

# $EL_\infty$ -algebras: Definition and Applications



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- [arXiv:2104.?????](#) with Leron Borsten and Hyungrok Kim

**YOU NEED MOTIVATION?**



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## Motivation: Five Questions

What is the algebraic structure underlying **Courant algebroids**?

Answers in the literature:

- **Roytenberg (2002)**:

An (exact) **Courant algebroid** is the symplectic dg-manifold

$$\mathcal{V}_2 = T^*[2]T[1]M, \quad \omega = dx^\mu \wedge dp_\mu + d\xi^\mu \wedge d\zeta^\mu,$$

$$Q = \{S, -\}, \quad S = \xi^\mu p_\mu$$

for  $M$  some manifold. **Dorfman** and **Courant** brackets:

$$[X, Y]_D = \{QX, Y\}, \quad [X, Y]_C = \frac{1}{2}(\{QX, Y\} - \{QY, X\})$$

- cf. **Rogers (2011)**:

- $[-, -]_C$  part of  $L_\infty$ -algebra
- $[-, -]_D$  part of dg-Leibniz algebra.

- **Is there more to it?**

This may be seen as a niche question, but:

- Courant algebroids underlie Hitchin's **Generalized Geometry**
- Application in supergravity: **Generalized tangent bundle**
- All generalized tangent bundles are **symplectic  $L_\infty$ -algebroids**
- Dorfman bracket structure relevant in **tensor hierarchies**
- Currently relevant: **Double** and **Exceptional Field Theory**.

In order to further understand supergravity:  
understand **symplectic  $L_\infty$ -algebroids!**

What is the alg. structure underlying **multisymplectic manifolds**?

Answers in the literature:

- **Multisymplectic/ $p$ -plectic manifold**  $(M, \omega)$ :

$$\omega \in \Omega^{p+1}(M), \quad d\omega = 0, \quad \iota_X \omega = 0 \Leftrightarrow X = 0$$

comes with Hamiltonian  $p - 1$ -forms  $\alpha$  with  $X_\alpha$ :

$$\iota_{X_\alpha} \omega = d\alpha$$

This leads to brackets:

$$[\alpha, \beta]_h = \mathcal{L}_{X_\alpha} \beta, \quad [\alpha, \beta]_s = \iota_{X_\alpha} \iota_{X_\beta} \omega$$

- cf. Baez, Hoffnung, Rogers (2008), Rogers (2011):
  - $[\alpha, \beta]_s$  is part of  $L_\infty$ -algebra
  - $[\alpha, \beta]_h$  for  $p = 2$  is part of **hemistrict Lie 2-algebra**.

Relation to symplectic  $L_\infty$ -algebroids:

- The  $L_\infty$ -algebra of  $(M, \omega)$  sits inside the  $L_\infty$ -algebra of  $\omega$ -twisted  $T^*[2]T[1]M$ .

Again, it sounds like a niche question, but:

- Higher version of **function algebras** over symplectic manifolds.
- Objects to be quantized in **higher geometric quantization**.
- Crucial for emergent geometry from noncommutative spaces:
  - Traditional NC geometry: mostly **Kähler manifolds**
  - More general: **Nambu-Poisson manifold**
  - Physical arguments for these.
  - More general: **multisymplectic manifolds**
  - **Abstract Nonsense** suggests these
- Also: gauge algebras for **M2-brane models** (fuzzy funnels)

In order to further understand emergent spacetimes  
as well as M2-brane models:  
understand **multisymplectic geometry!**

How to construct “good” curvatures for non-abelian gauge potentials in presence of  $B$ -field?

Answers in the literature:

- Bergshoeff et al. (1982), Chapline et al. (1983): Use Chern-Simons terms:

$$F = dA + \frac{1}{2}[A, A] , \quad H = dB + (A, dA) + \frac{1}{3}(A, [A, A])$$

- This is at odds with the “conventional” non-abelian gerbes:

$$F = dA + \frac{1}{2}[A, A] , \quad H = dB - \frac{1}{3}(A, [A, A])$$

- Sati, Schreiber (2009): Solution: twist definition of curvatures

$$F = dA + \frac{1}{2}[A, A] , \quad H = dB + (A, F) - \frac{1}{3}(A, [A, A])$$

- Where does  $(-, -)$  come from?



Observation:

- Traditional non-abelian gerbes: **fake curvature**  $\mathcal{F} = 0$
- All naive definitions **problematic** unless  $\mathcal{F} = 0$ :
  - Parallel transport
  - Topological invariants
  - Higher non-flat theories
- Then: can **gauge away** non-abelian parts.

Need this geometry:

- **Tensor hierarchies** of gauge supergravity
- **Heterotic supergravity**
- **6d superconformal field theories**

What is a **small cofibrant replacement** for the operad  $\mathcal{L}ie$  when working over finite characteristic?

Just briefly:

- **Operads** underlie and determined all kinds of algebras
- Cofibrant replacement: replace by **weakly equivalent object** with nice homotopy properties
- Over characteristic 0, can replace  $\mathcal{L}ie$  by  $\mathcal{L}ie_\infty$ , the cobar construction applied to the Koszul dual cooperad  $\mathcal{L}ie^!$ .
- Algebras over  $\mathcal{L}ie_\infty$  are just the usual  $L_\infty$ -algebras

Answers in the literature:

- Higher cobar-bar adjunction, but result **too big**.
- **Dehling (2017)**: Step-by-step construction
- Explicit for up to degree 3 (“**Weak Lie 3-algebras**”)

# Fifth question (mathematics)

Asked by **Loday (1993)**:

What do **Leibniz algebras** integrate to (“**coquecigrue problem**”)?

Recall:

- **Leibniz algebra**  $\mathfrak{a}$ :

$$\{-, -\} : \mathfrak{a} \otimes \mathfrak{a} \rightarrow \mathfrak{a},$$
$$\{a, \{b, c\}\} = \{\{a, b\}, c\} + \{b, \{a, c\}\}$$

Answer in the Literature:

- **Kinyon (2004)**: Lie racks

NB:

**Coquecigrue**: an imaginary creature regarded as an embodiment of absolute absurdity.





# Relevance of fourth and fifth questions

If we understood algebras properly, we should be able to answer such questions, at least in principle, by **Abstract Nonsense**.

All these questions:

- 1) Algebraic structure underlying **symplectic  $L_\infty$ -algebroids?**
- 2) Algebraic structure underlying **multisymplectic manifolds?**
- 3) Algebraic structure underlying **higher curvature forms?**
- 4) Cofibrant replacement of *Lie*?
- 5) How do you integrate Leibniz algebras?

have a simple, unifying answer:

*EL* $_\infty$ -algebras

**NOT SURE IF CATEGORY  
THEORY**

**OR JUST REGULAR ABSTRACT  
NONSENSE**

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# Starting point of construction

Hint: Start with **2-plectic manifolds** and **hemistrict Lie algebras**.

- Math. Objects: **Stuff + Structure + Structure Relations**
- E.g.: **Lie algebra**: Vector space  $V$  with Lie bracket  $[\cdot, \cdot]$ :  
 $[v, w] = -[w, v]$  and  $[u, [v, w]] + [v, [w, u]] + [w, [u, v]] = 0$
- **Internal categorification**:
  - “stuff”  $\rightarrow$  (small) **category**, objects and morphisms of “stuff”
  - “structure”  $\rightarrow$  **functors**
  - structure relations hold “**up to isomorphisms**”
  - functors satisfy **coherence axioms**
- **Weak Lie 2-algebra** is a category  $\mathcal{L}$ : **Roytenberg, 2007**
  - objects and morphisms form vector spaces
  - endowed with functor  $[\cdot, \cdot] : \mathcal{L} \times \mathcal{L} \rightarrow \mathcal{L}$
  - natural trafos:  $\text{Alt} : [v, w] \Rightarrow -[w, v]$   
 $\text{Jac} : [u, [v, w]] + [v, [w, u]] \Rightarrow -[w, [u, v]]$



Lie 2-algebras are equivalent to **differential graded algebras**  $\mathfrak{L}$  with

$$\varepsilon_2 : \mathfrak{L} \otimes \mathfrak{L} \rightarrow \mathfrak{L} , \quad |\varepsilon_2| = 0 , \quad \text{alt} : \mathfrak{L} \otimes \mathfrak{L} \rightarrow \mathfrak{L} , \quad |\text{alt}| = -1$$

Generalize, preserving differential compatibility:  **$h\mathcal{L}ie$ -algebras**

## $h\mathcal{L}ie$ -algebras

Graded vector space  $\mathfrak{L}$  with

$$\varepsilon_1 : \mathfrak{L} \rightarrow \mathfrak{L} , \quad |\varepsilon_1| = 1 , \quad \varepsilon_2^i : \mathfrak{L} \otimes \mathfrak{L} \rightarrow \mathfrak{L} , \quad |\varepsilon_2^i| = -i$$

such that

$$\varepsilon_1(\varepsilon_1(x_1)) = 0 ,$$

$$\varepsilon_1(\varepsilon_2^i(x_1, x_2)) = \pm \varepsilon_2^i(\varepsilon_1(x_1), x_2) \pm \varepsilon_2^i(x_1, \varepsilon_1(x_2)) + \varepsilon_2^{i-1}(x_1, x_2) \mp \varepsilon_2^{i-1}(x_2, x_1)$$

$$\varepsilon_2^i(\varepsilon_2^i(x_1, x_2), x_3) = \pm \varepsilon_2^i(x_1, \varepsilon_2^i(x_2, x_3)) \mp \varepsilon_2^i(x_2, \varepsilon_2^i(x_1, x_3)) \mp \varepsilon_2^{i+1}(x_2, \varepsilon_2^{i-1}(x_3, x_1))$$

$$\varepsilon_2^j(\varepsilon_2^i(x_1, x_2), x_3) = \pm \varepsilon_2^{i+1}(x_2, \varepsilon_2^{j-1}(x_3, x_1))$$

$$\varepsilon_2^i(\varepsilon_2^j(x_1, x_2), x_3) = \pm \varepsilon_2^j(x_1, \varepsilon_2^i(x_2, x_3)) \mp \varepsilon_2^i(x_2, \varepsilon_2^j(x_1, x_3)) \pm \varepsilon_2^{i+1}(x_3, \varepsilon_2^{j-1}(x_1, x_2))$$

Generalizes **hemistrict Lie 2-algs** and specializes **dg-Leibniz algs**.

Recall:

- Koszul dual of  $\mathcal{L}ie$  is  $\mathcal{C}om$
- Therefore:

$$\begin{array}{ccc} L_{\infty}\text{-algebras} & \leftrightarrow & \text{dg-com algebras} \\ \mu_i & \leftrightarrow & Q \end{array}$$

Here:

- Koszul dual of  $h\mathcal{L}ie$  is  $\mathcal{E}ilh$
- $\mathcal{E}ilh$  has products  $\mathcal{O}_i$  of degree  $i$
- Relations dual to the one shown before (generalised **Zinbiel**)

$EL_\infty$ -algebras

$EL_\infty$ -algebra are **homotopy  $h\mathcal{L}ie$ -algebras**.

That is, graded vector space  $\mathfrak{L}$  with higher products

$$\varepsilon_1 : \mathfrak{L} \rightarrow \mathfrak{L}, \quad |\varepsilon_1| = 1,$$

$$\varepsilon_2^i : \mathfrak{L} \otimes \mathfrak{L} \rightarrow \mathfrak{L}, \quad |\varepsilon_2^i| = -i$$

$$\varepsilon_3^{ij} : \mathfrak{L} \otimes \mathfrak{L} \otimes \mathfrak{L} \rightarrow \mathfrak{L}, \quad |\varepsilon_3^{ij}| = -i - j,$$

$$\vdots \quad \quad \quad \vdots$$

such that

$$\varepsilon_1(\varepsilon_1(x_1)) = 0,$$

$$\varepsilon_1(\varepsilon_2^i(x_1, x_2)) = \pm \varepsilon_2^i(\varepsilon_1(x_1), x_2) \pm \varepsilon_2^i(x_1, \varepsilon_1(x_2)) + \varepsilon_2^{i-1}(x_1, x_2) \mp \varepsilon_2^{i-1}(x_2, x_1)$$

$$\vdots \quad \quad \quad \vdots$$

amounting to  $Q^2 = 0$  in the corresponding dual  $\mathcal{E}ilh$ -algebra.

Note: if  $\varepsilon_k^I = 0$  for  $I \neq (0, 0, \dots, 0)$ , then this is  $L_\infty$ -algebra.

## They generalize:

- (dg) Lie algebras
- $L_\infty$ -algebras
- Roytenberg's hemistrict and semistrict Lie 2-algebras
- Dehlings weak Lie 3-algebras
- $\Rightarrow$  They are weak Lie  $\infty$ -algebras

## They specialize:

- Leibniz algebras
- homotopy Leibniz algebras

## Properties:

- Modified homotopy transfer (modified tensor trick)
- Minimal model and strictification theorems
- $EL_\infty$ -algebras antisymmetrize to  $L_\infty$ -algebras
- An  $L_\infty$ -algebras in each quasi-isomorphism class

- 1) Algebraic structure underlying **symplectic  $L_\infty$ -algebroids**?
  - Note: symplectic  $L_\infty$ -algebroids come with **dg-Lie algebra**
  - **Theorem**: Any dg-Lie algebra induces an  **$h\mathcal{L}ie$ -algebra**.
  - This  **$h\mathcal{L}ie$** -structure produces **Dorfman bracket**.
  - Antisymmetrization produces **Courant bracket**.
- 2) Algebraic structure underlying **multisymplectic manifolds**?
  - Note: multisymplectic manifold come with **dg-Lie algebra** extends the Lie algebra of Hamiltonian vector fields

$$L(M, \varpi) = \left( \underbrace{\Omega^0(M)}_{L(M, \varpi)_{-n}} \xrightarrow{d} \underbrace{\Omega^1(M)}_{L(M, \varpi)_{1-n}} \xrightarrow{d} \dots \xrightarrow{d} \underbrace{\Omega_{\text{Ham}}^{n-1}(M)}_{L(M, \varpi)_{-1}} \xrightarrow{\delta} \underbrace{\mathfrak{X}(M)}_{L(M, \varpi)_0} \right)$$

- Induced  **$h\mathcal{L}ie$** -bracket yields hemistrict brackets includes dg-Leibniz structure observed before.
- Antisymmetrization recovers  $L_\infty$ -algebra structure

- 3) Algebraic structure underlying **higher curvature forms**?
- $\exists$  **homotopy Maurer–Cartan theory**, but not much richer.
  - Need to extend to **Weil algebras** to define curvatures
  - In all examples: **adjusted Weil algebras** exist.
  - $EL_\infty$ -algebras allow for “**nice, good**” curvatures
  - **Conjecture**: in every quasi-isomorphism class, there is one representant with an **adjusted Weil algebra**.
  - $\Rightarrow$  **Good curvatures** for any higher gauge algebra.

NB:

- **Literature**: all kinds of variations of Leibniz algebras.
- Some get close to special cases of  $h\mathcal{L}ie$ .
- Gauge algebras: infinitesimal symmetries, need to **integrate!**
- $EL_\infty$ -algebras do this by **Abstract Nonsense**.

4) Cofibrant replacement of  $\mathcal{L}ie$ ?

- Yes:  $h\mathcal{L}ie_\infty$ .

5) How do you integrate Leibniz algebras?

- **Theorem:** Any Leibniz algebra is an  $h\mathcal{L}ie$ -algebra.
- Any  $h\mathcal{L}ie$ -algebra is a Lie 2-algebra and thus integrable by **Abstract Nonsense**.
- **Conjecture:** Any homotopy Leibniz algebra is an  $EL_\infty$ -algebra, and thus integrable by Abstract Nonsense.

- Constructed  $EL_\infty$ -algebras, special kind of **homotopy algebra**
- Generalize  $L_\infty$ -algebras
- They are higher Lie algebras, describe **infinitesimal symmetries**
- Appear in many contexts in maths/physics. In particular:
  - **Higher gauge theory**
  - **Tensor hierarchies** of supergravity



Thank You!