Towards Describing M5-branes with Loop Space

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Based on:

- CS, Commun. Math. Phys. 305 (2011) 513
- C. Papageorgakis and CS, JHEP 05 (2011) 099
- S. Palmer and CS, JHEP 10 (2011) 008

Motivation

Find an algorithm for the construction of self-dual string solutions

- Effective description of M2-branes proposed in 2007.
- This created lots of interest:
  BLG-model: >467 citations, ABJM-model: >605 citations
- Inspired by an idea by Basu-Harvey:
  Propose a lift of the Nahm eqn. describing D1-D3-system:
  Basu-Harvey eqn. describes M2-M5-brane system
- Nahm transform:
  go from Nahm eqn. to Bogomolny monopole eqn.
  switch perspective from D1-brane to D3-brane
- Is there a lift for this Nahm transform?
  go from BH eqn. to self-dual string eqn.
  switch perspective from M2-brane to M5-brane
- Such a transform would open up interesting possibilities:
  eff. description of M5-branes, new integrable structures, ...

(Alternative Approach: Gustavsson, 0802.3456)

Outline

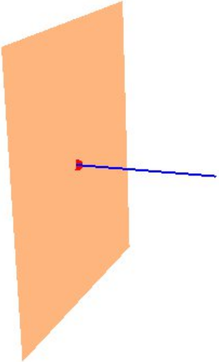
We will discuss the construction of monopoles and lift each ingredient to M-theory.

- Basu-Harvey lift of the Nahm equation and 3-Lie algebras
- Monopoles and self-dual strings
- Principal $\mathsf{U}(1)$-bundles, abelian gerbes and loop space
- ADHMN construction and its lift
- Examples of self-dual string solutions
- Non-abelian tensor multiplet on loop space

D1-D3-Branes and the Nahm Equation

D1-branes ending on D3-branes can be described by the Nahm equation.

dim	0	1	2	3	$\dots$	6
$D1$	$\times$					$\times$
$D3$	$\times$	$\times$	$\times$	$\times$		



N D1-branes ending on D3-branes:

A Monopole appears.

$X^i \in \mathfrak{u}(N)$: transverse fluctuations

Nahm equation: $(s = x^6)$

$$\frac{\mathrm{d}}{\mathrm{d}s} X^i + \varepsilon^{ijk}[X^j, X^k] = 0$$

Note $\mathsf{SO}(3)$-invariance.

Solution: $X^i = r(s) G^i$ with

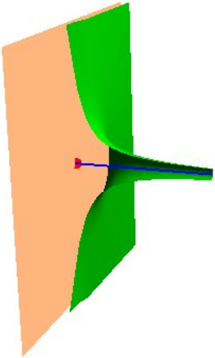
$$r(s) = \frac{1}{s}\ , \quad G^i = \varepsilon^{ijk}[G^j, G^k]$$

Nahm, Diaconescu, Tsimpis

D1-D3-Branes and the Nahm Equation

The D1-branes end on the D3-branes by forming a fuzzy funnel.

dim	0	1	2	3	…	6
$D1$	×					×
$D3$	×	×	×	×		



Solution: $X^i = r(s) G^i$

$$r(s) = \frac{1}{s}\ , \quad G^i = \varepsilon^{ijk}[G^j, G^k]$$

The D1-branes form a fuzzy funnel:

G^i form irrep of $\mathfrak{su}(2)$:
coordinates on fuzzy sphere S^2_F

D1-worldvolume polarizes: $2d \to 4d$

Myers

Lifting D1-D3-Branes to M2-M5-Branes

The lift to M-theory is performed by a T-duality and an M-theory lift

IIB	0	1	2	3	4	5	6
$D1$	×						×
$D3$	×	×	×	×			

T-dualize along x^5:

IIA	0	1	2	3	4	5	6
$D2$	×					×	×
$D4$	×	×	×	×		×	

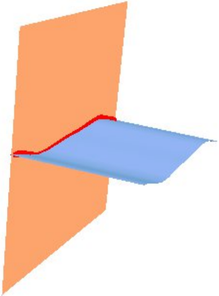
Interpret x^4 as M-theory direction:

M	0	1	2	3	4	5	6
$M2$	×					×	×
$M5$	×	×	×	×	×	×	

The Basu-Harvey Lift of the Nahm Equation

M2-branes ending on M5-branes yield a Nahm equation with a cubic term.

M	0	1	2	3	4	5	6
$M2$	×					×	×
$M5$	×	×	×	×	×	×	



Basu, Harvey, hep-th/0412310

A Self-Dual String appears.

Substitute SO(3)-inv. Nahm eqn.

$$\frac{\mathrm{d}}{\mathrm{d}s}X^i + \varepsilon^{ijk}[X^j, X^k] = 0$$

by the SO(4)-invariant equation

$$\frac{\mathrm{d}}{\mathrm{d}s}X^\mu + \varepsilon^{\mu\nu\rho\sigma}[X^\nu, X^\rho, X^\sigma] = 0$$

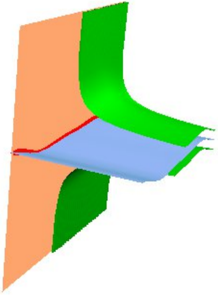
Solution: $X^\mu = r(s)G^\mu$ with

$$r(s) = \frac{1}{\sqrt{s}}\ , \quad G^\mu = \varepsilon^{\mu\nu\rho\sigma}[G^\nu, G^\rho, G^\sigma]$$

The Basu-Harvey Lift of the Nahm Equation

M2-branes ending on M5-branes yield a Nahm equation with a cubic term.

M	0	1	2	3	4	5	6
$M2$	×					×	×
$M5$	×	×	×	×	×	×	



Solution: $X^\mu = r(s) G^\mu$

$$r(s) = \frac{1}{\sqrt{s}}\ ,\ \ G^\mu = \varepsilon^{\mu\nu\rho\sigma}[G^\nu, G^\rho, G^\sigma]$$

The M2-branes form a fuzzy funnel:

G^μ form a rep of $\mathfrak{so}(4)$:
coordinates on fuzzy sphere S^3_F

M2-worldvolume polarizes: $3d \to 6d$

What is this triple bracket?

What is the algebra behind the triple bracket?

In analogy with Lie algebras, we can introduce 3-Lie algebras.

$$\frac{\mathrm{d}}{\mathrm{d}s}X^\mu + [A_s, X^\mu] + \varepsilon^{\mu\nu\rho\sigma}[X^\nu, X^\rho, X^\sigma] = 0\ , \quad X^\mu \in \mathcal{A}$$

Trivial: $\mathcal{A}$ is a vector space, $[\cdot,\cdot,\cdot]$ trilinear+antisymmetric.

▷ Gauge transformations from inner derivations:

Triple bracket forms a map $D : \mathcal{A} \wedge \mathcal{A} \to \mathrm{Der}(\mathcal{A}) =: \mathfrak{g}_\mathcal{A}$ via

$$D(A,B) \rhd C := [A,B,C]$$

Demand a "3-Jacobi identity," the fundamental identity:

$$\begin{aligned} D(A,B) \rhd (D(C,D) \rhd E) &:= [A,B,[C,D,E]] \\ &= [[A,B,C],D,E] + [C,[A,B,D],E] + [C,D,[A,B,E]] \end{aligned}$$

The inner derivations form indeed a Lie algebra:

$$[D(A,B), D(C,D)] \rhd E := D(A,B) \rhd (D(C,D) \rhd E) - D(C,D) \rhd (D(A,B) \rhd E)$$

Bracket closes due to fundamental identity.

Real and Hermitian 3-Algebras

There are two natural generalizations of 3-Lie algebras.

Real 3-Algebras

Almost same as 3-Lie algebras: triple bracket only antisymmetric in first two slots. S. Cherkis, CS, 0807.0808

Hermitian 3-Algebras

Start from a complex vector space $\mathcal{A}$. Bracket $[\,\cdot\,,\,\cdot\,;\,\cdot\,]$ satisfies

$$[A,B;C]=-[B,A;C],\ \ [\alpha A,B;C]:=\alpha[A,B;C],\ \ [A,B;\alpha C]:=\alpha^*[A,B;C]$$
$$[[C,D;E],A;B]-[[C,A;B],D;E]-[C,[D,A;B];E]+[C,D;[E,B;A]]=0$$

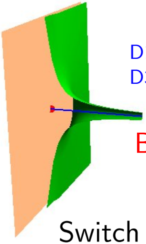
Aharony, Bergman, Jafferis, Maldacena, 0806.1218

Bagger, Lambert, 0807.0163

All our constructions generalize to these two types of 3-algebras.

Monopoles and Self-Dual Strings

Lifting monopoles to M-theory yields self-dual strings.



	0	1	2	3	4	5	6
D1	×						×
D3	×	×	×	×			

BPS configuration!

Switch perspective: D1→ D3:

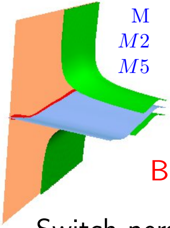
Bogomolny monopole eqn.:

$$F_{ij} = \varepsilon_{ijk}\nabla_k\Phi \;\Rightarrow\; \nabla^2\Phi = 0$$

Single D3: Dirac monopole

$$\Phi = \frac{1}{r} \;\Rightarrow\; r(s) = \frac{1}{s}$$

⇒ matching profile!



M	0	1	2	3	4	5	6
$M2$	×					×	×
$M5$	×	×	×	×	×	×	

BPS configuration!

Switch perspective: M2→ M5:

Self-dual string eqn.:

$$H_{\mu\nu\rho} = \varepsilon_{\mu\nu\rho\sigma}\partial_\sigma\Phi \;\Rightarrow\; \partial^2\Phi = 0$$

Only single M5 known:

$$\Phi = \frac{1}{r^2} \;\Rightarrow\; r(s) = \frac{1}{\sqrt{s}}$$

⇒ matching profile!

Dirac Monopoles and Principal $\mathsf{U}(1)$-bundles

Dirac monopoles are described by principal $\mathsf{U}(1)$-bundles over S^2.

Manifold M with cover $(U_i)_i$. Principal $\mathsf{U}(1)$-bundle over M:

$$\begin{gathered} F \in \Omega^2(M, \mathfrak{u}(1))\ , \\ A_{(i)} \in \Omega^1(U_i, \mathfrak{u}(1)) \text{ with } F = \mathrm{d}A_{(i)} \\ g_{ij} \in \Omega^0(U_i \cap U_j, \mathsf{U}(1)) \text{ with } A_{(i)} - A_{(j)} = \mathrm{d}\log g_{ij} \end{gathered}$$

Consider monopole in $\mathbb{R}^3$, but describe it on S^2 around monopole:

S^2 with patches U_+, U_-, $U_+ \cap U_- \sim S^1$: $g_{+-} = \mathrm{e}^{-\mathrm{i}n\phi},\ n \in \mathbb{Z}$

$$c_1 = \frac{\mathrm{i}}{2\pi} \int_{S^2} F = \frac{\mathrm{i}}{2\pi} \int_{S^1} A^+ - A^- = \frac{1}{2\pi} \int_0^{2\pi} n\mathrm{d}\phi = n$$

Monopole charge: n

Self-Dual Strings and Abelian Gerbes

Self-dual strings are described by abelian gerbes.

Manifold M with cover $(U_i)_i$. Abelian (local) gerbe over M:

$$H \in \Omega^3(M, \mathfrak{u}(1)) \ ,$$
$$B_{(i)} \in \Omega^2(U_i, \mathfrak{u}(1)) \text{ with } H = \mathrm{d}B_{(i)}$$
$$A_{(ij)} \in \Omega^1(U_i \cap U_j, \mathfrak{u}(1)) \text{ with } B_{(i)} - B_{(j)} = \mathrm{d}A_{ij}$$
$$h_{ijk} \in \Omega^0(U_i \cap U_j \cap U_k, \mathfrak{u}(1)) \text{ with } A_{(ij)} - A_{(ik)} + A_{(jk)} = \mathrm{d}h_{ijk}$$

Note: Local gerbe: principal $\mathsf{U}(1)$-bundles on intersections $U_i \cap U_j$.

Consider S^3, patches U_+, U_-, $U_+ \cap U_- \sim S^2$: bundle over S^2
Reflected in: $H^2(S^2, \mathbb{Z}) \cong H^3(S^3, \mathbb{Z}) \cong \mathbb{Z}$

$$\frac{\mathrm{i}}{2\pi} \int_{S^3} H = \frac{\mathrm{i}}{2\pi} \int_{S^2} B_+ - B_- = \ldots = n$$

Charge of self-dual string: n

Abelian Gerbes and Loop Space

By going to loop space, one can reduce differential forms by one degree.

Consider the following double fibration:

$$\begin{tikzcd} & \mathcal{L}M \times S^1 \arrow[dl, "ev"'] \arrow[dr, "pr"] & \\ M & & \mathcal{L}M \end{tikzcd}$$

Identify $T\mathcal{L}M = \mathcal{L}TM$, then: $x \in \mathcal{L}M \Rightarrow \dot{x}(\tau) \in T\mathcal{L}M$

Transgression

$$\mathcal{T} : \Omega^{k+1}(M) \to \Omega^k(\mathcal{L}M)~, \quad v_i = \oint \mathrm{d}\tau\, v_i^\mu(\tau) \frac{\delta}{\delta x^\mu(\tau)} \in T\mathcal{L}M$$

$$(\mathcal{T}\omega)_x(v_1(\tau), \ldots, v_k(\tau)) := \oint_{S^1} \mathrm{d}\tau\, \omega(v_1(\tau), \ldots, v_k(\tau), \dot{x}(\tau))$$

An abelian local gerbe over M is a principal $\mathsf{U}(1)$-bundle over $\mathcal{L}M$.

Transgressed Self-Dual Strings

By going to loop space, one can reduce differential forms by one degree.

Recall the self-dual string equation on $\mathbb{R}^4$:

$$H_{\mu\nu\kappa} = \varepsilon_{\mu\nu\kappa\lambda}\frac{\partial}{\partial x^\lambda}\Phi$$

Its transgressed form is an equation for a 2-form F on $\mathcal{L}\mathbb{R}^4$:

$$F(X,Y) = \oint \mathrm{d}\tau\,\varepsilon_{\mu\nu\kappa\lambda}\dot{x}^\kappa(\tau)\frac{\partial}{\partial x^\lambda}\Phi(x)\;X^\mu(\tau)Y^\nu(\tau)$$

or

$$F_{(\mu\sigma)(\nu\rho)} = \delta(\sigma-\rho)\varepsilon_{\mu\nu\kappa\lambda}\dot{x}^\kappa(\tau)\frac{\partial}{\partial x^\lambda}\Phi(x)$$

Extend to full non-abelian loop space curvature:

$$\begin{aligned}F^{\pm}_{(\mu\sigma)(\nu\tau)} = &\big(\varepsilon_{\mu\nu\kappa\lambda}\dot{x}^\kappa(\sigma)\nabla_{(\lambda\tau)}\Phi\big)_{(\sigma\tau)}\\ &\qquad\mp\big(\dot{x}_\mu(\sigma)\nabla_{(\nu\tau)}\Phi+\dot{x}_\nu(\sigma)\nabla_{(\mu\tau)}\Phi-\delta_{\mu\nu}\dot{x}^\kappa(\sigma)\nabla_{(\kappa\tau)}\Phi\big)_{[\sigma\tau]}\end{aligned}$$

where $\nabla_{(\mu\sigma)} := \oint \mathrm{d}\tau\delta x^\mu(\tau)\wedge\left(\frac{\delta}{\delta x^\mu(\tau)}+A_{(\mu\tau)}\right)$

Goal: Construct solutions to this equation.

The ADHMN Construction

There is a map between and solutions to the Nahm equations.

Nahm transform: Instantons on $T^4 \mapsto$ instantons on $(T^4)^*$

Roughly here:

T^4: $\begin{cases} 3 \text{ rad. } 0 \\ 1 \text{ rad. } \infty \ : \ \text{D1 WV} \end{cases}$ and $(T^4)^*$: $\begin{cases} 3 \text{ rad. } \infty \ : \ \text{D3 WV} \\ 1 \text{ rad. } 0 \end{cases}$

Introduce (twisted) "Dirac operators":

$$\slashed{\nabla}_{s,x} = -\mathbb{1}\tfrac{\mathrm{d}}{\mathrm{d}s} + \sigma^i \otimes (\mathrm{i}X^i + x^i \mathbb{1}_k) \ , \quad \bar{\slashed{\nabla}}_{s,x} := \mathbb{1}\tfrac{\mathrm{d}}{\mathrm{d}s} + \sigma^i \otimes (\mathrm{i}X^i + x^i \mathbb{1}_k)$$

Properties:

$\Delta_{s,x} := \bar{\slashed{\nabla}}_{s,x} \slashed{\nabla}_{s,x} > 0$, $[\Delta_{s,x}, \sigma^i] = 0 \Leftrightarrow X^i$ satisfy Nahm eqn.

Normalized zero modes: $\bar{\slashed{\nabla}}_{s,x} \psi_{s,x,\alpha} = 0$, $\mathbb{1} = \int_{\mathcal{I}} \mathrm{d}s \, \bar{\psi}_{s,x} \psi_{s,x}$ yield:

$$A_\mu := \int_{\mathcal{I}} \mathrm{d}s \, \bar{\psi}_{s,x} \frac{\partial}{\partial x^\mu} \psi_{s,x} \quad \text{and} \quad \Phi := -\mathrm{i} \int_{\mathcal{I}} \mathrm{d}s \, \bar{\psi}_{s,x} \, s \, \psi_{s,x}$$

This is a solution to the Bogomolny monopole equations!

Examples: Dirac monopoles

One can easily construct Dirac monopole solutions using the ADHMN construction.

Charge 1: Nahm eqn: $\partial_s X^i = 0$, so put $X^i = 0$. Zero mode:

$$\psi_+ = \mathrm{e}^{-sR}\frac{\sqrt{R+x^3}}{x^1 - \mathrm{i}x^2}\begin{pmatrix} x^1 - \mathrm{i}x^2 \\ R - x^3 \end{pmatrix}$$

Monopole solution:

$$\Phi^+ = -\frac{\mathrm{i}}{2R}\ ,\ \ A_i^+ = \frac{\mathrm{i}}{2(x^1+x^2)^2}\left(x^2\left(1-\frac{x^3}{R}\right), -x^1\left(1-\frac{x^3}{R}\right), 0\right)$$

Charge 2: Nahm eqn. nontrivial. Choose:

$$X^i = -\frac{1}{s}T^i \quad \text{with} \quad T^i = \frac{\sigma^i}{2\mathrm{i}} = -\bar{T}^i$$

Resulting solution:

$$\Phi^+ = -\frac{\mathrm{i}}{R}\ ,\ \ A_i^+ = \dots$$

Lift of the "Dirac operator"

There is a natural lift of the Dirac operator to M-theory.

Type IIB (twisted):

$$\slashed{\nabla}^{\mathrm{IIB}}_{s,x} = -\mathbb{1}\frac{\mathrm{d}}{\mathrm{d}s} + \sigma^i(\mathrm{i}X^i + x^i\mathbb{1}_k)$$

IIB	0	1	2	3	4	5	6
$D1$	×						×
$D3$	×	×	×	×			

Type IIA (twisted):

$$\slashed{\nabla}^{\mathrm{IIA}}_{s,x} = -\gamma_5\mathbb{1}_k\frac{\mathrm{d}}{\mathrm{d}s} + \gamma^4\gamma^i(X^i - \mathrm{i}x^i)$$

IIA	0	1	2	3	4	5	6
$D2$	×					×	×
$D4$	×	×	×	×		×	

M-theory (untwisted):

$$\slashed{\nabla}^{\mathrm{M}}_{s} = -\gamma_5\frac{\mathrm{d}}{\mathrm{d}s} + \tfrac{1}{2}\gamma^{\mu\nu}D(X^\mu, X^\nu)$$

M	0	1	2	3	4	5	6
$M2$	×					×	×
$M5$	×	×	×	×	×	×	

M-theory (twisted):

$$\slashed{\nabla}^{\mathrm{M}}_{s,x(\tau)} = -\gamma_5\frac{\mathrm{d}}{\mathrm{d}s} + \tfrac{1}{2}\gamma^{\mu\nu}\left(D(X^\mu, X^\nu) - \mathrm{i}\oint \mathrm{d}\tau\, x^\mu(\tau)\dot{x}^\nu(\tau)\right)$$

Lifted ADHMN Construction

The lifted ADHMN construction yields solutions to the loop space self-dual string eqns.

Recall: $\Delta^{\rm IIB}:=\bar{\slashed{\nabla}}^{\rm IIB}\slashed{\nabla}^{\rm IIB}$, $[\Delta^{\rm IIB},\sigma^i]=0\Leftrightarrow X^i$ satisfy Nahm eqn.

Here: $\Delta^{\rm M}:=\bar{\slashed{\nabla}}^{\rm M}\slashed{\nabla}^{\rm M}$, $[\Delta,\gamma^{\mu\nu}]=0\Leftarrow X^\mu$ satisfy BH eqn.

Recall extended self-dual string equation on loop space:

$$
\begin{aligned}
F^{\pm}_{(\mu\sigma)(\nu\tau)}=&\big(\varepsilon_{\mu\nu\kappa\lambda}\dot{x}^\kappa(\sigma)\nabla_{(\lambda\tau)}\Phi\big)_{(\sigma\tau)}\\
&\qquad\mp\big(\dot{x}_\mu(\sigma)\nabla_{(\nu\tau)}\Phi+\dot{x}_\nu(\sigma)\nabla_{(\mu\tau)}\Phi-\delta_{\mu\nu}\dot{x}^\kappa(\sigma)\nabla_{(\kappa\tau)}\Phi\big)_{[\sigma\tau]}
\end{aligned}
$$

From normalized, $\mathcal{A}$-valued zero modes $\psi_{s,x(\tau)}$ of $\bar{\slashed{\nabla}}^{\rm M}$ construct

$$
A_{(\mu\tau)}=\int{\rm d}s\,\bar{\psi}_{s,x}\frac{\delta}{\delta x^\mu(\tau)}\psi_{s,x}~,\quad \Phi=-{\rm i}\int{\rm d}s\,\bar{\psi}_{s,x}\,s\,\psi_{s,x}
$$

These fields solve the loop space self-dual string equation.

Verification of the Construction

Verifying the construction is rather straightforward.

The proof is easy and follows that of the ADHMN construction:

$$\begin{aligned}
F^{ab}_{(\mu\sigma)(\nu\tau)} &\overset{[\cdot]}{=} 2\int_{\mathcal{I}}\mathrm{d}s\,(\delta_{(\mu\sigma)}\bar\psi^a_{s,x},\delta_{(\nu\tau)}\psi^b_{s,x})+2\int_{\mathcal{I}}\mathrm{d}s\int_{\mathcal{I}}\mathrm{d}t\,(\bar\psi^a_{s,x},\delta_{(\mu\sigma)}\psi^c_{s,x})(\bar\psi^c_{t,x},\delta_{(\nu\tau)}\psi^b_{t,x})\\
&\overset{[\cdot]}{=} -2\int_{\mathcal{I}}\mathrm{d}s\int_{\mathcal{I}}\mathrm{d}t\,\Big(\delta_{(\mu\sigma)}\bar\psi^a_{s,x}\,,\,(\not{\nabla}_{s,x}G_x(s,t)\bar{\not{\nabla}}_{t,x})\,\delta_{(\nu\tau)}\psi^b_{t,x}\Big)\\
&\overset{[\cdot]}{=} 2\int_{\mathcal{I}}\mathrm{d}s\int_{\mathcal{I}}\mathrm{d}t\,\Big(\bar\psi^a_{s,x},\Big(\gamma^{\mu\kappa}\dot x^\kappa(\sigma)G_x(s,t)\gamma^{\nu\lambda}\dot x^\lambda(\tau)\Big)\,\psi^b_{t,x}\Big)\\
&\overset{[\cdot]}{=} 2\varepsilon_{\mu\nu\kappa\lambda}\int_{\mathcal{I}}\mathrm{d}s\int_{\mathcal{I}}\mathrm{d}t\,\Big(\bar\psi^a_{s,x},\,G_x(s,t)\gamma^{\kappa\rho}\gamma_5\dot x^\lambda(\sigma)\dot x^\rho(\tau)\psi^b_{t,x}\Big)\\
&\qquad+\int_{\mathcal{I}}\mathrm{d}s\int_{\mathcal{I}}\mathrm{d}t\,\Big(\bar\psi^a_{s,x},\,G_x(s,t)\,\Big(4\gamma^{\mu\lambda}\dot x^\nu(\sigma)\dot x^\lambda(\tau)-2\delta^{\mu\nu}\gamma^{\kappa\lambda}\dot x^\kappa(\sigma)\dot x^\lambda(\tau)\Big)\,\psi^b_{t,x}\Big)\\
&\overset{[\cdot]}{=} \mathrm{i}\varepsilon_{\mu\nu\kappa\lambda}\dot x^\kappa(\sigma)\int_{\mathcal{I}}\mathrm{d}s\,\Big((\nabla_{(\lambda\tau)}\bar\psi_{s,x})^a,\,s\,\psi^b_{s,x}\Big)+\Big(\bar\psi^a_{s,x},\,s\,(\nabla_{(\lambda\tau)}\psi_{s,x})^b\Big)\\
&\qquad\mp 2\mathrm{i}\dot x_\mu(\sigma)\int_{\mathcal{I}}\mathrm{d}s\,\Big((\nabla_{(\nu\tau)}\bar\psi_{s,x})^a,\,s\,\psi^b_{s,x}\Big)+\Big(\bar\psi^a_{s,x},\,s\,(\nabla_{(\nu\tau)}\psi_{s,x})^b\Big)\\
&\qquad\mp 2\mathrm{i}\dot x_\nu(\sigma)\int_{\mathcal{I}}\mathrm{d}s\,\Big((\nabla_{(\mu\tau)}\bar\psi_{s,x})^a,\,s\,\psi^b_{s,x}\Big)+\Big(\bar\psi^a_{s,x},\,s\,(\nabla_{(\mu\tau)}\psi_{s,x})^b\Big)\\
&\qquad\pm \mathrm{i}\delta_{\mu\nu}\dot x^\kappa(\sigma)\int_{\mathcal{I}}\mathrm{d}s\,\Big((\nabla_{(\kappa\tau)}\bar\psi_{s,x})^a,\,s\,\psi^b_{s,x}\Big)+\Big(\bar\psi^a_{s,x},\,s\,(\nabla_{(\kappa\tau)}\psi_{s,x})^b\Big)\\
&\overset{[\cdot]}{=} \Big(\varepsilon_{\mu\nu\kappa\lambda}\dot x^\kappa(\sigma)\nabla_{(\lambda\tau)}\Phi\mp\dot x_\mu(\sigma)\nabla_{(\nu\tau)}\Phi\mp\dot x_\nu(\sigma)\nabla_{(\mu\tau)}\Phi\pm\delta_{\mu\nu}\dot x^\kappa(\sigma)\nabla_{(\kappa\tau)}\Phi\Big)^{ab}\,.
\end{aligned}$$

Reduction to the ADHMN Construction

The lift reduces in the expected way to the ADHMN construction.

On $\mathcal{L}S^3 \subset \mathcal{L}\mathbb{R}^4$: $x^\mu x^\mu = \dot{x}^\mu \dot{x}^\mu = R^2$, $x^\mu \dot{x}^\mu = 0$.

Reduction (cf. Mukhi/Papageorgakis, 0803.3218):

$$\langle X^4 \rangle = \frac{r}{\ell_p^{3/2}} e_4 = g_{\mathrm{YM}} e_4 \ , \quad \dot{x}^4(\tau) = R \Rightarrow \dot{x}^i(\tau) = x^4(\tau) = 0$$

$$\nabla^{\mathrm{M}} = -\gamma_5 \frac{\mathrm{d}}{\mathrm{d}s} + \tfrac{1}{2}\gamma^{\mu\nu}\left(D(X^\mu, X^\nu) - \mathrm{i} \oint \mathrm{d}\tau x^\mu(\tau)\dot{x}^\nu(\tau) \right)$$

$$\to \ -\gamma_5 \frac{\mathrm{d}}{\mathrm{d}s} + \tfrac{1}{2}\gamma^{\mu\nu}\left(D(X^\mu, X^\nu) - 2\pi \mathrm{i} R x_0^\mu \delta_4^\nu \right)$$

$$= -\gamma_5 \frac{\mathrm{d}}{\mathrm{d}s} + R\gamma^{4i}\left(X^{i\alpha} D(e_\alpha, e_4) - \mathrm{i} x_0^i \right) + \ldots = \nabla^{\mathrm{IIA}} + \ldots$$

$$\frac{\mathrm{d}}{\mathrm{d}s} X^\mu = \tfrac{1}{3!}\varepsilon^{\mu\nu\rho\sigma}[X^\nu, X^\rho, X^\sigma] \quad \to \quad \frac{\mathrm{d}}{\mathrm{d}s} X^i = \tfrac{1}{2}\varepsilon^{ijk} R [X^j, X^k] + \ldots$$

$$F_{\mu\nu} = \varepsilon_{\mu\nu\rho\sigma} \dot{x}^\rho \frac{\partial}{\partial x^\sigma} \Phi_{\mathrm{SDS}} \quad \to \quad F_{ij} = \varepsilon_{ijk} \frac{\partial}{\partial x^k} R \Phi_{\mathrm{SDS}} + \ldots$$

Examples

Our examples reproduce the expected solutions.

Charge 1: Choose again trivial Nahm data. Introduce

$$y^{\mu\nu} := \oint \mathrm{d}\tau\, x^{[\mu}(\tau)\dot{x}^{\nu]}(\tau)\ , \quad r_\pm^2 := \tfrac{1}{2}\sqrt{(y^{\mu\nu} \pm \tfrac{1}{2}\varepsilon_{\mu\nu\kappa\lambda}y^{\kappa\lambda})^2}$$

The zero modes of the adjoint of the Dirac operators are:

$$\psi_{s,x} \sim \mathrm{e}^{-r_-^2 s} \begin{pmatrix} \mathrm{i}\left(r_-^2 + y^{12} - y^{34}\right) \\ y^{13} + y^{24} + \mathrm{i}(y^{23} - y^{14}) \\ 0 \\ 0 \end{pmatrix}$$

The solution then reads as $\Phi = \frac{\mathrm{i}}{2r_-^2}$ and

$$A(\sigma) = \frac{\mathrm{i}}{2r_-^2\,(r_-^2 + (y^{12} - y^{34}))} \begin{pmatrix} \dot{x}^3(\sigma)(y^{23} - y^{14}) + \dot{x}^4(\sigma)(y^{13} + y^{24}) \\ \dot{x}^4(\sigma)(y^{23} - y^{14}) - \dot{x}^3(\sigma)(y^{13} + y^{24}) \\ \dot{x}^1(\sigma)(y^{14} - y^{23}) + \dot{x}^2(\sigma)(y^{13} + y^{24}) \\ \dot{x}^2(\sigma)(y^{14} - y^{23}) - \dot{x}^1(\sigma)(y^{13} + y^{24}) \end{pmatrix}$$

This is indeed a solution.

Examples

Our examples reproduce the expected solutions.

Charge 2:

Nahm data:

$$X^\mu = \frac{e_\mu}{\sqrt{2s}}\ , \quad e_\mu \text{ generate } \mathcal{A}$$

Solution:

$$\Phi(x) = \frac{\mathrm{i}}{r_-^2}$$

As expected: twice the charge of the case $k = 1$.

Remarks

Our lift of the ADHMN construction is very natural and rather straightforward.

- The lift of the Dirac operator was natural considering the corresponding brane configurations.
- It is natural to go to loop space to describe self-dual strings.
- It can be trivially rendered nonabelian:
- The construction nicely involves the Basu-Harvey equation.
- It reduces naturally to the ADHMN construction.
- The construction does produce transgressed self-dual strings.
- It extends to real and hermitian 3-algebras (→ ABJM model)
- In these constructions: Gauge group on loop space is $G \times G$. cf. M5-brane models, e.g. Chu [1108.5131]
- Twistor space for self-dual strings can be found.

CS, M. Wolf [1111.nextweek]

The non-abelian tensor multiplet

A recently proposed 3-Lie algebra valued tensor-multiplet implies a transgression.

Recall the transgression map:

$$(\mathcal{T}\omega)_x(v_1(\tau),\dots,v_k(\tau)) := \int_{S^1} \mathrm{d}\tau\,\omega(v_1(\tau),\dots,v_k(\tau),\dot{x}(\tau))$$

Equations found by Lambert, Papageorgakis, 1007.2982:

$$\nabla^2 X^I - \tfrac{\mathrm{i}}{2}[\bar{\Psi},\Gamma_\nu\Gamma^I\Psi,C^\nu] - [X^J,C^\nu,[X^J,C_\nu,X^I]] = 0$$

$$\Gamma^\mu\nabla_\mu\Psi - [X^I,C^\nu,\Gamma_\nu\Gamma^I\Psi] = 0$$

$$\nabla_{[\mu}H_{\nu\lambda\rho]} + \tfrac{1}{4}\varepsilon_{\mu\nu\lambda\rho\sigma\tau}[X^I,\nabla^\tau X^I,C^\sigma] + \tfrac{\mathrm{i}}{8}\varepsilon_{\mu\nu\lambda\rho\sigma\tau}[\bar{\Psi},\Gamma^\tau\Psi,C^\sigma] = 0$$

$$F_{\mu\nu} - D(C^\lambda,H_{\mu\nu\lambda}) = 0$$

$$\nabla_\mu C^\nu = D(C^\mu,C^\nu) = 0$$

$$D(C^\rho,\nabla_\rho X^I) = D(C^\rho,\nabla_\rho\Psi) = D(C^\rho,\nabla_\rho H_{\mu\nu\lambda}) = 0$$

Factorization of $C^\rho = C\dot{x}^\rho$. Here, 3-Lie algebra transgression:

$$(\mathcal{T}\omega)_x(v_1(\tau),\dots,v_k(\tau)) := \int_{S^1} \mathrm{d}\tau\,D(\omega(v_1(\tau),\dots,v_k(\tau),\dot{x}(\tau)),C)$$

The non-abelian tensor multiplet on loop space

The corresponding equations can all be rewritten on loop space.

Transgression of matter fields (not in Huang, Huang, 1008.3834)

$$\mathring{X}^I(x(\tau)) := R\, D(C, X^I(x(\tau)))\,,\quad \mathring{\Psi}(x(\tau)) := \Gamma^\rho \dot{x}_\rho D(C, \Psi(x(\tau)))$$

Equations of motion (SYM-like):

$$\begin{aligned} \nabla^2 \mathring{X}^I + \tfrac{\mathrm{i}}{2}\tfrac{1}{R}\dot{x}^\nu[\bar{\mathring{\Psi}}, \Gamma_\nu \Gamma^I \mathring{\Psi}] + [\mathring{X}^J,[\mathring{X}^J,\mathring{X}^I]] &= 0\,, \\ \tfrac{1}{R}\Gamma^{\mu\nu}\dot{x}_\nu \nabla_\mu \mathring{\Psi} - \Gamma^I[\mathring{X}^I, \mathring{\Psi}] &= 0\,, \\ \nabla_\mu \mathring{F}^{\mu\nu} + [\mathring{X}^I, \nabla^\nu \mathring{X}^I] + \tfrac{\mathrm{i}}{2}\left([\bar{\mathring{\Psi}}, \Gamma^\nu \mathring{\Psi}] - \tfrac{2}{R^2}\dot{x}^\sigma \dot{x}^\nu [\bar{\mathring{\Psi}}, \Gamma_\sigma \mathring{\Psi}]\right) &= 0\,, \end{aligned}$$

Supersymmetry transformations (SYM-like):

$$\begin{aligned} \delta \mathring{X}^I &= \tfrac{1}{R}\mathrm{i}\bar{\varepsilon}\Gamma^I \dot{x}^\rho \Gamma_\rho \mathring{\Psi}\,, \\ \delta \mathring{A}_\mu &= \tfrac{1}{R^2}\mathrm{i}\bar{\varepsilon}\Gamma_{\mu\lambda}\Gamma_\rho \dot{x}^\lambda \dot{x}^\rho \mathring{\Psi}\,, \\ \delta \mathring{\Psi} &= \tfrac{1}{R}\Gamma^{\nu\mu}\dot{x}_\nu \Gamma^I \nabla_\mu \mathring{X}^I \varepsilon + \tfrac{1}{2}\Gamma_{\mu\nu}\mathring{F}^{\mu\nu}\varepsilon - \tfrac{1}{2}\Gamma^{IJ}[\mathring{X}^I, \mathring{X}^J]\varepsilon\,, \\ \delta C^\mu &= 0 \end{aligned}$$

Remarks

The loop space tensor multiplet fits well into the picture.

- Get SYM theory on loop space from the tensor multiplet
- BPS equation is essentially the self-dual string equation on loop space:
$$F_{\mu\nu} = \varepsilon_{\mu\nu\rho\sigma}\dot{x}^\rho\nabla^\sigma X^6$$
- We used a reduced loop space derivative: $\partial_\mu := \oint \mathrm{d}\tau \frac{\delta}{\delta x^\mu(\tau)}$
- Right direction, more work in progress to get rid of $\dot{x}$ etc.

Conclusions

Summary and Outlook.

Summary:

- ✓ Reformulation of self-dual string equation on loop space
- ✓ Generalized ADHMN construction for self-dual string
- ✓ Explicit constructions in many cases
- ✓ Reformulate non-abelian tensor multiplet eqns. on loop space
- ✓ Partially generalized ADHMN construction

Future directions:

- ▷ Find (2,0)-theory using loop space
- ▷ Study classical integrability in more detail (diff. dims., ...)
- ▷ Loop space quantization of S^3, cf. Dirac operator
- ▷ Quantization of S^3 via gerbes and 2-groupoids

Towards Describing M5-branes with Loop Space

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