$

- $Q^2 = p_M \eta^{MN}\frac{\partial}{\partial x^N} \neq 0 \to$ pre-NQ-manifold
- Derived brackets do not automatically form Lie 2-algebra, need restriction:

$$\begin{aligned}\{Q^2 f, g\} + \{Q^2 g, f\} &= 2\left(\frac{\partial}{\partial x^M} f\right)\eta^{MN}\left(\frac{\partial}{\partial x^N} g\right) = 0 \ ,\\ \{Q^2 X, f\} + \{Q^2 f, X\} &= 2\left(\frac{\partial}{\partial x^M} X\right)\eta^{MN}\left(\frac{\partial}{\partial x^N} f\right) = 0 \ ,\\ \{\{Q^2 X, Y\}, Z\}_{[X,Y,Z]} &= 2\theta^L\big((\partial^M X_L)(\partial_M Y^K)Z_K\big)_{[X,Y,Z]} = 0 \ .\end{aligned}$$

  μ_2 then reproduces C-bracket
  Weakening of strong section constraint $\partial^\mu \alpha \partial_\mu \beta = 0$
  $x_\mu = p^\mu = 0$ leads to generalized geometry with $Q^2 = 0$.
- Define action of L_∞-algebra on extended tensors:
  D-bracket, generalized Lie derivative
- Everything coordinate invariant, well defined, interpretation transparent

Advantages, e.g. global picture:

- Action of Lie 2-algebra on extended Tensors: ordinary Lie algebra
- Hohm-Zwiebach (2011): integrated this Lie algebra
- Berman, Cederwall, Perry (2014): Use this to glue together local patches
- Papadopoulos (2014): Only works if $H = \mathrm{d}B$ globally.
- From our perspective: clear.
- Generalized geometry:
  Generalized tangent vectors global: $B_i - B_j = \mathrm{d}A_{ij}$, $\tilde{B}_i - \tilde{B}_j = \mathrm{d}A_{ij}$
  implies: $X + \Lambda_i = X + \Lambda_j - \mathrm{d}A_{ij}$.
  Also: twist Q to

$$Q_T = \xi^\mu \frac{\partial}{\partial x^\mu} + p_\mu \frac{\partial}{\partial \zeta_\mu} + \frac{1}{3!} H_{\nu\mu_1\mu_2}\xi^{\mu_1}\xi^{\mu_2}\frac{\partial}{\partial \zeta_\nu} \ .$$

- Same needs to happen in Double Field Theory
- Preliminary study of twists etc.