

Field Theory on Fuzzy Spaces

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Essentially based on:

- [hep-th/0706.2493](https://arxiv.org/abs/hep-th/0706.2493), Denjoe O'Connor and CS.

Reasons for considering Fuzzy Geometry

Studying fuzzy geometry is well motivated.

Planck-Scale Structure of Spacetime

Smooth structure of spacetime probably not to arbitrary scales.
The most prominent modifications: **SUSY** and **Noncommutativity**.
Fuzzy Geometry: NC on compact symplectic Riemannian spaces
arises naturally in **string theory**

Regularization of Field Theories

Field theories on fuzzy spaces: **finite-dimensional matrix models**.
QFTs are **finite** and path integrals **well-defined**.
Advantages over lattice approach:
Isometries preserved, no fermion doubling, analytical handle
Numerical Simulations are easily done

Fuzzy Geometry as a Regulator

Naive regularization does not reproduce planar commutative limit.

Taking the commutative limit does not reproduce scalar ϕ^4 -theory.
UV/IR-mixing distorts the picture.

[Vaidya, Chu, Madore, Steinacker]

Modifications of the naïve model, however, could cure this problem.

[Dolan, O'Connor, Presnajder]

To do:

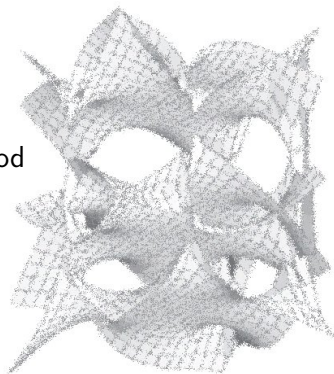
Obtain an **analytical** handle on fuzzy ϕ^4 -theory, in particular its **phase diagram** and study the effect of the proposed modifications.

(Gauge theory on the fuzzy sphere has recently been solved

[Steinacker, Szabo, hep-th/0701041]

scalar field theory appears to be simpler.)

- 1 The Fuzzy Sphere, Related Spaces, Symmetries
- 2 Fuzzy Scalar Field Theory
- 3 Known Results
- 4 Perturbative Expansion
- 5 Large N Limit & Saddle Point Method
- 6 Modification of the Model
- 7 Conclusions



The Fuzzy Sphere

Idea: Consider the spherical harmonics up to a certain angular momentum.

Quantization of the sphere:

As usually, do not quantize **space itself**, but **algebra of functions**.

Basis: Spherical harmonics Y_{lm} with $l = 0, \dots, \infty, m = -l, \dots, l$.

Quantization: Truncate angular momentum $l \leq L$

Product will not close any more: $Y_{l_1 \dots} Y_{l_2 \dots} = Y_{l_1 + l_2 \dots} + \dots$

However, deforming the product to the **star product**

$$[x^i \star, x^j] \sim i \varepsilon^{ijk} x^k,$$

where $x^i \in \mathbb{R}^3 \supset S^2$ yields a closed, truncated algebra.

The Fuzzy Sphere

(a) The truncated coordinate ring is mapped to an L -particle Hilbert space.

$$S^2 \cong \mathbb{C}P^1$$

The spherical harmonics Y_{lm} , $l \leq L$ can be written in terms of homogeneous coordinates z_α ($x^i \sim \bar{z}_\alpha \sigma_{\alpha\beta}^i z_\beta$) on $\mathbb{C}P^1$ in terms of

$$z_{\alpha_1} \dots z_{\alpha_L} \bar{z}_{\beta_1} \dots \bar{z}_{\beta_L}$$

with $\alpha_i, \beta_i = 1, 2$ due to the Hopf fibration

$$0 \rightarrow U(1) \rightarrow S^3 \rightarrow \mathbb{C}P^1 \rightarrow 0 .$$

Quantization as in flat case, $(z_\alpha, \bar{z}_\beta) \rightarrow (\hat{a}_\alpha^\dagger, \hat{a}_\beta)$:

$$\hat{a}_{\alpha_1}^\dagger \dots \hat{a}_{\alpha_L}^\dagger |0\rangle \langle 0| \hat{a}_{\beta_1} \dots \hat{a}_{\beta_L}$$

In this way, also fuzzy versions of $X \hookrightarrow \mathbb{C}P^n$. [CS, hep-th/0612124]

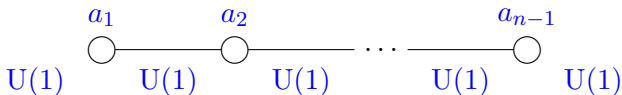
Fuzzy Flag Manifolds

(b) Explicite construction from coherent states/group theory.

More generally:

$$M = U(n) / (U(n_1) \times \dots \times U(n_k)) \quad (\text{Recall } S^2 \cong U(2) / (U(1) \times U(1)))$$

Consider irreducible representation ρ of $SU(n)$, extended to $U(n)$:



$n^2 - n$ “simple” raising and lowering operators $E_{\vec{\alpha}_i}^{\pm}$.

a_i : nontrivial actions $(E_{\vec{\alpha}_i}^-)^{a_i} |\mu\rangle \neq 0 = (E_{\vec{\alpha}_i}^-)^{a_i+1} |\mu\rangle$

n Cartan generators, isotropy group of $|\mu\rangle$: $H \supset U(1)^{\times n}$

One-To-One Correspondence:

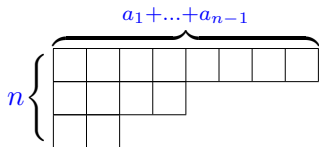
Coherent States $|p\rangle \in \rho \leftrightarrow p \in \text{Coset } U(n) / (U(n_1) \times \dots \times U(n_k))$

\Rightarrow Fuzzy Flag (Super) Manifolds [S. Murray, CS, hep-th/0611328]

Fuzzy Flag Manifolds

Young diagrams yield a Fock space construction of NC functions on fuzzy geometries.

Representation $\rho = (a_1, \dots, a_{n-1})$ corresponds to Young diagram



Fuzzy sphere: $\rho = (a_1) = L$ of $SU(2)$:

$$\overbrace{\begin{array}{|c|c|c|c|} \hline \square & \square & \square & \square \\ \hline \end{array}}^L \cong \text{span}(\hat{a}_{\alpha_1}^\dagger \dots \hat{a}_{\alpha_L}^\dagger |0\rangle), \quad \hat{\mathcal{L}}_i = \hat{a}_{\alpha}^\dagger \sigma_{\alpha\beta}^i \hat{a}_{\beta}$$

Isometry-preserving quantization of functions on S^2 via the rule:

$$f(p) = \langle p | \hat{f} | p \rangle,$$

$$\hat{f} \in \overbrace{\begin{array}{|c|c|c|c|} \hline \square & \square & \square & \square \\ \hline \end{array}}^L \otimes \overbrace{\begin{array}{|c|c|c|c|} \hline \square & \square & \square & \square \\ \hline \end{array}}^L \cong \text{span}(\hat{a}_{\alpha_1}^\dagger \dots \hat{a}_{\alpha_L}^\dagger |0\rangle \langle 0| \hat{a}_{\beta_1} \dots \hat{a}_{\beta_L})$$

The Fuzzy Sphere

(c) Geometric Quantization yields the same result.

$$S^2 \cong (\mathbb{C}P^1, \omega)$$

Take the line bundle $\mathcal{L} := \mathcal{O}(1)$ as the quantum line bundle.

Toeplitz quantization (Geometric quantization):

$$T^{(L)} : C^\infty(M) \rightarrow \text{End}(\Gamma(M, \mathcal{L}^{\otimes L})) .$$

The set of sections $\Gamma(\mathbb{C}P^1, \mathcal{L}^{\otimes L})$ is spanned by

$$z_{\alpha_1} \dots z_{\alpha_L}$$

The quantized algebra of functions is thus spanned by

$$z_{\alpha_1} \dots z_{\alpha_L} \frac{\partial}{\partial z_{\beta_1}} \dots \frac{\partial}{\partial z_{\beta_L}}$$

or, equivalently, by

$$\hat{a}_{\alpha_1}^\dagger \dots \hat{a}_{\alpha_L}^\dagger |0\rangle \langle 0| \hat{a}_{\beta_1} \dots \hat{a}_{\beta_L}$$

Fuzzy Scalar Field Theory: Definition.

Scalar field theory on the fuzzy sphere is a finite hermitian matrix model

Quantized algebra of functions on the fuzzy sphere:

$$\text{span}(\hat{a}_{\alpha_1}^\dagger \dots \hat{a}_{\alpha_L}^\dagger |0\rangle \langle 0| \hat{a}_{\beta_1} \dots \hat{a}_{\beta_L}) \cong \text{Mat}(L+1)$$

Laplacian and integration on the fuzzy sphere:

$$\mathcal{L}_i = i\varepsilon_{ijk} x^j \partial_k \rightarrow [L_i, \cdot], \quad \Delta \rightarrow C_2, \quad \int_{S^2} dA f \rightarrow \frac{4\pi R^2}{N} \text{tr}(\hat{f})$$

The action of real scalar field theory on the fuzzy sphere:

$$S \sim \text{tr} (a[L_i, \Phi][L_i, \Phi] + b\Phi^2 + c\Phi^4)$$

We define the partition function

$$Z = \int d\mu_D(\Phi) e^{-\text{tr} (a[L_i, \Phi][L_i, \Phi] + b\Phi^2 + c\Phi^4)}$$

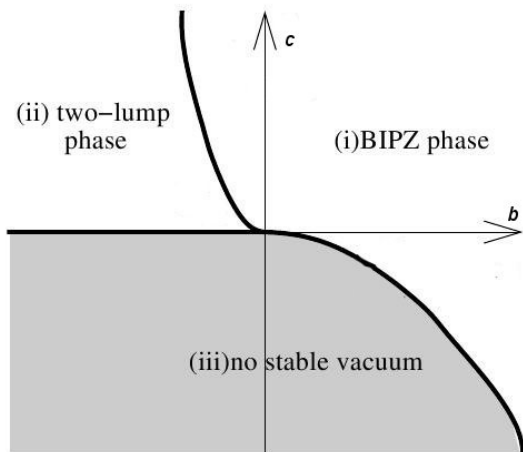
with the Dyson measure

$$d\mu_D(\Phi) = \prod_{i \leq j} d\Re(\Phi_{ij}) \prod_{i > j} d\Im(\Phi_{ij})$$

Known Results: Matrix Model Phase Diagram

The matrix model phase diagram suggests two phases.

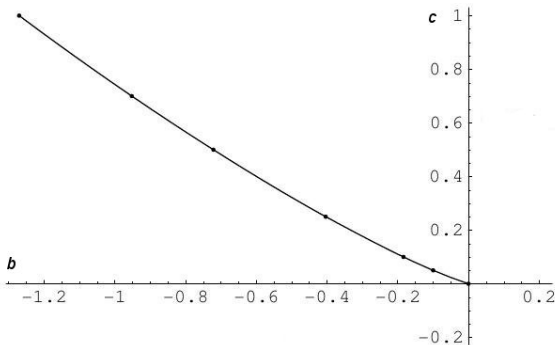
$$Z = \int d\mu_D(\Phi) e^{-\text{tr}(b\Phi^2+c\Phi^4)}, \quad 0 < \frac{dx}{d\lambda} < \infty, \quad c \sim b^2$$



Known Results: Phase Diagram for ϕ^4 -Theory on \mathbb{R}^2

The (lattice) model has two different phases.

$$Z = \int \mathcal{D}\phi e^{-\int d^2x \frac{1}{2}(\nabla\phi)^2 + b\phi^2 + c\phi^4}, \quad \langle\phi\rangle = 0, \quad \frac{c}{b} = \text{const.}$$



proof of existence: [Glimm, Jaffe, Spencer, 1974/1975]

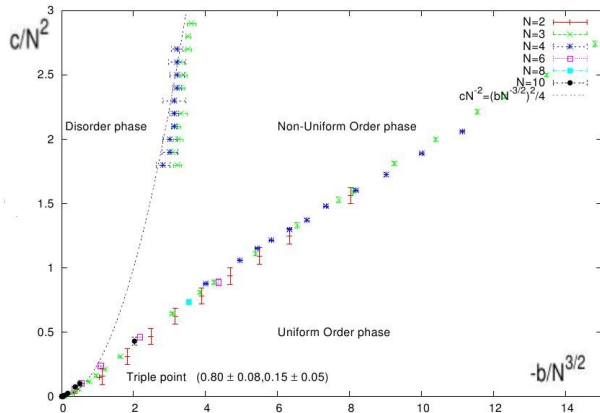
exact shape numerically: [Loinaz, Willey, hep-lat/9712008],

confirmed by [Lee, hep-th/9811117]

Known Results: Numerical Simulations

The phase diagram suggests a combination of the phases.

$$Z = \int d\mu_D(\Phi) e^{-\beta \text{tr} (a[L_i, \Phi][L_i, \Phi] + b\Phi^2 + c\Phi^4)}, \quad \chi = \frac{\partial}{\partial \beta^2} \log(Z)$$



[Flores, O'Connor, Martin, hep-th/0601012,
Panero, hep-th/0608202]

Known Results: ϕ^4 -Theory on \mathbb{R}_θ^2

The fuzzy model corresponds actually to regularized NC ϕ^4 theory.

- In ϕ^4 -theory on \mathbb{R}_θ^2 :
New phase predicted [Gubser, Sondhi, hep-th/0006119],
analytically confirmed [Chen, Wu, hep-th/0110134] and
numerically confirmed [Ambjorn, Catterall, hep-th/0209106].
 - Indications for the new phase also found by regularization
 ϕ^4 -theory with fuzzy spaces by [Steinacker, hep-th/0501174].
 - **Removal of new phase** would be an indicator of successful
regularization of **commutative** ϕ^4 -theory.
- ⇒ Better analytical handle on fuzzy ϕ^4 theory **necessary**.

Fuzzy Scalar Field Theory vs. HMMs

Fuzzy scalar field theory is significantly harder than matrix models usually considered.

$$\text{Fuzzy scalar field theory: } Z = \int d\mu_D(\Phi) e^{-\text{tr}(a[L_i, \Phi][L_i, \Phi] + b\Phi^2 + c\Phi^4)}$$

First example: **One-Hermitian Matrix Model**

$$Z = \int d\mu_D(\Phi) e^{-\text{tr}(b\Phi^2 + c\Phi^4)}$$

Solution: **splitting** $\Phi = \Omega\Lambda\Omega^\dagger$, $\Lambda = \text{diag}(\lambda_1, \dots, \lambda_N)$ as well as $\int d\mu_D(\Phi) = \int \prod_{i=1}^N d\lambda_i \Delta^2(\Lambda) \int d\mu_H(\Omega)$ yields

$$Z = \int \prod_{i=1}^N d\lambda_i e^{-2\sum_{i>j} \ln|\lambda_i - \lambda_j| - b\sum_i \lambda_i^2 + c\sum_i \lambda_i^4}$$

From here: **saddle point**, **orthogonal polynomials**, etc.

Fuzzy Scalar Field Theory vs. HMMs

Fuzzy scalar field theory is significantly harder than matrix models usually considered.

Fuzzy scalar field theory: $Z = \int d\mu_D(\Phi) e^{-\text{tr}(a[L_i, \Phi][L_i, \Phi] + b\Phi^2 + c\Phi^4)}$

Second example: **Hermitian matrix model with one external matrix**

$$Z = \int d\mu_D(\Phi) e^{-\text{tr}(V(A\Phi) + b\Phi^2 + c\Phi^4)}$$

Solution: splitting $\Phi = \Omega\Lambda\Omega^\dagger$, as well as **character expansion**

$$\exp(\text{tr}(V(A\Phi))) = \sum f_\rho \chi_\rho(A\Phi)$$

Orthogonality relation: $\int d\mu_H(\Omega) \chi_\rho(A\Omega^\dagger\Lambda\Omega) = \frac{1}{\dim(\rho)} \chi_\rho(A)\chi_\rho(\Lambda)$

Formula by **Itzykson and Di Francesco** [[hep-th/9212108](https://arxiv.org/abs/hep-th/9212108)]:

$$Z = \sum_{h_1 < \dots < h_N} \frac{\Pi(h^e - 1)!! h^o!!}{\Pi(h^e - h^o)} \chi_\rho(A)\chi_\rho(t)$$

Fuzzy Scalar Field Theory: Simplifications

Symmetry arguments yield simplifications.

$$\text{Fuzzy scalar field theory: } Z = \int d\mu_D(\Phi) e^{-\text{tr}(a[L_i, \Phi][L_i, \Phi] + b\Phi^2 + c\Phi^4)}$$

$$N = 2: Z = \int d\lambda_1 d\lambda_2 (\lambda_1 - \lambda_2)^2 e^{-(a(\lambda_1 - \lambda_2)^2 + b(\lambda_1^2 + \lambda_2^2) + c(\lambda_1^4 + \lambda_2^4))}$$

Symmetries:

$$1. d\mu_D(\Phi) = d\mu_D(\Omega\Phi\Omega^\dagger) \Rightarrow \int d\mu_D(\Phi) e^{-S} = \int d\mu_D(\Phi) e^{-S_0}$$

$$S_0 = \sum_n s_n \text{tr}(\Phi^n) + \sum_{n,m} s_{nm} \text{tr}(\Phi^n) \text{tr}(\Phi^m) + \dots$$

$$2. d\mu_D(\Phi) f(\Phi) \sim d^{N^2} \Phi^\mu f(\Phi^\mu \tau_\mu) \Rightarrow S_0 = \sum_n s_n (\text{tr}(\Phi^2))^n$$

$$3. [L_i, \mathbb{1}] = 0, \lambda \leftrightarrow -\lambda \Rightarrow \text{tr}([L_i, \Phi]^2) \sim \left(\sum_{i>j} (\lambda_i - \lambda_j)^{2m_k} \right)^{n_l}$$

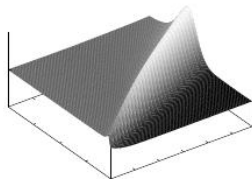
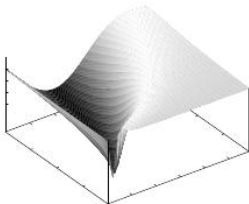
Perturbative expansion: Motivation

A high temperature expansion yields a useful approximation.

Idea: Treat the **kinetic term** perturbatively.

Motivation:

- **Hopping parameter expansion** successfully used on the lattice.
- Specific heat up to $O(a^8)$ for $N = 2$:



- **Group theoretical considerations** allow everything else to be treated **exactly**.

Perturbative expansion: Principles

The angular variables can be integrated out in the perturbative series.

Introduce $K_{ab} := \text{tr}([L_i, \tau_a][L_i, \tau_b])$, $\Phi^a = \text{tr}(\tau^a \Omega \Lambda \Omega^\dagger)$. Then:

$$e^{a\Phi^a K_{ab} \Phi^b} = 1 + a\Phi^a K_{ab} \Phi^b + \frac{a^2}{2} \Phi^a K_{ab} \Phi^b \Phi^c K_{cd} \Phi^d + \dots$$

To integrate over $d\mu_H(\Omega)$ we need to compute terms like

$$\int d\mu_H(\Omega) K_{ab} \text{tr}(\tau^a \Omega \Lambda \Omega^\dagger) \text{tr}(\tau^b \Omega \Lambda \Omega^\dagger)$$

Recall:
$$\int d\mu_H(\Omega) [\rho(\Omega)]_{ij} [\rho^\dagger(\Omega)]_{kl} = \frac{1}{\dim(\rho)} \delta_{il} \delta_{jk}$$

$$\text{tr}((\tau^a \Omega \Lambda \Omega^\dagger) \otimes (\tau^b \Omega \Lambda \Omega^\dagger)) = \text{tr}((\tau^a \otimes \tau^b)(\Omega \otimes \Omega)(\Lambda \otimes \Lambda)(\Omega^\dagger \otimes \Omega^\dagger))$$

Thus:
$$\int d\mu_H(\Omega) K_{ab} \Phi^a \Phi^b = K_{ab} \sum_{\rho} \frac{1}{\dim(\rho)} \text{tr}_{\rho}(\tau^a \otimes \tau^b) \text{tr}_{\rho}(\Lambda \otimes \Lambda)$$

Perturbative expansion: Results

Up to $\mathcal{O}(a^2)$, the perturbative expansion is easily doable.

After much **group theory** and algebra, we obtain:

$$\int d\mu_H(\Omega) K_{ab} \Phi^a \Phi^b = \frac{1}{2} N \sum_{i>j} (\lambda_i - \lambda_j)^2$$

$$\int d\mu_H(\Omega) K_{ab} \Phi^a \Phi^b K_{cd} \Phi^c \Phi^d = \frac{(2 \operatorname{tr} K^2 + (\operatorname{tr} K)^2)}{N^2(N^4 - 10N^2 + 9)} (\alpha_1 A_1 + \alpha_2 A_2) \\ + \frac{1}{N(-36 + N^2(-7 + N^2)^2)} (\beta_1 A_1 + \beta_2 A_2) K^\top K ,$$

where

$$A_1 = \sum_{i>j} (\lambda_i - \lambda_j)^4 \quad \text{and} \quad A_2 = \left(\sum_{i>j} (\lambda_i - \lambda_j)^2 \right)^2$$

(Confirmation: Structure correct, limit $N = 2$ is valid.)

Large N expansion

The large N expansion is rewritten in terms of a multi-matrix model.

In the large N limit, we have:

$$\int d\mu_H(\Omega) K_{ab} \Phi^a \Phi^b = \frac{1}{2} N \sum_{i>j} (\lambda_i - \lambda_j)^2$$

$$\int d\mu_H(\Omega) (K_{ab} \Phi^a \Phi^b)^2 = -\frac{N^2}{2} \sum_{i>j} (\lambda_i - \lambda_j)^4 + \frac{N^2}{4} \left(\sum_{i>j} (\lambda_i - \lambda_j)^2 \right)^2$$

After **re-exponentiating** the terms (still exact to $\mathcal{O}(a^2)$)

$$S = \sum_i (b\lambda_i^2 + c\lambda_i^4) + \sum_{i>j} \left(-\frac{a}{2} N (\lambda_i - \lambda_j)^2 + \frac{a^2}{4} N^2 (\lambda_i - \lambda_j)^4 - 2 \ln |\lambda_i - \lambda_j| \right)$$

Saddle point approximation

The saddle point approximation gives a rough picture of what is going on.

Rewrite: $\lambda_i \rightarrow \lambda\left(\frac{i}{N}\right) = \lambda(x)$, $0 < x < 1$, $\sum_{i=0}^N \rightarrow N \int_0^1 dx$

Rescale: $a = N^{\theta_a} \tilde{a}$, $b = N^{\theta_b} \tilde{b}$, $c = N^{\theta_c} \tilde{c}$, $\lambda(x) = N^{\theta_\lambda} \tilde{\lambda}(x)$

Partition function: $Z = \int \mathcal{D}\lambda \exp(-N^2 \tilde{S})$

Action:

$$\tilde{S} = \int_0^1 dx \left(\tilde{b} \tilde{\lambda}^2(x) + \tilde{c} \tilde{\lambda}^4(x) + \int_0^1 dy \left(-\frac{\tilde{a}}{4} (\tilde{\lambda}(x) - \tilde{\lambda}(y))^2 + \frac{\tilde{a}^2}{8} (\tilde{\lambda}(x) - \tilde{\lambda}(y))^4 - \ln |\tilde{\lambda}(x) - \tilde{\lambda}(y)| \right) \right)$$

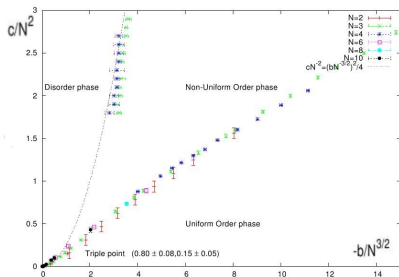
Saddle point solution (one symmetric cut $[-\delta, \delta]$): $u(\tilde{\lambda}) =$

$$\left(4\tilde{b} - \tilde{a} + 12\pi\tilde{a}^2 c_2 + 4 \left(\tilde{c} + \frac{\pi\tilde{a}^2}{2} \right) \delta^2 + 8 \left(\tilde{c} + \frac{\pi\tilde{a}^2}{2} \right) \tilde{\lambda}^2 \right) \sqrt{\delta^2 - \tilde{\lambda}^2}$$

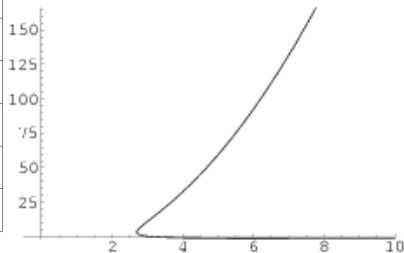
Saddle point approximation

New phase lies within the region of the two-cut solution.

The boundary of the **region of validity** of the one-cut solution is consistent with the data.



Numerics, fuzzy ϕ^4 -theory



Analytical results, multi-trace MM

Triple pt: $(-2.3 \pm 0.2, 0.52 \pm 0.02)$ Turning pt: $(-2.7, 0.25)$

\Rightarrow Turning point corresponds to triple point.

Modification of the model

The proposed modification of the model moves the triple point in the right direction.

Modify the action, assuming momentum-dependent **wave function regularization** $\mathcal{Z}_L(C_2) \approx 1 + \kappa C_2$:

$$\tilde{S} = \text{tr} (a\Phi(C_2 + \kappa C_2 C_2)\Phi + b\Phi^2 + c\Phi^4)$$

This implies the following modification in our analysis

$$K_{ab} \rightarrow \check{K}_{ab} := K_{ab} + \kappa K_{ac} K_{cb} \quad \text{and} \quad \check{a} = a(1 + \frac{2}{3}\kappa(N^2 - 1))$$

Rescaling of κ to keep highest order term yields $\check{a} = a(1 + \frac{2}{3}\tilde{\kappa})$.

\Rightarrow The triple point moves off to infinity for increasing $\tilde{\kappa}$.

We achieved the following:

- formulated a **generalized character expansion** technique
- reformulated fuzzy ϕ^4 -theory as **multi-trace matrix model**
- preliminary analysis of the approximation **looks promising**

Future directions:

- Examine all possible **one-** and **two-cut solutions**; this should yield a (full) explanation of the phase diagram
- Examine **modifications** of the model for extended solutions
- Extract **more information** from the expansion
- Study **other models** (**other fuzzy spaces**) with this technique
- Relation to **$c > 1$ string theories?**

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