Towards an M5-Brane Model: Progress Report

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Based on:

- (1705.02353, 1712.06623,) 1908.08086, with Lennart Schmidt
- 1911.06390, with Hyungrok Kim
- 2020.?????, with Dominik Rist and Miro van der Worp

Conjecture

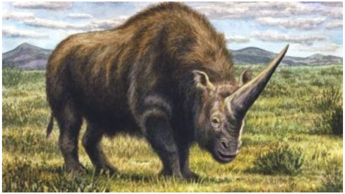
The (2,0)-theory is classically a higher gauge theory.

I know, I know ...

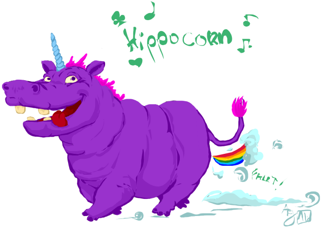
> *"... by hunting for unicorns we may find other creatures that are useful in understanding the theory more generally."*
>
> *Neil Lambert*

Wish:

Reality:



or



Motivation

Conjecture

The (2,0)-theory is classically a higher gauge theory.

Supporting the conjecture:

- Plenty of evidence and heuristic arguments
- Have: higher bundles with connections
- Have: (1,0)-interacting actions

Problems:

- Mathematical gaps (most of this talk)
- Physical problems (hopefully soon)

Outline

Summary:

We resolve a number of mathematical issues,
making interesting observations along the way.

Rough perspective:

- Need higher analogue of connections for higher gauge theory
- Physics should see higher gauge algebra up to equivalence
- Need metric on higher gauge algebra for actions
- Need definition of higher parallel transport
- All this now clarified.

Summary:

We resolve a number of mathematical issues, making interesting observations along the way.

Detailed perspective:

- Non-abelian gerbes with connection are locally abelian
- Adjustment resolves this ⇒ String structures with connection
  - Quickly deriving BRST complexes and adjustments.
- Metric string structures with connctions in general
- Replacing PST by Sen's mechanism
- Adjusted higher parallel transport
  - Relation between 2-crossed modules and L_∞-algebras
- Actions from tree-level scattering amplitudes

Outline

Alternative Summary:

Adjustments to naive definitions

"Category theory is the subject where you can leave the definitions as exercises."

John Baez

... but sometimes you have to find hidden extra solutions!

Need for adjusting higher gauge theory

Summary:

- Higher gauge theories come with fake curvatures, e.g.:

$$\mathcal{F} = \mathrm{d}A + \tfrac{1}{2}[A, A] + \mu_1(B)$$

- Consistency of usual definition requires fake flatness: $\mathcal{F} = 0$
- This is too severe a restriction, unsuitable for (2,0)-theory.

Higher connection locally, over $\mathbb{R}^d$: 1908.08086, 1810.06278

- Lie 2-group (crossed module) $(\mathsf{H} \xrightarrow{\mathsf{t}} \mathsf{G}, \rhd)$, $(\mathfrak{h} \xrightarrow{\mathsf{t}} \mathfrak{g}, \rhd)$
- Potential forms: $A \in \Omega^1(\mathbb{R}^d, \mathfrak{g})$, $B \in \Omega^2(\mathbb{R}^d, \mathfrak{h})$
- Fake flatness: $\mathcal{F} := \mathrm{d}A + \frac{1}{2}[A, A] + \mathsf{t}(B) = 0$
- Gauge transformations: $g \in \Omega^0(\mathbb{R}^d, \mathsf{G})$, $\Lambda \in \Omega^1(\mathbb{R}^d, \mathfrak{h})$

$$A \mapsto \tilde{A} = g^{-1}Ag + g^{-1}\mathrm{d}g + \mathsf{t}(\Lambda_1)$$
$$B \mapsto \tilde{B} = g^{-1} \rhd B + \mathrm{d}\Lambda_1 + \tilde{A} \rhd \Lambda_1 + \tfrac{1}{2}[\Lambda_1, \Lambda_1]$$

- A and gauge transformations restrict to $\mathsf{G}^\circ = \mathsf{G}/\mathrm{im}(\mathsf{t})$
- $F = 0$ and non-abelian Poincaré lemma: gauge with $\tilde{A}^\circ = 0$,
- $\tilde{A} \in \mathrm{im}(\mathsf{t})$, gauge away with Λ-transformation: $\tilde{\tilde{A}} = 0$
- connection is abelian with $\tilde{\tilde{B}} \in \ker(\mathsf{t})$!
- NA gerbes of Breen/Messing or Aschieri/Cantini/Jurčo: only good for higher Chern–Simons theory, but not (2,0)-theory

Problems with the fake curvature condition

Fake curvature, e.g.:

- Lie 2-algebra (2-term L_∞-algebra) $\mathfrak{h} \xrightarrow{\mu_1} \mathfrak{g}$
- Potential forms: $A \in \Omega^1(\mathbb{R}^d, \mathfrak{g})$, $B \in \Omega^2(\mathbb{R}^d, \mathfrak{h})$
- Fake flatness: $\mathcal{F} := \mathrm{d}A + \frac{1}{2}[A, A] + \mu_1(B) = 0$

Appears:

- For $\mu_3 \neq 0$: infinitesimal gauge transformations do not close:
$$[\delta_{c_0}, \delta_{c_1}]A = \delta_{[c_0,c_1]}A + \tfrac{1}{2}\mu_3(\mathcal{F}, A, A)$$
- For $\mu_3 = 0$, finite gauge transformations do not compose
- Self-duality $H = \star H$ requires $\mathcal{F} = 0$ for covariance:
$$H \to \tilde{H} = g \rhd H - \mathcal{F} \rhd \Lambda$$
- Parallel transport requires $\mathcal{F} = 0$ for reparam. invariance

One more basic problem

- Higher gauge algebra as L_∞-algebras $\mathfrak{g}$
- Appropriate equivalence: quasi-isomorphisms
- Dually: Chevalley–Eilenberg algebra $\mathsf{CE}(\mathfrak{g}) = (\odot^\bullet \mathfrak{g}[1]^*, Q)$
  - differential graded commutative algebra (dgca)
  - E.g. Lie algebra $\mathfrak{g}$, $\mathsf{CE}(\mathfrak{g})$ generated by $t^\alpha : \mathfrak{g}[1] \to \mathbb{R}$

$$Q = -\tfrac{1}{2} f^\gamma_{\alpha\beta} t^\alpha t^\beta \frac{\partial}{\partial t^\gamma} \qquad Q^2 = 0 \;\leftrightarrow\; \text{Jacobi identity}$$

- Also important:
  - Weil algebra $\mathsf{W}(\mathfrak{g})$, dual of $\mathfrak{inn}(\mathfrak{g}) = \mathfrak{g} \to \mathfrak{g}[1]$, generated by t^α and $\hat{t}^\alpha$.
  - $\mathsf{W}_\mathrm{h}(\mathfrak{g}) \subset \mathsf{W}(\mathfrak{g})$ generated by $\hat{t}^\alpha$
  - Inv. polynomials $\mathsf{inv}(\mathfrak{g}) \subset \mathsf{W}_\mathrm{h}(\mathfrak{g})$ with $Q_\mathsf{W}(\mathsf{inv}(\mathfrak{g})) \subset \mathsf{W}_\mathrm{h}(\mathfrak{g})$
- Definition of $\mathsf{inv}(\mathfrak{g})$ not compatible with quasi-isomorphisms

All of this is resolved when using connections on string structures.

Summary:

- Some L_∞-algebras: can deform ("adjust") definition of $\mathsf{W}(\mathfrak{g})$
- In gauge theory: curvatures change $\Rightarrow$ gauge trafos. change
- Prime example: String structures
  - Spin bundle with trivialization of Chern–Simons 2-gerbe
  - Particular non-abelian principal 3-bundle
  - Corresponds to spin bundle on loop space
  - Can carry connections, allow adjustment
    Killingback 1987, Witten 1988, Urs&Hisham&Stasheff 2009, ...

Adjustment of Weil algebras and curvatures

Adjusted Weil algebra

- Deformation of the action of differential Q_{W} of dga $\mathsf{W}(\mathfrak{g})$
- Deformation vanishes for generators $\hat{t}^\alpha \to 0$
- BRST complex closes
- Curvatures get deformed $\Rightarrow$ gauge trafos. deformed

Quick way of deriving BRST complex (generalized AKSZ):

$$\mathscr{A} : \mathsf{W}(\mathfrak{g}) \to \Omega^\bullet(U) \quad \in \mathsf{Map}(\Sigma, X)$$

$$Q_{\rm BRST}\mathscr{A} := {\rm d} \circ \mathscr{A} - \mathscr{A} \circ Q_{\mathsf{W}}$$

Put antifields and extra ghosts to zero.

For example: Lie algebra $\mathfrak{g}$:

$$\mathscr{A}(t^\alpha) = \Lambda_0^\alpha + A^\alpha + A^{+\alpha} + \Lambda_0^{+\alpha} + \dots$$

$$\mathscr{A}(\hat{t}^\alpha) = \vartheta_0^\alpha + \vartheta_1^\alpha + F^\alpha + F^{+\alpha} + \dots$$

We obtain for Q_{BRST} action:

$$
\begin{aligned}
\Lambda_0^\alpha &\mapsto \tfrac{1}{2} f^\alpha_{\beta\gamma} \Lambda_0^\beta \Lambda_0^\gamma - \vartheta_0^\alpha \\
A^\alpha &\mapsto \mathrm{d}\Lambda_0^\alpha + f^\alpha_{\beta\gamma} A^\beta \Lambda_0^\gamma - \vartheta_1^\alpha \\
A^{+\alpha} &\mapsto \mathrm{d}A^\alpha + f^\alpha_{\beta\gamma}(\Lambda_0^\beta A^{+\gamma} + \tfrac{1}{2} A^\beta A^\gamma) - F^\alpha \\
\Lambda_0^{+\alpha} &\mapsto \mathrm{d}A^{+\alpha} + f^\alpha_{\beta\gamma}(\Lambda_0^\beta \Lambda_0^{+\gamma} + A^\beta A^{+\gamma}) - F^{+\alpha} \\
\vartheta_0^\alpha &\mapsto f^\alpha_{\beta\gamma} \Lambda_0^\beta \vartheta_0^\gamma \\
\vartheta_1^\alpha &\mapsto \mathrm{d}\vartheta_0^\alpha + f^\alpha_{\beta\gamma}(\Lambda_0^\beta \vartheta_1^\gamma + A^\beta \vartheta_0^\gamma) \\
F^\alpha &\mapsto \mathrm{d}\vartheta_1^\alpha + f^\alpha_{\beta\gamma}(A^\beta \vartheta_1^\gamma + A^{+\beta} \vartheta_0^\gamma - F^\beta \Lambda_0^\gamma) \\
F^{+\alpha} &\mapsto \mathrm{d}F^\alpha + f^\alpha_{\beta\gamma}(\Lambda_0^\beta F^{+\alpha} + A^\beta F^\gamma + A^{+\beta} \vartheta_1^\gamma + \Lambda_0^{+\beta} \vartheta_0^\gamma)
\end{aligned}
$$

- Truncation $\vartheta_0^\alpha, \dots \to 0$ is consistent
- Yields kinematical data:
  potential, curvature, gauge transformations, Bianchi identities

Extended AKSZ-formalism for general Lie 2-algebra

Start from Lie 2-algebra $\mathfrak{g} = (\mathfrak{g}_{-1} \to \mathfrak{g}_0)$.

Extended AKSZ has maps:

$$\mathscr{A}(t^\alpha) = \Lambda_0^\alpha + A^\alpha + A^{+\alpha} + \Lambda_0^{+\alpha} + \dots$$
$$\mathscr{A}(\hat{t}^\alpha) = \vartheta_0^\alpha + \vartheta_1^\alpha + \mathcal{F}^\alpha + \mathcal{F}^{+\alpha} + \dots$$
$$\mathscr{A}(r^a) = \Sigma_0^a + \Lambda_1^a + B^a + B^{+a} + \Lambda_1^{+a} + \dots$$
$$\mathscr{A}(\hat{r}^a) = \Theta_0^a + \Theta_1^a + \Theta_2^a + H^a + H^{+a} + \dots$$

Q_{BRST} yields usual potentials, curvatures, gauge trafos., Bianchi:

$$\delta A = \mathrm{d}\Lambda_0 + \mu_2(A, \Lambda_0) + \mu_1(\Lambda_1) \qquad \delta B = \mathrm{d}\Lambda_1 + \mu_2(A, \Lambda_1) + \mu_2(\Lambda_0, B) - \tfrac{1}{2}\mu_3(A, A, \Lambda_0)$$
$$\mathcal{F} = \mathrm{d}A + \tfrac{1}{2}\mu_2(A, A) + \mu_1(B) \qquad H = \mathrm{d}B + \mu_2(A, B) - \tfrac{1}{3!}\mu_3(A, A, A)$$
$$\mathrm{d}\mathcal{F} = -\mu_2(A, \mathcal{F}) + \mu_1(H) \qquad \mathrm{d}H = -\mu_2(A, H) + \mu_2(\mathcal{F}, B) - \tfrac{1}{2}\mu_3(A, A, \mathcal{F})$$

But: truncation $\Theta_2^a = 0$ is inconsistent, unless fake flatness $\mathcal{F} = 0$

$$-\tfrac{1}{2}\mu_3(\Lambda_0, \Lambda_0, \mathcal{F}) + \mu_2(\mathcal{F}, \Sigma_0) = 0$$

Adjustment for skeletal string algebra

- Skeletal string Lie 2-algebra: $\mathfrak{string}_{\rm sk}(\mathfrak{g}) = (\mathbb{R} \to \mathfrak{g})$
- For later: Chern–Simons 2-gerbe visible:

$$\hat{\mathfrak{g}}_{\rm sk} := \left(\mathbb{R}_q \xrightarrow{\rm id} \mathbb{R}_r \longrightarrow \mathfrak{g}_t \right) := \left(\mathbb{R}[2] \xrightarrow{\rm id} \mathbb{R}[1] \longrightarrow \mathfrak{g}_t \right) \;\approx\; \mathfrak{g}$$

- Unadjusted action of $Q_{\rm W}$:

$$t^\alpha \mapsto -\tfrac{1}{2} f^\alpha_{\beta\gamma} t^\beta t^\gamma + \hat{t}^\alpha \qquad r \mapsto \tfrac{1}{3!} f_{\alpha\beta\gamma} t^\alpha t^\beta t^\gamma + q + \hat{r} \qquad q \mapsto \hat{q}$$

$$\hat{t}^\alpha \mapsto -f^\alpha_{\beta\gamma} t^\beta \hat{t}^\gamma \qquad \hat{r} \mapsto -\tfrac{1}{2} f_{\alpha\beta\gamma} t^\alpha t^\beta \hat{t}^\gamma - \hat{q} \qquad \hat{q} \mapsto 0$$

- Adjusted action of $Q_{\rm W}$

$$t^\alpha \mapsto -\tfrac{1}{2} f^\alpha_{\beta\gamma} t^\beta t^\gamma + \hat{t}^\alpha \qquad r \mapsto \tfrac{1}{3!} f_{\alpha\beta\gamma} t^\alpha t^\beta t^\gamma - \chi_{\rm sk}(t, \hat{t}) + q + \hat{r} \qquad q \mapsto \hat{q}$$

$$\hat{t}^\alpha \mapsto -f^\alpha_{\beta\gamma} t^\beta \hat{t}^\gamma \qquad \hat{r} \mapsto \chi_{\rm sk}(\hat{t}, \hat{t}) - \hat{q} \qquad \hat{q} \mapsto 0$$

- Adjustment governed by Killing form $\kappa_{\alpha\beta}$:

$$\chi_{\rm sk}(t, \hat{t}) := \kappa_{\alpha\beta} t^\alpha \hat{t}^\beta \quad \text{and} \quad \chi_{\rm sk}(\hat{t}, \hat{t}) := \kappa_{\alpha\beta} \hat{t}^\alpha \hat{t}^\beta$$

Adjusted kinematical data

Gauge potentials:

$$(A, B, C) \in (\Omega^1(U.\mathfrak{g}) \oplus \Omega^2(U) \oplus \Omega^3(U)$$

Curvatures:

$$F := \mathrm{d}A + \tfrac{1}{2}[A, A] \qquad G := \mathrm{d}C$$

$$H := \mathrm{d}B - \tfrac{1}{3!}\mu_3(A, A, A) + \chi_{\mathrm{sk}}(A, F) - C$$

$$= \mathrm{d}B + \underbrace{(A, \mathrm{d}A) + \tfrac{1}{3}(A, [A, A])}_{\mathrm{cs}(A)} - C$$

Bianchi identities:

$$\mathrm{d}F + [A, F] = 0 \qquad \mathrm{d}G = 0$$

$$\mathrm{d}H - \chi_{\mathrm{sk}}(F, F) + G = \mathrm{d}H - (F, F) + G = 0$$

Gauge transformations:

$$\delta A = \mathrm{d}\Lambda_0 + \mu_2(A, \Lambda_0) \qquad \delta C = \mathrm{d}\Sigma$$

$$\delta B = \mathrm{d}\Lambda_1 + (\Lambda_0, F) - \tfrac{1}{2}\mu_3(A, A, \Lambda_0) - \Sigma$$

$$\delta F = -\mu_2(F, \Lambda_0) \qquad \delta H = 0 \qquad \delta G = 0$$

Metric extension

Summary:

- For actions, need metric extension
- Metric on L_∞-algebras: cyclic structure $\leftrightarrow$ symplectic form
- String algebra not symplectic, cf. $\hat{\mathfrak{g}}_{\rm sk} = \left(\mathbb{R}_q \xrightarrow{\rm id} \mathbb{R}_r \longrightarrow \mathfrak{g}_t \right)$
- Construct $T^*[-2]\hat{\mathfrak{g}}_{\rm sk}$, imitating BV-formalism

Schmidt, CS, 2017

Metric extension

Result:

$$\hat{\mathfrak{g}}^{\omega}_{\rm sk} = \begin{pmatrix} \mathfrak{g}^*_v & \xrightarrow{\mu_1=\mathrm{id}} & \mathfrak{g}^*_u & & \mathbb{R}^*_s & \xrightarrow{\mu_1=\mathrm{id}} & \mathbb{R}^*_p \\ & & \oplus & & \oplus & & \oplus \\ & & \mathbb{R}_q & \xrightarrow{\mu_1=\mathrm{id}} & \mathbb{R}_r & & \mathfrak{g}_t \end{pmatrix}$$

- Allows for adjustment
- Kinematical data:

$$A \in \Omega^1(U) \otimes (\mathfrak{g} \oplus \mathbb{R}^*) \qquad B \in \Omega^2(U) \otimes (\mathbb{R} \oplus \mathbb{R}^*)[1]$$

$$C \in \Omega^3(U) \otimes (\mathfrak{g}^* \oplus \mathbb{R})[2] \qquad D \in \Omega^4(U) \otimes \mathfrak{g}^*[3]$$

Adjusted curvatures:

$$\begin{aligned} F &= \mathrm{d}A + \tfrac{1}{2}\mu_2(A,A) + \mu_1(B) && \in \ \Omega^2(U) \otimes (\mathfrak{g} \oplus \mathbb{R}^*) \\ H &= \mathrm{d}B - \tfrac{1}{3!}\mu_3(A,A,A) + \chi_{\rm sk}(A,F) - \mu_1(C) && \\ &= \mathrm{d}B + (A, \mathrm{d}A) + \tfrac{1}{3}(A, \mu_2(A,A)) - \mu_1(C) && \in \ \Omega^3(U) \otimes (\mathbb{R} \oplus \mathbb{R}^*)[1] \\ G &= \mathrm{d}C + \mu_2(A,C) + \tfrac{1}{2}\mu_3(A,A,B) + \mu_1(D) && \in \ \Omega^4(U) \otimes (\mathfrak{g}^* \oplus \mathbb{R})[2] \\ I &= \mathrm{d}D + \mu_2(A,D) + \chi_{\rm sk}(F,C) + \tfrac{1}{2}\chi_{\rm sk}(A,A,H) && \\ &\quad + \chi_{\rm sk}(F,A,B) && \in \ \Omega^5(U) \otimes \mathfrak{g}^*[3] \end{aligned}$$

(1,0)-action

Summary:

- $\exists$ general (1,0)-action in 6d with "mysterious" gauge structure
  Samtleben, Sezgin, Wimmer, 2011
- specializes to adjusted metric string structures
- PST part can be replaced by Sen's mechanism

Recap: classically well-defined (1,0)-action

- 6d (1,0)-action derived from tensor hierarchies in SUGRA
  Samtleben, Sezgin, Wimmer, 2011
- Action can be specialized to adjusted metric string structure
  Schmidt, CS, 2017a
- Field content:
  - (1,0) tensor multiplet (ϕ, χ^i, B), values in $\mathbb{R}^2$, $\phi = \phi_s + \phi_r$, ...
  - (1,0) vector multiplet (A, λ^i, Y^{ij}), values in $\mathfrak{g} \oplus \mathbb{R}$
  - C-field (3-form), values in $\mathbb{R} \oplus \mathfrak{g}^*$
  - D-field (4-form), values in $\mathfrak{g}^*$
- Action (schematically):

$$S = \int_{\mathbb{R}^{1,5}} \Big(\mathcal{H}_r \wedge *\mathcal{H}_s + \mathrm{d}\phi_r \wedge *\mathrm{d}\phi_s - *\bar{\chi}_r \partial\!\!\!/ \chi_s + \mathcal{H}_s \wedge *(\bar{\lambda}, \gamma_{(3)}\lambda) + *(Y, \bar{\lambda})\chi_s$$
$$+ \phi_s \big((\mathcal{F}, *\mathcal{F}) - *(Y, Y) + *(\bar{\lambda}, \nabla\!\!\!\!/ \lambda) \big) + (\bar{\lambda}, \mathcal{F}) \wedge *\gamma_{(2)}\chi_s$$
$$+ \mu_1(C) \wedge \mathcal{H}_s + B_s \wedge (\mathcal{F}, \mathcal{F}) + B_s \wedge ([A, A], [A, A]) \Big)$$

- Supersymmetric completion by matter & PST sector
  Schmidt, CS, 2017b

Problem with PST-action

$$\mathcal{L}_{\text{PST}} = \tfrac{1}{2}\langle \iota_V \mathscr{H}^+, \mathscr{H}^+ \rangle \wedge v + \tfrac{1}{\phi_s}(\iota_V * \mathscr{G}^+, * \iota_V * \mathscr{G}^+)$$

Alternative: Sen's mechanism

- Consider $\beth \in \Omega^3_+(\mathbb{R}^{1,5})$ with action

$$S = \int \left(\tfrac{1}{2}\mathrm{d}B \wedge \star\mathrm{d}B - \mathrm{d}B \wedge \beth + \mathcal{L}_{\text{int}}(\beth, \Psi) \right)$$

- Equations of motion:

$$\mathrm{d}(\star\mathrm{d}B - \beth) = 0 \quad \mathrm{d}B - \star\mathrm{d}B + \frac{\delta \int \mathcal{L}_{\text{int}}(\beth, \Psi)}{\delta \beth} = 0 \quad \frac{\delta \int \mathcal{L}_{\text{int}}(\beth, \Psi)}{\delta \Psi} = 0$$

- Self dual forms (first decouples, second stays):

$$\mathrm{d}B + \star\mathrm{d}B - \beth \quad \text{and} \quad \mathrm{d}B + \star\mathrm{d}B + \beth$$

- Idea: replace PST by Sen, cf. Neil's talk

Desired equations of motion (full duality in 6d):

$$\mathscr{H}^+ := *\mathcal{H} - \mathcal{H} - \chi_{\rm sk}(\bar\lambda, \gamma_{(3)}\lambda) = 0$$
$$\mathscr{G}^+ := \mathcal{G} - \chi_{\rm sk}(*\mathcal{F}, \phi) + 2\chi_{\rm sk}(\bar\lambda, *\gamma_{(2)}\chi) = 0$$
$$\mathscr{I}^+ := \mathcal{I} + \mu_2(\chi_{\rm sk}(\bar\lambda, \phi), \gamma^\mu\lambda)\mathrm{vol}_\mu + \text{matter} = 0$$

Lagrangian (with new fields for every curvature):

$$\mathcal{L} = \mathcal{L}_{\beth} + \mathcal{L}_{\daleth} + \mathcal{L}_{\gimel} + \mathcal{L}_{\rm vector} + \mathcal{L}_{\rm hyper}$$

For example:

$$\mathcal{L}_{\beth} = -\mathscr{H}^+ \wedge \beth^s - \frac{1}{2}\mathscr{H}^+ \wedge \star\mathscr{H}^+$$

- extension of Sen's mechanism
- fully replaces PST
- SUSY and other checks still incomplete, but should work

Strict metric string structures

Summary:

- Several models (quasi-isomorphic) for string Lie 2-algebra
- Extreme cases: skeletal (above) and strict (now)
- Physics shouldn't care!
- Should have metric adjustment also for strict model
- $\Rightarrow$ constraints on action

Two extreme models of the string Lie 2-algebra

Skeletal/minimal model:

$$\begin{array}{rcl} \mathfrak{string}_{\rm sk}(\mathfrak{g}) & = & \left(\begin{array}{ccc} \mathbb{R} & \xrightarrow{\mu_1} & \mathfrak{g} \\ r & \xmapsto{\mu_1} & 0 \end{array} \right) \end{array}$$

$$\mu_2\colon \mathfrak{g}\wedge\mathfrak{g}\to\mathfrak{g} \qquad \mu_2(a_1,a_2)=[a_1,a_2]$$

$$\mu_3\colon \mathfrak{g}\wedge\mathfrak{g}\wedge\mathfrak{g}\to\mathbb{R} \qquad \mu_3(a_1,a_2,a_3)=\big\langle a_1,[a_2,a_3]\big\rangle$$

Strict model:

$$\begin{array}{rcl} \mathfrak{string}_{\rm lp}(\mathfrak{g}) & = & \left(\begin{array}{ccc} \hat{L}_0\mathfrak{g} & \xrightarrow{\mu_1} & P_0\mathfrak{g} \\ (\lambda,r) & \xmapsto{\mu_1} & \lambda \end{array} \right) \end{array}$$

$$\mu_2: \quad P_0\mathfrak{g}\wedge P_0\mathfrak{g}\to P_0\mathfrak{g} \qquad (\gamma_1,\gamma_2)\mapsto[\gamma_1,\gamma_2]$$

$$\mu_2: \quad P_0\mathfrak{g}\otimes\hat{L}_0\mathfrak{g}[1]\to\hat{L}_0\mathfrak{g}[1] \qquad (\gamma,(\lambda,r))\mapsto\left([\gamma,\lambda]\,,\ -2\int_0^1 \mathrm{d}\tau\big\langle\gamma(\tau),\dot{\lambda}(\tau)\big\rangle\right)$$

Result after metric extension and adjustment

$$\hat{\mathfrak{g}}^\omega_{\mathrm{lp}} = \left(\begin{array}{ccccccc}
\mathfrak{g}^*_v & \xrightarrow{\partial^*} & (P_0\mathfrak{g})^*_u & \twoheadrightarrow & (L_0\mathfrak{g})^*_s & & \\
 & & & & \oplus & & \\
 & & \oplus & & \mathbb{R}^*_{s_0} & \xrightarrow{\mathrm{id}} & \mathbb{R}^*_p \\
 & & & & \oplus & & \\
 & & \mathbb{R}_q & \xrightarrow{\mathrm{id}} & \mathbb{R}_{r_0} & & \oplus \\
 & & & & \oplus & & \\
 & & & & (L_0\mathfrak{g})_r & \hookrightarrow & (P_0\mathfrak{g})_t
\end{array}\right)$$

Kinematical data:

$A \in \Omega^1(U) \otimes (P_0\mathfrak{g} \oplus \mathbb{R}^*)$ $\quad$ $B \in \Omega^2(U) \otimes (L_0\mathfrak{g} \oplus (L_0\mathfrak{g})^* \oplus \mathbb{R} \oplus \mathbb{R}^*)[1]$

$C \in \Omega^3(U) \otimes ((P_0\mathfrak{g})^* \oplus \mathbb{R})[2]$ $\quad$ $D \in \Omega^4(U) \otimes \mathfrak{g}^*[3]$

Adjusted curvatures:

$$\begin{aligned}
F &= \mathrm{d}A + \tfrac{1}{2}\mu_2(A,A) + \mu_1(B) && \in \Omega^2(U) \otimes (P_0\mathfrak{g} \oplus \mathbb{R}^*) \\
H &= \mathrm{d}B + \mu_2(A,B) - \chi_{\mathrm{lp}}(A,F) - \mu_1(C) && \in \Omega^3(U) \otimes (L_0\mathfrak{g} \oplus (L_0\mathfrak{g})^* \oplus \mathbb{R} \oplus \mathbb{R}^*)[1] \\
G &= \mathrm{d}C + \mu_2(A,C) + \mu_2(B,B) + \mu_1(D) + \\
&\quad + \chi_{\mathrm{lp}}(A,H) - \chi_{\mathrm{lp}}(F,B) && \in \Omega^4(U) \otimes ((P_0\mathfrak{g})^* \oplus \mathbb{R})[2] \\
I &= \mathrm{d}D + \mu_2(A,D) + \chi_{\mathrm{lp}}(F,C) - \\
&\quad - \chi_{\mathrm{lp}}(A,H) - \chi_{\mathrm{lp}}(F,\chi_{\mathrm{lp}}(A,B)) && \in \Omega^5(U) \otimes \mathfrak{g}^*[3]
\end{aligned}$$

Adjusted parallel transport

Summary:

- Ordinary higher parallel transport requires fake flatness $\mathcal{F} = 0$
- Can define an adjustment by extending functorial definition

Ordinary parallel transport

Usual functorial perspective on parallel transport (locally!):

$$\Phi\colon \mathcal{P}U \longrightarrow \mathsf{BG} \qquad \begin{array}{ccc} \mathsf{paths} & \xrightarrow{\Phi_1} & \mathsf{G} \\ \downarrow\downarrow & & \downarrow\downarrow \\ U & \xrightarrow{\Phi_0} & * \end{array}$$

- Modulo technicalities (thin homotopy, sitting instances)
- Composition of paths $\Rightarrow$ multiplication of group elements
- Connection: $g = \mathbb{1} + \iota_X A$ for inf. paths in direction X
- Conversely: $g(\gamma) = \mathrm{P}\exp \int_\gamma A$
- Readily extends to higher gauge theory:
  - Higher path groupoid
  - Higher gauge group, as one-object higher groupoid
  - But: requires fake curvatures to vanish!

Towards adjusted parallel transport

- Ordinary parallel transport: $\Phi\colon \mathcal{P}U \longrightarrow \mathsf{BG}$
- This "sees" connections, but we adjust only curvatures!
- Short exact sequence of groupoids:

$$* \longrightarrow \left(\begin{array}{c}\mathsf{G}\\ \downdownarrows\\ \mathsf{G}\end{array}\right) \hookrightarrow \mathsf{Inn(G)} \longrightarrow \left(\begin{array}{c}\mathsf{G}\\ \downdownarrows\\ *\end{array}\right) \longrightarrow *$$

- $\mathsf{Inn(G)}$ is integration of Lie 2-algebra corresponding to $\mathsf{W}(\mathfrak{g})$
- Derived parallel transport functor:

$$\begin{array}{ccccc}
\mathcal{P}U & \hookrightarrow & \mathcal{P}_{(2)}U & & \\
\downarrow{\scriptstyle \Phi} & & \downarrow{\scriptstyle \mathit{\Phi}} & \searrow^{\mathit{\Phi}_{\rm curv}} & \\
\mathsf{BG} & \hookrightarrow & \mathsf{BInn(G)} & \overset{\Pi}{\twoheadrightarrow} & \mathsf{BBG}
\end{array}$$

- $\mathit{\Phi}$ fully determined/equivalent to Φ

Preliminary remarks

- Need to find $\mathsf{BInn}(\mathscr{G})$ and thus $\mathsf{Inn}(\mathscr{G})$ for 2-group $\mathscr{G}$
- Note: $\mathfrak{inn}(\mathfrak{g})$ is the Lie 2-algebra dual to Weil algebra $\mathsf{W}(\mathfrak{g})$.
- Thus: integrate an adjusted Weil algebra
- Finite constructions $\rightarrow$ use strict model!
- General adjustment possible
- Simple example to illustrate construction.

Example for adjusted Weil algebra

For simplicity, drop central extension: $\mathfrak{g}_{\rm lp} := (L_0\mathfrak{g} \to P_0\mathfrak{g}) \cong \mathfrak{g}$:

- Adjusted Weil algebra with generators $t^{\alpha\tau}, \hat{t}^{\alpha\tau}, r^{\alpha\tau}, \hat{r}^{\alpha\tau}$.
- Action of differential $Q_{\mathsf{W}_{\rm adj}}$:

$$t^{\alpha\tau} \mapsto -\tfrac{1}{2} f^\alpha_{\beta\gamma} t^{\beta\tau} t^{\gamma\tau} - r^{\alpha\tau} + \hat{t}^{\alpha\tau} \qquad \hat{t}^{\alpha\tau} \mapsto -f^\alpha_{\beta\gamma} t^{\beta\tau} \hat{t}^{\gamma\tau} + \chi^{\alpha\tau}(t,\hat{t}) + \hat{r}^{\alpha\tau}$$

$$r^{\alpha\tau} \mapsto -f^\alpha_{\beta\gamma} t^{\beta\tau} r^{\gamma\tau} + \chi^{\alpha\tau}(t,\hat{t}) + \hat{r}^{\alpha\tau} \qquad \hat{r}^{\alpha\tau} \mapsto 0$$

with

$$\chi^{\alpha\tau}(t,\hat{t}) := f^\alpha_{\beta\gamma}(t^{\beta\tau}\hat{t}^{\gamma\tau} - \ell(\tau) t^{\beta 1}\hat{t}^{\gamma 1})$$

- Coordinate change: $\hat{t}^{\alpha\tau} \to \tilde{t}^{\alpha\tau} = \hat{t}^{\alpha\tau} - r^{\alpha\tau}$, others remain:

$$t^{\alpha\tau} \mapsto -\tfrac{1}{2} f^\alpha_{\beta\gamma} t^{\beta\tau} t^{\gamma\tau} + \tilde{t}^{\alpha\tau} \qquad \tilde{t}^{\alpha\tau} \mapsto -f^\alpha_{\beta\gamma} t^{\beta\tau} \tilde{t}^{\gamma\tau}$$

$$r^{\alpha\tau} \mapsto f^\alpha_{\beta\gamma}(t^{\beta\tau}\tilde{t}^{\gamma\tau} - \ell(\tau) t^{\beta 1}\tilde{t}^{\gamma 1}) + \hat{r}^{\alpha\tau} \qquad \hat{r}^{\alpha\tau} \mapsto 0$$

- Results in a strict 3-term L_∞-algebra

$$\begin{array}{cccccccc} \mathsf{W}_{\rm adj}(L_0\mathfrak{g} \to P_0\mathfrak{g}) & \leftrightarrow (& L_0\mathfrak{g} & \xrightarrow{\mu_1} & L_0\mathfrak{g}' \rtimes P_0\mathfrak{g}' & \xrightarrow{\mu_1} & P_0\mathfrak{g} &) \\ & & \lambda & \xrightarrow{\mu_1} & (\lambda, 0) & & & \\ & & & & (\lambda,\gamma) & \xrightarrow{\mu_1} & \gamma & \end{array}$$

The 2-crossed module of Lie algebras

Recall: Hypercrossed modules

- Correspond to simplicial groups
- Higher groups with trivial associator
- No further obstruction to integration.
- crossed modules $\leftrightarrow$ strict 2-term L_∞-algebras
- 2-crossed module $\xrightarrow{\text{forget}}$ strict 3-term L_∞-algebra
- Some strict 3-term L_∞-algebras $\xrightarrow{\text{extend}}$ 2-crossed modules

Here:

- Strict 3-term L_∞-algebra from $\mathsf{W}_{\mathsf{adj}}(L_0\mathfrak{g} \to P_0\mathfrak{g})$ is 2cm!
- Extension data, the Peiffer lifting given by adjustment data:

$$\{(\lambda_1, \gamma_1), (\lambda_2, \gamma_2)\} = \chi_{\rm lp}(\lambda_1 + \gamma_1, \lambda_2 + \gamma_2)$$

- That is, one should look at EL_∞-algebras... Roytenberg

- Strict Lie 2-algebra $\mathfrak{g}_{\rm lp} := (L_0\mathfrak{g} \xrightarrow{\mu_1} P_0\mathfrak{g})$
- Adjusted inner derivations $\mathfrak{inn}(\mathfrak{g}_{\rm lp})$:

$$\mathfrak{inn}(\mathfrak{g}_{\rm lp}) := (L_0\mathfrak{g} \xrightarrow{\mathsf{t}} L_0\mathfrak{g}' \rtimes P_0\mathfrak{g}' \xrightarrow{\mathsf{t}} P_0\mathfrak{g})$$

- This is a 2-crossed module of Lie algebras ...
- ... and integrates to 2-crossed module of Lie groups:

$$\mathsf{Inn}_{\rm adj,}(\mathsf{G}_{\rm lp}) := (L_0\mathsf{G} \xrightarrow{\mathsf{t}} L_0\mathsf{G} \rtimes P_0\mathsf{G} \xrightarrow{\mathsf{t}} P_0\mathsf{G})$$

Adjusted higher parallel transport

Ordinary derived parallel transport for 2-group $\mathscr{G}$:

$$
\begin{array}{ccccc}
\mathcal{P}_{(2)}U & \hookrightarrow & \mathcal{P}_{(3)}U & & \\
\downarrow{\scriptstyle \Phi} & & \downarrow{\scriptstyle \Phi} & \searrow^{\Phi_{\rm curv}} & \\
\mathsf{B}\mathscr{G} & \hookrightarrow & \mathsf{BInn}(\mathscr{G}) & \xrightarrow{\Pi} & \mathsf{BB}\mathscr{G}
\end{array}
$$

Consistency: fake curvature condition $\mathcal{F}=0$

Adjusted parallel transport functor for string group:

$$
\begin{array}{ccccc}
 & & \mathcal{P}_{(3)}U & & \\
 & & \downarrow{\scriptstyle \Phi^{\rm adj}} & \searrow^{\Phi^{\rm adj}_{\rm curv}} & \\
\mathsf{BString}_{\rm lp} & \hookrightarrow & \mathsf{BInn}_{\rm adj}(\mathsf{String}_{\rm lp}) & \xrightarrow{\Pi} & \mathsf{BBString}_{\rm lp}
\end{array}
$$

Consistency: Bianchi identity $\nabla F=0$

Note: no "ordinary" functor without curvatures!

(2,0)-theory from amplitudes

Summary:

- Tree-level scattering amplitudes for $(2,0)$-theory
- This should allow us to reconstruct some action.

An action exists

Homotopy Algebras and Perturbative Quantum Field Theory

- Classical Lagrangian field theory $\xleftrightarrow{\text{BV-formalism}}$ L_∞-algebra
- L_∞-algebra come with minimal models
- Minimal models encode tree-level scattering amplitudes
- Any L_∞-algebra quasi$\cong$ to a strict one ($\mu_i = 0$ for $i \geq 3$)

Many people, cf. 1912.06695

For the $(2,0)$-theory:

- Have tree-level scattering amplitudes from bootstrap

  Geyer, Mason, 1901.00134
- Yield a minimal model
- This is quasi-isomorphic to a strict L_∞-algebra
- So we obtain an action with cubic vertices.
- Goal: construct the strict and other actions.

Summary and outlook

Done:

- Naive higher gauge theory is abelian
- Corrections/Adjustment possible for connections
- Metric string structures derived
- Have action with higher connections, relation to SUGRA
- PST can be replaced by Sen's mechanism
- Adjusted higher parallel transport
- Amplitudes: indication for an action.

Soon to come:

- Completion of calculations for Sen's mechanism
- General actions for general metric string structures
- Actions derived from amplitudes
- Nahm transforms for M2-M5-brane systems

Towards an M5-Brane Model: Progress Report

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