Lie 2-algebra models

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1. Introduction

- Current formulations of string theory **background dependent**
- Ways out:
  - **IKKT** (string theory)
  - **BFSS** (M-theory in infinite-momentum-frame)
- Nice picture:
  - background geometry + dynamics **emerge**
  - based on symplectic manifolds and Poisson/Lie algebras
- **However**, get only GR restricted to Kähler manifolds
- D-brane actions and M-theory suggests ways out: categorification
- We need:
  - Higher symplectic and Poisson/Lie algebra analogon
  - A generalized IKKT model or corresponding EOMS.
- Also: To test our ideas, we should generalize/deform/categorify

2. IKKT model

- Regularization of Green-Schwarz action of superstring in Schild gauge.
- Alternatively: full dimensional reduction of 10d SYM $\nabla_\mu \to A_\mu$, $\int \to -$:

$$S = \int \mathrm{d}^{10}x \quad \mathrm{tr}\,\left(F_{\mu\nu}F^{\mu\nu} - \tfrac{\mathrm{i}}{2}\bar{\psi}\Gamma^\mu\nabla_\mu\psi\right)$$

- This yields a 0d matrix model:

$$S_{\mathrm{IKKT}} = \alpha\,\mathrm{tr}\,\left(-\tfrac{1}{4}\,[A_\mu, A_\nu]^2 - \tfrac{1}{2}\,\bar{\psi}\Gamma^\mu[A_\mu,\psi] + \beta\mathbb{1}\right)\ .$$

- Allow for deformation terms

$$S_{\mathrm{def}} = S_{\mathrm{IKKT}} + \,\mathrm{tr}\,\left(-\tfrac{1}{2}\sum_\mu m^2_{1,\mu}A_\mu A_\mu + \tfrac{\mathrm{i}}{2}m_2\bar{\psi}\psi + c_{\mu\nu\kappa}A^\mu[A^\nu, A^\kappa]\right)\ ,$$

- EOM:

$$-\alpha[A^\mu,[A_\mu,A_\nu]] - m^2_{1,\mu}A_\nu + 3c_{\nu\kappa\lambda}[A^\kappa,A^\lambda] = 0$$

- Suggested as background indept. definition of type IIB superstring theory

3. Quantization of symplectic manifolds

- Classical Phase space:
  - Symplectic manifold (M, ω)
  - Associative algebra of observables $\mathcal{C}^\infty(M)$
  - Additional Poisson (Lie) algebra structure:

$$\iota_{X_f}\omega = \mathrm{d}f \ , \quad \{f, g\} := \iota_{X_f}\iota_{X_g}\omega \ .$$

- Quantization:
  - Quantum Hilbert space $\mathcal{H}$
  - Associative alg. of observables $\mathsf{End}\,(\mathcal{H})$
  - Lie algebra structure: $[A, B]$
  - quantization map $\hat{-} : \mathcal{C}^\infty(M) \to \mathsf{End}\,(\mathcal{H})$
  - $\hat{f}\hat{g} = \widehat{fg} + \mathcal{O}(\hbar)$
  - $[\hat{f}, \hat{g}] = -\mathrm{i}\hbar\widehat{\{f, g\}} + \mathcal{O}(\hbar^2)$
  - Quantization: Lie algebra homomorphism up to $\mathcal{O}(\hbar)$

- Examples:
  - $\mathbb{R}^2$, $\omega = \mathrm{d}x^1 \wedge \mathrm{d}x^2$, $\{f, g\} = \varepsilon^{ab}(\partial_a f)(\partial_b g)$
    Moyal plane: $[\hat{x}^1, \hat{x}^2] = -\mathrm{i}\hbar\mathbb{1}$
  - S^2, $\omega = \sin\theta \mathrm{d}\theta \wedge \mathrm{d}\phi$, $\{f, g\} = \frac{\varepsilon^{ab}}{\sin\theta}(\partial_a f)(\partial_b g)$
    Fuzzy sphere: $S^2 \hookrightarrow \mathbb{R}^3$: $[\hat{x}^i, \hat{x}^j] = -\mathrm{i}\hbar\varepsilon^{ijk}\hat{x}^k$

4. Emergence of spacetime and dynamics

- EOMs of IKKT model:

$$-\alpha[A^\mu, [A_\mu, A_\nu]] - m_{1,\mu}^2 A_\nu + 3c_{\nu\kappa\lambda}[A^\kappa, A^\lambda] = 0$$

- Solutions:
  - For $m_{1,\nu} = c_{\mu\nu\kappa} = 0$:
    $[\hat{x}^i, \hat{x}^j] = -\mathrm{i}\hbar\mathbb{1}$: Moyal plane and products thereof
  - For $m_{1,\nu} = 0$, $c_{ijk} \sim \varepsilon_{ijk}$:
    $[\hat{x}^i, \hat{x}^j] = -\mathrm{i}\hbar\varepsilon^{ijk}\hat{x}^k$, Fuzzy sphere

- Dynamics:
  - Linear perturb. around solution: $A_\mu \to X_\mu + a_\mu$.
  - For IKKT: NC Yang Mills with eom: $[X^\mu, [X_\mu, a_\mu]] = 0$ etc.

5. M-theory "lift"

- There are Schild-type actions of higher dimensional branes $\Rightarrow L_\infty$-algebra
- B-field of string theory becomes C-field of M-theory
  or: 2-forms (symplectic) become 3-forms (2-plectic), lead to L_∞-algebra
- often: going to M-theory means to categorify $\Rightarrow L_\infty$-algebra
- L_∞-algebras actually all over the place in string/M-theory:
  - BV-quantization
  - String Field Theory
  - Courant Algebroids in Double Field Theory

6. L_∞-algebras

- From NQ-manifolds (BV-quantization, AKSZ-formalism):
  - $\mathbb{N}^*$-graded vector space $V = \oplus_{i=1}^{\infty} V_i$
  - Vector field Q of degree 1 with $Q^2 = 0$
- Example: $V = V_1 \oplus V_2$. Coordinates ξ^α, p_a Then

$$Q = m^{a\alpha} p_a \frac{\partial}{\partial \xi^\alpha} + m^\alpha_{\beta\gamma} \xi^\beta \xi^\gamma \frac{\partial}{\partial \xi^\alpha} + m^b_{a\alpha} \xi^\alpha p_b \frac{\partial}{\partial p_a} + m_{a\alpha\beta\gamma} \xi^\alpha \xi^\beta \xi^\gamma \frac{\partial}{\partial p_a}$$

- On dual spaces V_i^* with coordinates ξ_α, p^a we have brackets:

$$\mu_1(p^a) = m^{a\alpha}\xi_\alpha \ , \qquad \mu_2(\xi_\alpha, \xi_\beta) = m^\gamma_{\alpha\beta}\xi_\gamma \ ,$$
$$\mu_2(\xi_\alpha, p^a) = m^a_{b\alpha} p^b \ , \qquad \mu_3(\xi_\alpha, \xi_\beta, \xi_\gamma) = m_{a\alpha\beta\gamma} p^a$$

  From $Q^2 = 0$: higher homotopy relations, e.g.

$$\mu_1(\mu_2(\xi, p)) = \mu_2(\xi, \mu_1(p)) \ , \quad \mu_2(\mu_1(p_1), p_2) = \mu_2(p_1, \mu_1(p_2)) \ ,$$
$$\mu_1(\mu_3(\xi_1, \xi_2, \xi_3)) = -\mu_2(\mu_2(\xi_1, \xi_2), \xi_3) - \mu_2(\mu_2(\xi_3, \xi_1), \xi_2) - \mu_2(\mu_2(\xi_2, \xi_3), \xi_1) \ ,$$

- Examples:
  - Lie algebra $\mathfrak{g}$ (Chevalley-Eilenberg) $V = \mathfrak{g}^*[1]$
  - String Lie 2-algebra $V = \mathbb{R}[2] \oplus \mathfrak{g}^*[1]$,
    $\mu_1(r) = 0$, $\mu_2(g_1, g_2) = [g_1, g_2]$ $\mu_3(g_1, g_2, g_3) = \mathrm{tr}\,(g_1[g_2, g_3])$
- Metric on L_∞-algebra from symplectic structure on NQ-manifold
- Extension to algebroids: Add a manifold in degree 0.

7. Quantized 2-plectic manifolds

- 2-plectic manifold (M, ϖ) comes with a Lie 2-algebra:
  - Hamiltonian one-forms: $\iota_{X_\alpha}\omega = \mathrm{d}\alpha$
  - $V^* = \mathcal{C}^\infty(M) \oplus \Omega^1_{\mathrm{Ham}}(M)$
  - $\pi_1 = \mathrm{d}$, $\pi_2(\alpha, \beta) = \iota_{X_\alpha}\iota_{X_\beta}\varpi$, $\pi_3(\alpha, \beta, \gamma) = \iota_{X_\alpha}\iota_{X_\beta}\iota_{X_\gamma}\varpi$.
  - Example: Heisenberg Lie 2-algebra of $\mathbb{R}^3$:
    $\xi_i = \frac{1}{2}\varepsilon_{ijk}x^j\mathrm{d}x^k$, $\pi_2(\xi_i, \xi_j) = \varepsilon_{ijk}\mathrm{d}x^k$ and $\pi_3(\xi_i, \xi_j, \xi_k) = -\varepsilon_{ijk}$.
- Associative product on L unknown, problems ... loop space etc.
- Quantization: Lie 2-algebra homomorphism up to $\mathcal{O}(\hbar)$. Restrict to:

$$\mu_1(\hat{X}) = -\mathrm{i}\hbar\,\widehat{\pi_1(X)} + \mathcal{O}(\hbar)\ ,\quad \mu_2(\hat{X}, \hat{Y}) = -\mathrm{i}\hbar\,\widehat{\pi_2(X, Y)} + \mathcal{O}(\hbar^2)\ ,$$
$$\mu_3(\hat{X}, \hat{Y}, \hat{Z}) = -\mathrm{i}\hbar\,\widehat{\pi_3(X, Y, Z)} + \mathcal{O}(\hbar^2)\ .$$

8. Homogeneous Lie 2-algebra models

- Fields $X^a = p^a + \xi^a$ in metric Lie 2-algebra

$$\begin{aligned} S_\infty :=& \tfrac{1}{2}m_{ab}\left(X^a, X^b\right) + \tfrac{1}{3}c_{abc}\left(X^a, \mu_2(X^b, X^c)\right) + \tfrac{1}{4}\left(\mu_2(X^a, X^b), \mu_2(X^a, X^b)\right) \\ =& \tfrac{1}{2}m_{ab}\left(p^a, p^b\right) + \tfrac{1}{2}m_{ab}\left(\xi^a, \xi^b\right) + \tfrac{1}{3}c_{abc}\left(\xi^a, \mu_2(\xi^b, \xi^c)\right) + \\ &+ \tfrac{1}{4}\left(\mu_2(\xi^a, \xi^b), \mu_2(\xi^a, \xi^b)\right)\ . \end{aligned}$$

- Solutions:
  - $m = c = 0$: quantized $\mathbb{R}^3$
  - $m \neq 0$ or $c \neq 0$: quantized S^3
- Remarkable:
  - This is exactly how the solutions appear in the IKKT model.
  - For $V^* = V_1^* = \mathfrak{u}(N)$, we recover IKKT model.
  - Quantized solutions reduce nicely from 2-plectic to symplectic

9. Optional: Inhomogeneous Lie 2-algebra models

- IKKT obtained from SYM reduced to 0d, what about M2-brane models?
- Lie 2-algebra from "3-Lie algebras are special differential crossed modules"
- Allowing fields $X \in V$ and $Y \in W$ (and superpartners $\Psi \in X$)

$$\begin{aligned} S_{\mathrm{M2}} =& \tfrac{1}{6}\varepsilon^{ijk}\langle Y^i, \mu_2(Y^j, Y^k)\rangle - \tfrac{1}{2}\langle \mu_2(Y^i, X^a), \mu_2(Y^i, X^a)\rangle + \tfrac{\mathrm{i}}{2}\langle \bar{\Psi}, \mu_2(\Gamma^i Y^i, \Psi)\rangle \\ &- \tfrac{\mathrm{i}}{4}\langle \bar{\Psi}, \mu_2(\mu_2^*(X^a, X^b), \Gamma_{ab}\Psi)\rangle - \tfrac{1}{12}\langle \mu_2(\mu_2^*(X^a, X^b), X^c), \mu_2(\mu_2^*(X^a, X^b), X^c)\rangle \end{aligned}$$

- For L a differential crossed module with $\mathsf{t} = 0$: (generalized) BLG model.

10. Optional: Background field expansion

- Higher gauge theory with one- and two-forms A, B,

$$\begin{aligned}\mathcal{F} :=& \mathrm{d}A + \tfrac{1}{2}\mu_2(A,A) - \mu_1(B) = 0 \ ,\\ \mathcal{H} :=& \mathrm{d}B + \mu_2(A,B) + \tfrac{1}{6}\mu_3(A,A,A) \ .\end{aligned}$$

- Consider BF-theory:

$$S_{\mathrm{BF}} = \int_{\mathbb{R}^3} \langle \lambda_1, \mathcal{F}\rangle + \langle \lambda_0, H\rangle \ , \tag{10.1}$$

- Dimensionally reduce: $A \to Y_i$, $B \to X_{ij}$

$$\begin{aligned}S_{0d} =& \varepsilon^{ijk}\langle \lambda_i, \mu_2(Y_j,Y_k) - \tfrac{1}{2}\mu_1(X_{jk})\rangle + 2\mathrm{i}\hbar\langle \lambda_i, \widehat{\mathrm{d}x^i}\rangle\\ &+ \varepsilon^{ijk}\langle \lambda, \tfrac{1}{2}\mu_2(Y_i,X_{jk}) + \tfrac{1}{6}\mu_3(Y_i,Y_j,Y_k)\rangle - \mathrm{i}\hbar\langle\lambda, \mathbb{1}\rangle \ .\end{aligned}$$

- Solutions: Heisenberg 2-algebra of $\mathbb{R}^3$.
  Moreover: $\mathrm{d}x^i \wedge \pi_2(\xi_i, \alpha) = \mathrm{d}\alpha$, $\xi_i \to \partial_i$.

- Expand around background: $Y_i = \hat{\xi}_i + \hat{A}_i$ and $X_{ij} = 0 + \hat{B}_{ij}$:

$$\begin{aligned}\hat{\mathcal{F}}_{ij} &= \mu_2(\hat{\xi}_i, \hat{A}_j) - \mu_2(\hat{\xi}_j, \hat{A}_i) + \mu_2(\hat{A}_i, \hat{A}_j) - \mu_1(\hat{B}_{ij}) \ ,\\ \hat{H}_{ijk} &= \tfrac{1}{2}\mu_2(\hat{\xi}_{[i}, \hat{B}_{jk]}) + \tfrac{1}{2}\mu_2(\hat{A}_{[i}, \hat{B}_{jk]}) + \tfrac{1}{6}\mu_3(\hat{\xi}_i + \hat{A}_i, \hat{\xi}_j + \hat{A}_j, \hat{\xi}_k + \hat{A}_k) - \mathrm{i}\hbar\hat{\mathbb{1}}\end{aligned}$$

- This is BF-theory on quantum $\mathbb{R}^3$.

11. Summary

- Started a study of 0d Lie 2-algebra models
- Structure easier than higher gauge theory
- Recovered essentially everything known classically about the IKKT model
- Models comprise IKKT-model and reductions of M2-brane models
- Quantization via one-forms reasonable
- Also: compatibility with loop space
- Extension to L_∞-algebra models
- Functorial assignment of Poisson algebra

Future directions

- Full geometric quantization of S^3
- Use as gauge Lie algebra in BLG-model