Extended Riemannian Geometry and Double Field Theory

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Based on:

- arXiv:1611.02772 with Andreas Deser

Motivation

Clarifying the mathematical structures underlying Double Field Theory.

Aim: Clarify mathematical structures behind Double Field Theory

Double Field Theory (DFT)

- target space formulations useful (SUSY YM, SUGRA)
- T-duality very interesting

$\Big\}$ DFT

Tseytlin, Siegel, Hull, Zwiebach, . . .

- Success, e.g.: DFT action on $\mathbb{R}^{2D}$:

$$S = \int \mathrm{d}^{2D}x \; \mathrm{e}^{-2d} \Big(\tfrac{1}{8}\mathcal{H}_{MN}\partial^M\mathcal{H}_{KL}\partial^N\mathcal{H}^{KL} - \tfrac{1}{2}\mathcal{H}_{MN}\partial^M\mathcal{H}_{KL}\partial^L\mathcal{H}^{KN} - 2\partial^M d\partial^N\mathcal{H}_{MN} + 4\mathcal{H}_{MN}\partial^M d\partial^N d \Big)$$

$\mathcal{H}_{MN}$: generalized metric, d: DFT dilaton

- Reproduces SUGRAs previously not derived from string theory
- sensible? strange truncation of string modes? seems to work

Motivation

Clarifying the mathematical structures underlying Double Field Theory.

- Massless string modes: metric g, 2-form B, dilaton ϕ.
- g and B transform differently, combine in generalized metric:

$$\mathcal{H}_{MN} = \begin{pmatrix} g_{\mu\nu} - B_{\mu\kappa} g^{\kappa\lambda} B_{\lambda\nu} & B_{\mu\kappa} g^{\kappa\nu} \\ -g^{\mu\kappa} B_{\kappa\nu} & g^{\mu\nu} \end{pmatrix} \quad \text{on} \quad T^*M \oplus TM$$

- Manifest T-duality: double space, coords.: $x^M = (x^\mu, x_\mu)$
- Level matching: Fields ϕ in DFT satisfy $\Box\phi = 0$
- Algebra of fields: strong section condition:

$$\partial^M \phi \partial_M \psi = 0$$

- Generalized Lie derivative wrt. $X = X^\mu \frac{\partial}{\partial x^\mu} + X_\mu \frac{\partial}{\partial x_\mu}$:

$$\hat{\mathcal{L}}_X \mathcal{H}_{MN} = X^P \partial_P \mathcal{H}_{MN} + (\partial_M X^P - \partial^P X_M)\mathcal{H}_{PN} + (\partial_N X^P - \partial^P X_N)\mathcal{H}_{MP}$$

- $\hat{\mathcal{L}}_X$ form Lie algebra, but not representation of Lie algebra

$$\hat{\mathcal{L}}_X \hat{\mathcal{L}}_Y - \hat{\mathcal{L}}_Y \hat{\mathcal{L}}_X = \hat{\mathcal{L}}_{\mu_2(X,Y)} \ ,$$

- $\mu_2(X,Y)$ known as C-bracket (actual interpretation later)

Motivation

Clarifying the mathematical structures underlying Double Field Theory.

Issues to address in a mathematical formulation:

- Issue 1: Algebraic structure behind C-bracket
- Issue 2: Section condition is ugly/coordinate dependent:

$$\partial^M \phi \, \partial_M \psi = 0$$

- Issue 3: Section condition is often too strong
- Issue 4: Reasonable version of Doubled Riemannian Geometry

$$\begin{aligned}\widehat{\Gamma}_{MNK} \;=\; & -2(P\partial_M P)_{[NK]} - 2\big(\bar{P}_{[N}{}^P \bar{P}_{K]}{}^Q - P_{[N}{}^P P_{K]}{}^Q\big)\partial_P P_{QM} \\ & + \frac{4}{D-1}\big(P_{M[N}P_{K]}{}^Q + \bar{P}_{M[N}\bar{P}_{K]}{}^Q\big)\big(\partial_Q d + (P\partial^P P)_{[PQ]}\big)\end{aligned}$$

- Issue 5: Global formulation of DFT
- Issue 6: Same for Exceptional field theory (M-theory analogue)

Outline of Arguments

- Massless string modes: metric g, 2-form B, dilaton ϕ.
- Recall: B-field belongs to gerbe/categorified principal bundle
- Generalized Geometry: Courant/Lie 2-algebroid $TM \oplus T^*M$
- Captures Lie 2-algebra of symmetries (gauge+diffeos)
- In particular: Courant/Dorfman brackets
- Similarly: GR coupled to n-form fields
- Picture allows for an extension to pre-Lie 2-algebroids
- Lie 2-algebra with C- and D-brackets
- Categorified Jacobi relations $\Leftrightarrow$ weakened section condition
- Fully algebraic and coordinate independent picture
- Know how to twist pre-Lie 2-algebroids
- Potential global picture (work in progress)

Part I : Generalized Geometry

Recall: Generalized Geometry

Generalized Geometry provides a nice geometric description of the B-field.

Ingredients: massless string modes: metric g, 2-form B, dilaton ϕ.

- Well-known: B belongs to connective structure of a gerbe
  Gawedzki 1987, Freed&Witten 1999

Gerbes - Intuitive Picture

- Smooth Manifold $\longrightarrow$ smooth categories $\mathcal{M}_1 \rightrightarrows \mathcal{M}_0$
- Smooth morphisms between manifolds $\longrightarrow$ smooth functors
- Structure group $\longrightarrow$ category, product on objects, morphisms
- There are obvious categorified notions of:
  fibration, surjection, diffeomorphism, free, transitive action
- Define categorified principal bundles in obvious manner
- Gerbe: categorified principal bundle for $\mathsf{U}(1) \rightrightarrows *$

Recall: Čech Description of a Gerbe

Generalized Geometry provides a nice geometric description of the B-field.

Surjective submersion

Manifold M, cover $Y = \sqcup_i U_i \twoheadrightarrow M$, $Y^{[2]} = \sqcup_{i,j} U_i \cap U_j$, $\ldots$

Principal U(1)-bundle over M

$$\mathsf{U}(1) \to P \to M \ , \quad g_{ij} g_{jk} = g_{ik} \ , \quad A_i = g_{ij}^{-1}(A_j + \mathrm{d}) g_{ij}$$

Symmetries: $\mathsf{U}(1)$ gauge symmetry and diffeomorphisms.

U(1)-gerbe $\mathscr{G}$ over M

$$\begin{tikzcd}
\mathsf{U}(1) \arrow[dd, shift left] \arrow[dd, shift right] & & \mathscr{G}_1 \arrow[dd, shift left] \arrow[dd, shift right] & & Y^{[2]} \arrow[dd, shift left] \arrow[dd, shift right] & & M \arrow[dd, shift left] \arrow[dd, shift right] \\
& {} \arrow[r] & {} & {} \arrow[r] & {} & {} \arrow[r, "\cong"] & {} \\
* & & \mathscr{G}_0 & & Y & & M
\end{tikzcd}$$

$$h_{ijk} h_{ik\ell} = h_{ij\ell} h_{jk\ell} \ , \ A_{ij} - A_{ik} + A_{jk} = \mathrm{d} \log(h_{ijk}) \ , \ B_i - B_j = \mathrm{d} A_{ij}$$

Categorified spaces ⇒ categorified symmetries!

Generalized Geometry, Local Description

Generalized Geometry provides a nice geometric description of the B-field.

Ordinary geometry: “Local” Atiyah Algebroid for U(1)-bundle

$$0 \rightarrow M \times \mathrm{i}\mathbb{R} \xrightarrow{\hookrightarrow} TM \oplus \mathrm{i}\mathbb{R} \xrightarrow{\mathrm{pr}} TM \rightarrow 0$$

- Gauge potential $A : TM \rightarrow \mathrm{i}\mathbb{R}$ is defined by section of pr
- Infinitesimal symm. (diffeos+gauge): sections of $TM \oplus \mathrm{i}\mathbb{R}$

Generalized Geometry: Courant algebroid for U(1)-gerbe

$$0 \rightarrow T^*M \xrightarrow{\hookrightarrow} TM \oplus T^*M \xrightarrow{\mathrm{pr}} TM \rightarrow 0$$

- $g + B : TM \rightarrow T^*M$ again from section of pr
- Infinitesimal symm. (diffeos+gauge): sections of $TM \oplus T^*M$

not complete picture:

$$\begin{tikzcd}
\bullet \arrow[r, bend left=50, "X+\Lambda", ""{name=U, below}] \arrow[r, bend right=50, "X+\Lambda+\mathrm{d}f"', ""{name=D}] & \bullet \arrow[Rightarrow, from=U, to=D, "f"]
\end{tikzcd}$$

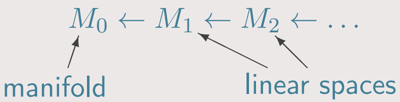
more later

NQ-Manifolds

NQ-manifolds, known from BRST quantization, provide very useful language.

N-manifolds, NQ-manifold

- $\mathbb{N}_0$-graded manifold with coordinates of degree $0, 1, 2, \ldots$



$$M_0 \leftarrow M_1 \leftarrow M_2 \leftarrow \ldots$$

manifold

linear spaces

- NQ-manifold: vector field Q of degree 1, $Q^2 = 0$
- Physicists: think ghost numbers, BRST charge, SFT

Examples:

- Tangent algebroid $T[1]M$, $\mathcal{C}^\infty(T[1]M) \cong \Omega^\bullet(M)$, $Q = \mathrm{d}$
- Lie algebra $\mathfrak{g}[1]$, coordinates ξ^a of degree 1:

$$Q = -\tfrac{1}{2} f_{ab}^c \xi^a \xi^b \frac{\partial}{\partial \xi^c} \quad , \quad \text{Jacobi identity} \Leftrightarrow \; Q^2 = 0$$

L_∞-Algebras, Lie 2-Algebras

NQ-manifolds provide an easy definition of L_∞-algebras.

Lie n-algebroid or n-term L_∞-algebroid:

$$M_0 \leftarrow M_1 \leftarrow M_2 \leftarrow \ldots \leftarrow M_n \leftarrow * \leftarrow * \leftarrow \ldots$$

Lie n-algebra or n-term L_∞-algebra:

$$* \leftarrow M_1 \leftarrow M_2 \leftarrow \ldots \leftarrow M_n \leftarrow * \leftarrow * \leftarrow \ldots$$

Important example: Lie 2-algebra

- Graded vector space: $W[1] \leftarrow V[2]$, Category: $V \oplus W \rightrightarrows W$
- Coordinates: w^a of degree 1 on $W[1]$, v^i of degree 2 on $V[2]$
- Most general vector field Q of degree 1:

$$Q = -m_i^a v^i \frac{\partial}{\partial w^a} - \tfrac{1}{2} m_{ab}^c w^a w^b \frac{\partial}{\partial w^c} - m_{ai}^j w^a v^i \frac{\partial}{\partial v^j} - \tfrac{1}{3!} m_{abc}^i w^a w^b w^c \frac{\partial}{\partial v^i}$$

- Induces “brackets”/“higher products”:

$$\mu_1(\tau_i) = m_i^a \tau_a \ , \quad \mu_2(\tau_a, \tau_b) = m_{ab}^c \tau_c \ , \quad \ldots \ , \quad \mu_3(\tau_a, \tau_b, \tau_c) = m_{abc}^i \tau_i$$

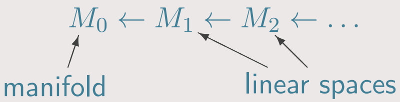
- $Q^2 = 0 \Leftrightarrow$ Homotopy Jacobi identities, e.g. $\mu_1(\mu_1(-)) = 0$
- Failure of Jacobi identity: $\mu_2(x, \mu_2(y, z)) + \ldots = \mu_1(\mu_3(x, y, z))$

Symplectic NQ-Manifolds

Symplectic NQ-manifolds are very convenient to work with.

Symplectic NQ-manifold

- $\mathbb{N}_0$-graded manifold, vector field Q of degree 1, $Q^2 = 0$



$$M_0 \leftarrow M_1 \leftarrow M_2 \leftarrow \dots$$

manifold

linear spaces

- symplectic NQ-manifold: ω nondegenerate, closed, $\mathcal{L}_Q \omega = 0$

Examples of symplectic NQ-manifolds:

- Metric Lie algebra:

$$(\mathfrak{g}[1], \omega = g_{\alpha\beta} \mathrm{d}\xi^\alpha \wedge \mathrm{d}\xi^\beta) \ , \quad Q = -\tfrac{1}{2} f^c_{ab} \xi^a \xi^b \frac{\partial}{\partial \xi^c}$$

- Symplectic manifolds (symplectic Lie 0-algebroids):

$$(M, \omega) \ , \quad Q = 0$$

- Poisson manifolds (symplectic Lie 1-algebroids):

$$T^*[1]M \ , \quad \omega = \mathrm{d}p_\mu \wedge \mathrm{d}x^\mu \ , \quad Q = \pi^{\mu\nu} p_\mu \frac{\partial}{\partial x^\nu}$$

- Courant algebroids (symplectic Lie 2-algebroids, more later)

The L_∞-Algebra of a Symplectic L_∞-algebroid

Every symplectic L_∞-algebroid comes with an L_∞-algebra.

An underappreciated/widely unknown fact:

All symplectic Lie n-algebroids $\mathcal{M}$ come with Lie n-algebra $\mathsf{L}(\mathcal{M})$:

$$\begin{tikzcd}
\mathcal{C}^\infty_0(\mathcal{M}) \arrow[r, "Q"] & \mathcal{C}^\infty_1(\mathcal{M}) \arrow[r, "Q"] & \dots \arrow[r, "Q"] & \mathcal{C}^\infty_{n-2}(\mathcal{M}) \arrow[r, "Q"] & \mathcal{C}^\infty_{n-1}(\mathcal{M}) \\
\mathsf{L}_{n-1}(\mathcal{M}) \arrow[r, "Q"] & \mathsf{L}_{n-2}(\mathcal{M}) \arrow[r, "Q"] & \dots \arrow[r, "Q"] & \mathsf{L}_1(\mathcal{M}) \arrow[r, "Q"] & \mathsf{L}_0(\mathcal{M})
\end{tikzcd}$$

$$\mu_1(\ell) = \begin{cases} 0 & \ell \in \mathcal{C}^\infty_{n-1}(\mathcal{M}) = \mathsf{L}_0(\mathcal{M}) \\ Q\ell & \text{else} \end{cases}$$

$$\mu_2(\ell_1, \ell_2) = \tfrac{1}{2}\big(\{\delta\ell_1, \ell_2\} \pm \{\delta\ell_2, \ell_1\}\big)~, \qquad \delta(\ell) = \begin{cases} Q\ell & \ell \in \mathsf{L}_0(\mathcal{M}) \\ 0 & \text{else} \end{cases}$$

$$\mu_3(\ell_1, \ell_2, \ell_3) = -\tfrac{1}{12}\big(\{\{\delta\ell_1, \ell_2\}, \ell_3\} \pm \dots\big)$$

Roytenberg, Rogers, Fiorenza, Getzler

- "Derived brackets" (Kosmann-Schwarzbach, Voronov, ...)
- Important class of examples next

The Courant Algebroid as a Symplectic NQ-Manifold

The key geometric structure behind generalized geometry is locally $T^*[2]T[1]M$.

The symplectic NQ-manifold $T^*[2]T[1]M$

Local description, choose $M = \mathbb{R}^D$ as base manifold.

$$\begin{array}{ccccccc} T^*[2]T[1]M = \mathbb{R}^D & \oplus & \mathbb{R}^D[1] & \oplus & \mathbb{R}^D[1] & \oplus & \mathbb{R}^D[2] \\ x^\mu & & \xi^\mu & & \zeta_\mu & & p_\mu \end{array}$$

$$Q = \xi^\mu \frac{\partial}{\partial x^\mu} + p_\mu \frac{\partial}{\partial \zeta_\mu} \qquad \omega = \mathrm{d}x^\mu \wedge \mathrm{d}p_\mu + \mathrm{d}\xi^\mu \wedge \mathrm{d}\zeta_\mu$$

- functions on $T^*[2]T[1]M$ of degree 0: functions on M
- functions on $T^*[2]T[1]M$ of degree 1: sections of $TM \oplus T^*M$
- Poisson bracket induced by ω, e.g. on degree-1 functions:

$$\{X + \alpha, Y + \beta\} = \iota_X \alpha + \iota_Y \beta$$

Associated Algebraic Structures of Courant Algebroid

The key geometric structure behind generalized geometry is locally $T^*[2]T[1]M$.

- $\mathcal{M}=T^*[2]T[1]\mathbb{R}^D=\mathbb{R}^D\oplus\mathbb{R}^{2D}[1]\oplus\mathbb{R}^D[2]$, Q, ω, $\{-,-\}$
- Graded vector space:

$$\begin{aligned}\mathcal{C}^\infty_0(\mathcal{M})\rightarrow\mathcal{C}^\infty_1(\mathcal{M})&=\mathcal{C}^\infty(\mathbb{R}^D)\rightarrow\Gamma(T\mathbb{R}^D\oplus T^*\mathbb{R}^D)\\&=\mathsf{L}_1(\mathcal{M})\rightarrow\mathsf{L}_0(\mathcal{M})\end{aligned}$$

- Derived brackets from differential $\delta(\ell)=\begin{cases}Q\ell & \ell\in\mathsf{L}_0(\mathcal{M})\\ 0 & \text{else}\end{cases}$
- Leibniz algebra from Dorfman bracket $\nu_2(\ell_1,\ell_2)=\{\delta\ell_1,\ell_2\}$
- Lie 2-alg. from Courant bracket $\mu_2(\ell_1,\ell_2)=\{\delta\ell_{[1},\ell_{2]}\}$
- This is the symmetry Lie 2-algebra of corresponding gerbe!
- Categorical picture:

$$\Big(\mathcal{C}^\infty(\mathbb{R}^D)\oplus\Gamma(T\mathbb{R}^D\oplus T^*\mathbb{R}^D)\Big)\rightrightarrows\Gamma(T\mathbb{R}^D\oplus T^*\mathbb{R}^D)\,,$$

$$\begin{tikzcd}\bullet \arrow[r, bend left=50, "X+\Lambda"{name=U}] \arrow[r, bend right=50, "X+\Lambda+\mathrm{d}f"'{name=D}] & \bullet \arrow[Rightarrow, from=U, to=D, "f"]\end{tikzcd}$$

Part of Larger Family: GR + n-Form Potentials

The above picture is part of a larger story, involving GR coupled to n-form potentials.

More generally: Couple GR to n-form gauge potential

Vinogradov algebroids $\mathcal{V}_n(M)$

- Locally as a vector bundle: $\mathcal{V}_n(M) := T^*[n]T[1]M$
- coords: $(x^\mu, \xi^\mu, \zeta_\mu, p_\mu)$ of degrees $(0, 1, n-1, n)$
- Homological vector field: $Q = \xi^\mu \frac{\partial}{\partial x^\mu} + p_\mu \frac{\partial}{\partial \zeta_\mu}$
- Symplectic form: $\omega = \mathrm{d}x^\mu \wedge \mathrm{d}p_\mu + \mathrm{d}\xi^\mu \wedge \mathrm{d}\zeta_\mu$

- Geometric picture: principal n-bundle or $n-1$-gerbe
- diffeos+gauge: degree $n-1$ functions of $\mathcal{V}_n(M)$.
- Higher Courant/Dorfman brackets
- Full Symmetrie Lie n-algebra: Lie n-algebra of $\mathcal{V}_n(M)$

How to extend this to Double Field Theory?

Part II : Double Field Theory / Extended Geometry

Weakening the Definition of Symplectic NQ-Manifolds

There are two properties that offer themselves to a weakening.

Requirements

- Geometry built from (doubled) spacetime
- Symmetry Lie n-algebra structure from derived brackets
- Reduction to Vinogradov algebroids of Generalized Geometry
- a few further points, related to global picture

Need to keep: graded vector bundle, symplectic form, $|Q| = 1$

Note: $Q^2 = 0$ not necessary everywhere for L_∞-algebra:

$$\mathsf{L}_{n-1}(\mathcal{M}) \xrightarrow{Q} \mathsf{L}_{n-2}(\mathcal{M}) \xrightarrow{Q} \dots \xrightarrow{Q} \mathsf{L}_1(\mathcal{M}) \xrightarrow{Q} \mathsf{L}_0(\mathcal{M}) \longrightarrow 0$$

Instead: something like $\{Q^2 -, -\} = 0$.

(Exceptional field theory: Q does not have to be vector field.)

Pre-NQ-Manifolds and L_∞-Structures

All relevant notions can be reasonably extended.

Definition: Symplectic pre-NQ-Manifold of degree n

Symplectic N-manifold $(\mathcal{M},\omega)$ of degree n, i.e. $|\omega|=n$, compatible vector field Q of degree 1, i.e. $|Q|=1$ and $\mathcal{L}_Q\omega=0$.

Definition: L_∞-structure

A subset $\mathsf{L}(\mathcal{M})$ of the functions $\mathcal{C}^\infty(\mathcal{M})$ such that the derived brackets close and form an L_∞-algebra.

Theorem

$(\mathcal{M},\omega)$ of degree 2.
$\mathsf{L}(\mathcal{M})=\mathsf{L}_1(\mathcal{M})\oplus\mathsf{L}_0(\mathcal{M})\subset\mathcal{C}^\infty(\mathcal{M})$, derived brackets close.
$\mathsf{L}(\mathcal{M})$ is L_∞-structure iff for all $f,g\in\mathsf{L}_1(\mathcal{M})$, $X,Y,Z\in\mathsf{L}_0(\mathcal{M})$:

$$\{Q^2f,g\}+\{Q^2g,f\}=0\ ,\quad \{Q^2X,f\}+\{Q^2f,X\}=0$$

$$\{\{Q^2X,Y\},Z\}_{[X,Y,Z]}=0$$

Application to DFT: Weakened Section Condition

This framework is readily applied to reproduce DFT's strong section condition.

- Start: $M=\mathbb{R}^D$, double: T^*M, coords. $x^M=(x^\mu,x_\mu)$
- $\mathcal{V}_2(T^*M)=T^*[2]T[1](T^*M)$, coords. (x^M,ξ^M,ζ_M,p_M)
- $\omega=\mathrm{d}x^M\wedge\mathrm{d}p_M+\mathrm{d}\xi^M\wedge\mathrm{d}\zeta_M\,,\qquad Q=\sqrt{2}\big(\xi^M\frac{\partial}{\partial x^M}+p_M\frac{\partial}{\partial\zeta_M}\big)$
- Crucial to GenGeo/DFT: T-duality group $\mathsf{O}(D,D)$
- Reduce structure group $\mathsf{GL}(2D,\mathbb{R})$ to $\mathsf{O}(D,D)$ by introducing

$$\eta_{MN}=\eta^{MN}=\left(\begin{array}{cc}0&\mathbb{1}\\ \mathbb{1}&0\end{array}\right)$$

- New coordinates:

$$\theta^M=\frac{1}{\sqrt{2}}(\xi^M+\eta^{MN}\zeta_N)\qquad\qquad\beta^M=\frac{1}{\sqrt{2}}(\xi^M-\eta^{MN}\zeta_N)$$

- Polarize by putting $\beta^M\overset{!}{=}0$:

$$Q=\theta^M\frac{\partial}{\partial x^M}+p_M\eta^{MN}\frac{\partial}{\partial\theta^N}\,,\quad Q^2=p_M\eta^{MN}\frac{\partial}{\partial x^N}\neq 0$$

- We obtain a symplectic pre-NQ-manifold, $\mathcal{E}_2(M)$.

Application to DFT: Weakened Section Condition

This framework is readily applied to reproduce DFT's strong section condition.

$$\begin{array}{ccccccc} \mathcal{E}_2(\mathbb{R}^D) & = & \mathbb{R}^{2D} & \oplus & \mathbb{R}^{2D}[1] & \oplus & \mathbb{R}^{2D}[2] \\ & & x^M & & \theta^M & & p_M \end{array}$$

$$Q = \theta^M \frac{\partial}{\partial x^M} + p_M \eta^{MN} \frac{\partial}{\partial \theta^N} \qquad \omega = \mathrm{d}x^M \wedge \mathrm{d}p_M + \tfrac{1}{2}\eta_{MN} \mathrm{d}\theta^M \wedge \mathrm{d}\theta^N$$

Proposition

Let $\mathsf{L} = \mathsf{L}_0 \oplus \mathsf{L}_1$ be L_∞-structure on $\mathcal{E}_2(\mathbb{R}^D)$, $f, g \in \mathsf{L}_1$ and $X = X_M\theta^M, Y = Y_M\theta^M, Z = Z_M\theta^M \in \mathsf{L}_0$. Then:

$$\{Q^2 f, g\} + \{Q^2 g, f\} = 2\partial^M f \ \partial_M g = 0$$

$$\{Q^2 X, f\} + \{Q^2 f, X\} = 2\partial^M X \ \partial_M f = 0$$

$$\{\{Q^2 X, Y\}, Z\}_{[X,Y,Z]} = 2\theta^L \big((\partial^M X_L)(\partial_M Y^K) Z_K\big)_{[X,Y,Z]} = 0$$

Note: Fulfilled for $\partial^M \phi \ \partial_M \psi = 0 \Rightarrow$ Weakened section condition

Relation to Generalized Geometry

Extended Geometry is sandwiched between Generalized Geometries.

The picture we have:

- Symmetries of DFT are described by L_∞-structure on $\mathcal{E}_2(\mathbb{R}^D)$
- Extended Geometry $\mathcal{E}_2(\mathbb{R}^D)$: polarization of GenGeo
- Reduce Ext. Geo. → GenGeo: Choice of L_∞-structure
- Example:

$$\mathsf{L} = \left\{ F \in \mathcal{C}^\infty(\mathcal{E}_2(\mathbb{R}^D)) | \frac{\partial}{\partial x_\mu} F = 0 \right\}$$

$$Q = \theta^\mu \frac{\partial}{\partial x^\mu} + p_\mu \frac{\partial}{\partial \theta^\mu} \qquad \omega = \mathrm{d}x^\mu \wedge \mathrm{d}p_\mu + \mathrm{d}\theta^\mu \wedge \mathrm{d}\theta_\mu$$

$\Rightarrow$ Vinogradov algebroid $\mathcal{V}_2(\mathbb{R}^D)$ of Generalized Geometry

Action of Symmetries

An action of the symmetry Lie n-algebra can be defined.

Observations:

- Action of Lie algebra $\mathfrak{g}$ on manifold M: hom. $\mathfrak{g} \to \mathfrak{X}(M)$
- N-manifold $\mathcal{M}$, $\mathfrak{X}(\mathcal{M})$ is $\mathbb{N}_0$-graded Lie algebra, L_∞-algebra

Definition

Action of L_∞-algebra L on manifold $\mathcal{M}$: L_∞-morph. $\mathsf{L} \to \mathfrak{X}(\mathcal{M})$.

Note: $\mathcal{M} = \mathcal{V}_2(M)$ only encodes forms, not symmetric tensors.

Extension of Poisson bracket

$$\begin{aligned}\{-,-\}:\ \mathcal{C}^\infty(\mathcal{M}) \times T(\mathcal{M}) &\to T(\mathcal{M})\\ \{f, g \otimes h\} &:= \{f,g\} \otimes h + (-1)^{(n-|f|)|g|} g \otimes \{f,h\}\end{aligned}$$

Definition: Extended tensors

Let L be L_∞-structure on $\mathcal{M}$. Extended tensors are elts. of $T(\mathcal{M})$ such that elements of L act on it via $X \rhd t := \{\delta X, t\}$.

Action Functionals

Action functionals are readily constructed from the generalized metric.

- Extended tangent bundle has structure group GL$(m, \mathbb{R})$
- Reduce to subgroup H by coboundary: $h_{ij} = \gamma_i h_{ij} \gamma_j$
- Factor out H-equivalent coboundaries: $\mathcal{H}_i = \gamma_i^* \gamma_i$.

Action functionals for generalized metric $\mathcal{H}$, dilaton d:

$$S = \int_M \mathrm{e}^{-2d}\,\mathrm{d}^D x \Big(c_0 \mathcal{H}_{MN} \partial^M \mathcal{H}_{KL} \partial^N \mathcal{H}^{KL} + c_1 \mathcal{H}_{MN} \partial^M \mathcal{H}_{KL} \partial^L \mathcal{H}^{KN}$$
$$+ c_2 \mathcal{H}^{MN} (\mathcal{H}^{KL} \partial_M \mathcal{H}_{KL})(\mathcal{H}^{RS} \partial_N \mathcal{H}_{RS}) + c_3 \mathcal{H}^{MN} \mathcal{H}^{PQ} (\mathcal{H}^{RS} \partial_P \mathcal{H}_{RS})(\partial_M \mathcal{H}_{NQ})$$
$$+ c_4 \partial^M d \partial^N \mathcal{H}_{MN} + c_5 \mathcal{H}_{MN} \partial^M d \partial^N d \Big)$$

Examples:

- No reduction, γ_i are vielbeins, $\mathcal{H} = (g_{\mu\nu})$, $\Rightarrow$ GR
- GL$(D+1, \mathbb{R}) \rightarrow$ O$(D, 1)$, yields GR + 1-form gauge potential

$$\mathcal{H}^{mn} = \begin{pmatrix} g^{\mu\nu} & -g^{\mu\nu} A_\nu \\ -A_\mu g^{\mu\nu} & 1 + g^{\mu\nu} A_\mu A_\nu \end{pmatrix}$$

- GL$(2D, \mathbb{R}) \rightarrow$ O(D, D) yields GR + 2-form gauge potential
  Generalized Metric of Generalized Geometry.

Comments on the Global Picture

The global picture seems to be in reach now.

- Surjective submersion $Y \twoheadrightarrow M$, $Y^{[2]} := Y \times_M Y$, $Y^{[n]} := \ldots$
- For example, $Y = \sqcup_i U_i$, $Y^{[2]} = \sqcup_{i,j} U_i \cap U_j$
- Gerbe is a principal $\mathsf{U}(1)$-bundle over $Y^{[2]}$ + data over $Y^{[3]}$
- Trivial gerbe: $Y = M$, $Y^{[2]} = M$, $\mathscr{G} = M \times \mathsf{U}(1) = M \times S^1$.
- T-duality on $M \times S^1$: trivial gerbe, $H = \mathrm{d}B$ globally.

Non-trivial gerbe in Generalized Geometry

- $\mathcal{V}_2(M) = T^*[2]T[1]M$ and $Q = \xi^\mu \frac{\partial}{\partial x^\mu} + p_\mu \frac{\partial}{\partial \zeta_\mu}$ only locally
- Assume nontrivial gerbe $H, B_{(i)}, A_{(ij)}, h_{(ijk)}$
- Hitchin's Generalized tangent bundle over patches i, j: $(X_{(i)}, \Lambda_{(i)}) = (X_{(j)}, \Lambda_{(j)} + \iota_X \mathrm{d}A_{(ij)})$
- $Q = \xi^\mu \frac{\partial}{\partial x^\mu} + p_\mu \frac{\partial}{\partial \zeta_\mu} - \frac{1}{3!}(\frac{\partial}{\partial x^\nu} H) \frac{\partial}{\partial p_\nu} + \frac{1}{2!} H_{\nu\mu_1\mu_2} \xi^{\mu_1} \xi^{\mu_2} \frac{\partial}{\partial \zeta_\nu}$

Comments on the Global Picture: Double Field Theory

The global picture seems to be in reach now.

- L_∞-algebra of symmetries of DFT acts as Lie algebra
- Lie algebra can be integrated Hohm, Zwiebach, 2012
- Proposal: Patch local descriptions by finite DFT symmetries Berman, Cederwall, Perry, 2014
- Papadopoulos 2014: This only works for trivial gerbes
- No surprise, DFT reduces to GenGeo, where we need to twist!
- Need to twist C-/D-bracket, just as in GenGeo.
- Twist can be defined and studied in our framework.
- We recover twists of Generalized Geometry as special cases.
- Integrate twisted action
- Global picture: Patch together with twisted transformations!

Conclusions

Summary and Outlook.

Summary:

- ✓ Full algebraic and geometric picture for local DFT
- ✓ Picture is extension from GR coupled to n-form fields
- ✓ weakened section condition from algebra
- ✓ twist of symmetry Lie 2-algebra
- ✓ Initial studies of global picture and Riemannian Geometry

Soon to come:

- ▷ Exceptional Field Theory (M-theory)
- ▷ Full Extended Riemannian Geometry
- ▷ Global Picture

Extended Riemannian Geometry and Double Field Theory

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