Higher Gauge Theory and M-theory

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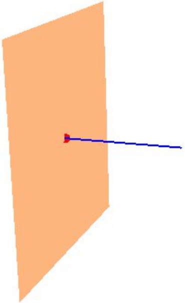
Based on work w. S Palmer, G Demessie, B Jurčo, M Wolf, P Ritter, R Szabo:

- Higher Gauge Theory: 1203.5757, 1308.2622, 1311.1977, 1406.5342
- Integrability: 1105.3904, 1205.3108, 1305.4870, 1312.5644, 1403.7185
- Geometric quantization: 1211.0395, 1308.4892

Why Higher Gauge Theory?

(2,0) theory should capture parallel transport of self-dual strings.

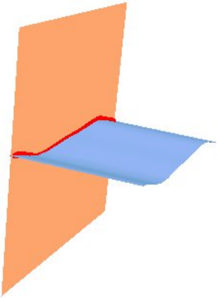


D-branes

- D-branes interact via strings.
- Effective description: theory of endpoints
- Parallel transport of these: Gauge theory

M5-branes

- M5-branes interact via M2-branes.
- Eff. description: theory of self-dual strings
- Parallel transport: Higher gauge theory
- (2,0) theory a higher gauge theory (HGT)?



Lifting Things to M-Theory

Things are not straightforward but do look very promising.

So why not write down an HGT action and be done?

Things are more complicated...

- Higher gauge theory is a very young area (since ~2002).
- Very few actions known for higher gauge theory.
- More groundwork needed (2-vector spaces, ...)

However, what we can see so far is very encouraging:

- M2-brane models (BLG/ABJM) are HGTs
- Integrability of BPS subsectors via ADHM-type constructions
- Twistor descriptions of HGTs
- Higher monopole and instanton solutions
- IKKT model has a clear categorified analogue
- (1,0)-models from tensor hierarchies are HGTs
- ...

Higher Gauge Theory

Higher gauge theory describes parallel transport of extended objects.

Parallel transport of particles in representation of gauge group G:

- holonomy functor hol : path $p \mapsto \mathsf{hol}(p) \in \mathsf{G}$
- $\mathsf{hol}(p) = P\exp(\int_p A)$, P: path ordering, trivial for $\mathsf{U}(1)$.

Parallel transport of strings with gauge group $\mathsf{U}(1)$:

- map hol : surface $s \mapsto \mathsf{hol}(s) \in \mathsf{U}(1)$
- $\mathsf{hol}(s) = \exp(\int_s B)$, B: connective structure on gerbe.

Nonabelian case:

- much more involved!
- no straightforward definition of surface ordering
- solution: Categorification!

see Baez, Huerta, 1003.4485

We will need to use some very simple notions of category theory, an esoteric subject noted for its difficulty and irrelevance.
G. Moore and N. Seiberg, 1989

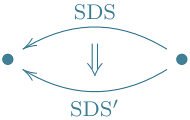
What does categorification mean?
One of Jeff Harvey's questions to identify the "generation PhD>1999" at Strings 2013.

Parallel Transport Along Surfaces

Categorification eliminates the need for surface ordering.

- Consider self-dual strings:
  endpoints: objects
  string: morphisms of a category.

- Parallel transport along surface:
  morphism between morphisms



- This yields a 2-category: objects, 1-morphisms, 2-morphisms

We need to categorify all notions in gauge theory.

Example: Internal Categorification of Lie Algebras

A weak Lie 2-algebra is the internal categorification of a Lie algebra.

Consider the category Vect: linear maps $\rightrightarrows$ vector spaces.

A category $\mathscr{C}$ internal to Vect is a structure

$$C_1 \ \rightrightarrows \ C_0 \hookrightarrow C_1 \ , \quad \circ : C_1 \times_{C_0} C_1 \to C_1$$

- C_0 and C_1 are vector spaces or objects in Vect
- $\rightrightarrows$ and $\hookrightarrow$ are linear maps or morphisms in Vect
- composition is a binary morphism in Vect
- the usual axioms of a category hold
- Obvious: internal functors, internal natural transformations

2-Vector space

A 2-vector space is a category in Vect.

Baez, Crans 2003

Example: Internal Categorification of Lie Algebras

A weak Lie 2-algebra is the internal categorification of a Lie algebra.

Lie algebra

A Lie algebra is vector space with a bilinear morphism $[-,-]$, which is totally antisymmetric and satisfies the Jacobi identity.

Lie 2-algebra

A Lie 2-algebra is a 2-vector space with a bilinear functor $[-,-]$, which is totally antisymmetric up to isomorphisms and satisfies the Jacobi identity up to isomorphisms.

That is, we have internal natural isomorphisms:

- Alt : $[v,w] \Rightarrow -[w,v]$
- Jac : $[u,[v,w]] + [v,[w,u]] \Rightarrow -[w,[u,v]]$

Similarly for other mathematical notions ("stuff+structures"):
Sets $\rightarrow$ categories, structure maps $\rightarrow$ functors

Semistrict Lie 2-Algebras: 2-Term L_∞-Algebras

A semistrict Lie 2-algebra is equivalent to a 2-term strong-homotopy Lie algebra.

Further Restrictions of Lie 2-algebras:

- Alt $=\mathrm{id}$: semistrict
- Jac $=\mathrm{id}$: hemistrict
- Alt $=$ Jac $=\mathrm{id}$: strict

2-term L_∞-algebra/strong homotopy Lie algebra

Semistrict case: $\mathsf{s},\mathsf{t}:C_1\rightrightarrows C_0$ cat. equiv. to $\ker(\mathsf{s})\xrightarrow{\mathsf{t}}C_0$.

More generally:

Semistrict Lie n-algebras $\leftrightarrow$ n-term strong homotopy Lie algebras:

- Graded vector space: $L_{-n}\xrightarrow{\mu_1}\dots\xrightarrow{\mu_1}L_1\xrightarrow{\mu_1}L_0\xrightarrow{\mu_1}0$
- Higher "brackets" $\mu_n:L^{\wedge n}\to L$ of degree $2-n$
- Higher/Homotopy Jacobi identities, e.g.
  $\mu_1^2=0\ ,\quad \mu_1(\mu_2(\ell_1,\ell_2))=\pm\mu_2(\mu_1(\ell_1),\ell_2)\pm\mu_2(\mu_1(\ell_2),\ell_2)$
  $\mu_2(\mu_2(\ell_1,\ell_2),\ell_3)+\text{cycl.}=\pm\mu_1(\mu_3(\ell_1,\ell_2,\ell_3))$
- Known from: BV-quant., string FT, deformation quant., ...

Application:

Local Higher Gauge Theory

Local Higher Gauge Theory with L_∞-Algebras

Homotopy Maurer-Cartan equations determine higher gauge theory.

Homotopy Maurer-Cartan equations (BV-quant., SFT)

Define curvatures. $F = \mathrm{d}A + \frac{1}{2}[A, A] = 0$ generalizes to

$$\mu_1(\phi) + \tfrac{1}{2}\mu_2(\phi, \phi) + \ldots = \sum_{i=1}^{\infty} \frac{(-1)^{i(i+1)/2}}{i!}\mu_i(\phi, \cdots, \phi) = 0$$

Gauge transformations $\delta A = \mathrm{d}\alpha + [A, \alpha]$ generalizes to

$$\delta\phi = \mu_1(\lambda) + \mu_2(\phi, \lambda) + \ldots = \sum_{i=1}^{\infty} \frac{(-1)^{i(i-1)/2}}{(i-1)!}\mu_i(\lambda, \phi, \cdots, \phi)$$

- Note: L_∞-algebra $\tilde{L} \to L = \Omega^\bullet(M) \otimes \tilde{L}$, degrees add.
- HMC equations for semistrict Lie 2-algebra:
  - $\phi = A + B \in L_1 = \Omega^1(M) \otimes \tilde{L}_0 \oplus \Omega^2(M) \otimes \tilde{L}_{-1}$
  - EOMs: $$\mathcal{F} = \mathrm{d}A + \tfrac{1}{2}\mu_2(A, A) - \mu_1(B) = 0$$ $$\mathcal{H} = \mathrm{d}B + \mu_2(A, B) + \tfrac{1}{3!}\mu_3(A, A, A) = 0$$

Local Higher Gauge Theories

The most interesting higher gauge theories for us live in 6 and 4 dimensions.

- "Fake curvature": $\mathcal{F} = \mathrm{d}A + \frac{1}{2}\mu_2(A, A) - \mu_1(B) = 0$
  Vanishing makes parallel transport reparam. invariant.
  Rumour: $\mathcal{F} = 0 \Rightarrow$ theory abelian. This is false!
- 3-form curvature: $\mathcal{H} = \mathrm{d}B + \mu_2(A, B) + \frac{1}{3!}\mu_3(A, A, A) = 0$
  This describes a flat bundle, we can generalize this.

Gauge part of (2,0) theory

If (2,0) theory on $\mathbb{R}^{1,5}$ is a higher gauge theory, then gauge part is:

$$\mathcal{H} = *\mathcal{H}\ , \quad \mathcal{F} = 0\ .$$

Non-Abelian Self-Dual Strings

BPS equation for (2,0) theory on $\mathbb{R}^4$ ($\sim$ monopoles in 4d SYM)

$$\mathcal{H} = *\left(\mathrm{d}\Phi + \mu_2(A, \Phi)\right)\ , \quad \mathcal{F} = 0\ .$$

Later: solutions, categorified $\mathsf{SU}(2)$-Instanton/-monopole

Application:

Categorified IKKT model

Lie 2-algebra models

There is a categorified analogue of the IKKT matrix model.

Recall: IKKT-model arises from regularizing the Schild-action

$$\int \mathrm{d}\sigma^2 \left(-\tfrac{1}{4}\{X_\mu, X_\nu\}^2 + \ldots\right) \quad \Rightarrow \quad \mathrm{tr}\left(-\tfrac{1}{4}\left[X_\mu, X_\nu\right]^2 + \ldots\right)$$

- Possibly background independent form of type IIB
- Quantum geometries emerge as classical solutions
- Expanding around solutions: NC Yang-Mills theory
- Problem: Get only Kähler geometries

Reformulate p-brane actions with Nambu-Poisson brackets

$$\int \mathrm{d}\sigma^{p+1} \left(-\tfrac{1}{4}\{X_{\mu_1}, X_{\mu_2}, \ldots, X_{\mu_{p+1}}\}^2 + \ldots\right)$$

Question: What is the right regularization of this?

Lie 2-algebra models

There is a categorified analogue of the IKKT matrix model.

Answer for $p = 2$: fields on $\mathbb{R}^0$ with values in a Lie 2-algebra.

- Nambu-Poisson struct. induces product on $\wedge^2 \mathcal{C}^\infty(M)$.
- NP struct. $\Rightarrow$ multisymplectic 3-forms $\Rightarrow$ Lie 2-algebra:

$$\mu_1(f) = \mathrm{d}f \ , \quad \iota_{X_\alpha}\varpi = \mathrm{d}\alpha \ , \quad \alpha \in \Omega^1(M) \ ,$$
$$\mu_2(\alpha,\beta) = \iota_{X_\alpha}\iota_{X_\beta}\varpi \ , \quad \mu_3(\alpha,\beta,\gamma) = \iota_{X_\alpha}\iota_{X_\beta}\iota_{X_\gamma}\varpi$$

- Define model $S = -\frac{1}{4}\langle \mu_2(X_\mu, X_\nu), \mu_2(X_\mu, X_\nu)\rangle + \dots$
- IKKT model is a special case
- More general quantum spaces ($S^3_\hbar$, $\mathbb{R}^3_\hbar$, ...) as solutions

P Ritter & CS, 1308.4892

Categorification of Principal Bundles

Restricting to $\mathsf{Alt} = \mathsf{Jac} = \mathrm{id}$ in a weak Lie 2-algebra yields:

Differential Lie crossed modules / Lie crossed modules

Pair of Lie algebras $(\mathfrak{g}, \mathfrak{h})$, written as $(\mathfrak{h} \xrightarrow{\mathsf{t}} \mathfrak{g})$ with:

- left automorphism action $\triangleright : \mathfrak{g} \times \mathfrak{h} \to \mathfrak{h}$
- group homomorphism $\mathsf{t} : \mathfrak{h} \to \mathfrak{g}$

$$\mathsf{t}(g \triangleright h) = [g, \mathsf{t}(h)] \quad \text{and} \quad \mathsf{t}(h_1) \triangleright h_2 = [h_1, h_2]$$

- Finite version: Lie crossed module $(\mathsf{H} \xrightarrow{\mathsf{t}} \mathsf{G})$

Simplest examples:

- Lie group G, Lie crossed module: $(1 \xrightarrow{\mathsf{t}} \mathsf{G})$.
- Abelian Lie group G, Lie crossed module: $\mathsf{BG} = (\mathsf{G} \xrightarrow{\mathsf{t}} 1)$.

More involved:

- Automorphism 2-group of Lie group G: $(\mathsf{G} \xrightarrow{\mathsf{t}} \mathsf{Aut}(\mathsf{G}))$

Categorical Description of Principal Bundles

Descent data for principal bundles is encoded in a functor.

The cover $\sqcup_a U_a$ of a manifold M encoded in the Čech groupoid:

$$\check{\mathscr{C}}(U) \;:\; \bigsqcup_{a,b} U_{ab} \rightrightarrows \bigsqcup_a U_a \;, \quad U_{ab} \circ U_{bc} = U_{ac} \;.$$

Principal G-bundle

Transition functions are nothing but a functor $g : \check{\mathscr{C}}(U) \to \mathsf{BG}$

$$\begin{array}{ccc} \sqcup\, U_{ab} & \xrightarrow{\;g_{ab}\;} & \mathsf{G} \\ \downarrow\downarrow & & \downarrow\downarrow \\ \sqcup\, U_a & \xrightarrow{\;*\;} & * \end{array} \qquad\qquad g_{ab} g_{bc} = g_{ac}$$

Equivalence relations: natural isomorphisms.

Similarly: Higher bundles (involves n-categories)

Principal 2-Bundles with Connective Structures

Higher gauge theory is the dynamical theory of principal 2-bundles.

Object	Principal G-bundle	Principal $(\mathsf{H} \xrightarrow{\mathsf{t}} \mathsf{G})$-bundle
Cochains	(g_{ab}) valued in G	(g_{ab}) valued in G, (h_{abc}) valued in H
Cocycle	$g_{ab}g_{bc} = g_{ac}$	$\mathsf{t}(h_{abc})g_{ab}g_{bc} = g_{ac}$ $h_{acd}h_{abc} = h_{abd}(g_{ab} \rhd h_{bcd})$
Coboundary	$g_a g'_{ab} = g_{ab}g_b$	$g_a g'_{ab} = \mathsf{t}(h_{ab})g_{ab}g_b$ $h_{ac}h_{abc} = (g_a \rhd h'_{abc})h_{ab}(g_{ab} \rhd h_{bc})$
gauge pot.	$A_a \in \Omega^1(U_a) \otimes \mathfrak{g}$	$A_a \in \Omega^1(U_a) \otimes \mathfrak{g}$, $B_a \in \Omega^2(U_a) \otimes \mathfrak{h}$
Curvature	$F_a = \mathrm{d}A_a + A_a \wedge A_a$	$\mathcal{F}_a = \mathrm{d}A_a + A_a \wedge A_a - \mathsf{t}(B_a) \overset{!}{=} 0$ $\mathcal{H}_a = \mathrm{d}B_a + A_a \rhd B_a$
Gauge trafos	$\tilde{A}_a := g_a^{-1}A_a g_a + g_a^{-1}\mathrm{d}g_a$	$\tilde{A}_a := g_a^{-1}A_a g_a + g_a^{-1}\mathrm{d}g_a + \mathsf{t}(\Lambda_a)$ $\tilde{B}_a := g_a^{-1} \rhd B_a + \tilde{A}_a \rhd \Lambda_a + \mathrm{d}\Lambda_a - \Lambda_a \wedge \Lambda_a$

Remarks:

- A principal $(1 \xrightarrow{\mathsf{t}} \mathsf{G})$-bundle is a principal G-bundle.
- A principal $(\mathsf{U}(1) \xrightarrow{\mathsf{t}} 1) = \mathsf{BU}(1)$-bundle is an abelian gerbe.
- Gauge part of (2,0) theory even clear for general space.

Application:

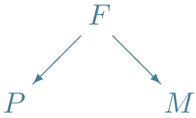
Constructing Superconformal (2,0) Theories using Twistor Spaces

Twistor Description of Higher Yang-Mills Fields

Using twistor spaces, one can map holomorphic data to solutions to field equations.

Recall the principle of the Penrose-Ward transform:

- We construct a double fibration



  P: twistor space, F: correspondence space

- $H^n(P, \mathfrak{S})$ (e.g. vector bundles) $\overset{1:1}{\longleftrightarrow}$ sols. to field equations.
- Our new contributions:
  - Use non-abelian gerbes
  - New twistor space
- Can describe in this way:
  - 6d (2,0) superconformal equations of motion
  - self-dual strings

Known Examples of Twistor Descriptions

For Yang-Mills theories and its BPS subsectors, there is a wealth of twistor descriptions.

$$\mathbb{C}P^3_\circ \leftarrow \mathbb{C}^4 \times \mathbb{C}P^1 \rightarrow \mathbb{C}^4$$

Instantons
hol. vector bundle

$$T\mathbb{C}P^1 \leftarrow \mathbb{C}^3 \times \mathbb{C}P^1 \rightarrow \mathbb{C}^3$$

Monopoles
hol. vector bundle

$$P^{5|6} \leftarrow \mathbb{C}^{4|12} \times \mathbb{C}P^1 \times \mathbb{C}P^1 \rightarrow \mathbb{C}^{4|12}$$

Super Yang-Mills
hol. vector bundle

$$P^6 \leftarrow \mathbb{C}^6 \times \mathbb{C}P^3 \rightarrow \mathbb{C}^6$$

abelian $\mathcal{H} = *\mathcal{H}$
hol. gerbe

Hughston, Murray, Eastwood, CS & M Wolf, Mason et al.

Note: last twistor space reduces nicely to the above ones.

New Results

New: Penrose-Ward transform for self-dual tensor multiplet.

$$\begin{array}{ccc} & \mathbb{C}^{6|16} \times \mathbb{C}P^3 & \\ \swarrow & & \searrow \\ P^{6|4} & & \mathbb{C}^{6|16} \end{array}$$

non-abelian self-dual tensor multiplet

hol. principal 2-bundle

hol. principal 3-bundle

CS & M Wolf, 1205.3108, 1305.4870

Note:

- $P^{6|4}$ is a straightforward SUSY generalization of P^6
- EOMs, abelian: $\mathcal{H} = \star\mathcal{H}$, $\mathcal{F} = 0$, $\nabla\!\!\!/\,\psi = 0$, $\Box\phi = 0$
- $\mathcal{N} = (2,0)$ SC non-abelian tensor multiplet EOMs!
- EOMs on superspace, remain to be boiled down (expected).

New twistor space parameterizing hyperplanes in $\mathbb{C}^4$:

$$\begin{array}{ccc} & \mathbb{C}^4 \times \mathbb{C}P^1 \times \mathbb{C}P^1 & \\ \swarrow & & \searrow \\ P^3 & & \mathbb{C}^4 \end{array}$$

self-dual strings
hol. principal 2-bundle
hol. principal 3-bundle

CS & M Wolf, 1111.2539, 1205.3108, 1305.4870

Note:

- The hyperplane twistor space P^3 is the total space of the line bundle $\mathcal{O}(1,1) \to \mathbb{C}P^1 \times \mathbb{C}P^1$.
- The spheres $\mathbb{C}P^1 \times \mathbb{C}P^1$ parameterize an α- and a β-plane.
- The span of both is a hyperplane.
- Nonabelian self-dual string equations: $\mathcal{H} = *\mathrm{d}_A\Phi$, $\mathcal{F} = 0$.
- Reduces nicely to the monopole twistor space: $\mathcal{O}(2) \to \mathbb{C}P^1$.

Problem

No explicit examples of solutions arising from the previous constructions have been found.

Further extensions to generalize the group structures:

- Lie 2-crossed modules (strict Lie 3-algebras)
  CS & M Wolf, 1305.4870
- Semistrict gauge groups (using monoidal categories)
  B Jurco, CS & M Wolf, 1403.7185
- ∞-groupoids (using simplicial manifolds, Kan complexes)
  B Jurco, CS & M Wolf, 150?.????
- String-group bundles (using stacky Lie groups)
  CS & G Demessie, 150?.????

Context:

The ABJM Model as a Higher Gauge Theory

The ABJM Model as a Higher Gauge Theory

The ABJM model can be completed to a higher gauge theory.

- Most dualities in string theory between Yang-Mills theories.
- And in M-theory? M2-branes: Chern-Simons-matter theories
  M5-branes: Tensor-multiplet theories
- These can be put on equal footing. S Palmer&CS, 1311.1997

Step 1: The ABJM gauge structures / hermitian 3-Lie algebras

- form differential crossed modules. S Palmer&CS, 1203.5757
- but: $\mathsf{t} = 0$, thus $F = \mathsf{t}(B) = 0$.
- Recall: Lie algebra $\mathfrak{g}$ has inner derivation dcm $\mathfrak{g} \xrightarrow{\mathsf{t}} \mathfrak{g}$
- $\mathfrak{h} \xrightarrow{\mathsf{t}} \mathfrak{g}$ lifted to inner derivation Lie 3-algebra $\mathfrak{h} \xrightarrow{\mathsf{t}} \mathfrak{g} \ltimes \mathfrak{h} \xrightarrow{\mathsf{t}} \mathfrak{g}$

Explicitly:

$$\begin{pmatrix} 0 & \mathfrak{gl}(N,\mathbb{C}) \\ 0 & 0 \end{pmatrix} \xrightarrow{\mathsf{t}} \begin{pmatrix} \mathfrak{u}(N) & \mathfrak{gl}(N,\mathbb{C}) \\ 0 & \mathfrak{u}(N) \end{pmatrix} \xrightarrow{\mathsf{t}} \begin{pmatrix} \mathfrak{u}(N) & 0 \\ 0 & \mathfrak{u}(N) \end{pmatrix}$$

The ABJM model can be completed to a higher gauge theory.

Step 2: Implement the fake curvature conditions

- Here, we are working with a strict Lie 3-algebra.
- Gauge potentials: A, B, C. Curvatures: F, H, G.
- Conditions $\mathcal{F} = F - \mathsf{t}(B) = 0$, $\mathcal{H} = H - \mathsf{t}(C) = 0$
- Action:

$$S_{\rm ABJM} = \int_{\mathbb{R}^{1,2}} \mathrm{tr}\Big(\tfrac{k}{4\pi}\eta\; A \wedge (\mathrm{d}A + \tfrac{1}{3}[A,A]) - \nabla Z^\dagger_A \wedge *\nabla Z^A - *\mathrm{i}\bar{\psi}^A \wedge \nabla\!\!\!\!/\, \psi_A\Big) + V$$

$$S_{\rm HGT} = S_{\rm ABJM} + \int_{\mathbb{R}^{1,2}} \mathrm{tr}\Big(\lambda_1^\dagger \wedge (F - \mathsf{t}(B)) + \lambda_2^\dagger (H - \mathsf{t}(C)) + \lambda_3^\dagger \mathsf{t}(\lambda_2)\Big)$$

- This yields ABJM eoms + fake curvature constraints

Application:

Higher Monopole and Instanton Solutions

The BPST instanton can be conveniently written using quaternions.

Recall the quaternionic form of the elementary instanton on S^4:

Conformal geometry of S^4

Describe S^4 by $\mathbb{H} \cup \{\infty\}$. Coordinates: $x = x^1 + \mathrm{i}x^2 + \mathrm{j}x^3 + \mathrm{k}x^4$.
Conformal transformations:

$$x \mapsto (ax+b)(cx+d)^{-1}\ , \quad a,b,c,d \in \mathbb{H}$$

SU(2)-Instanton:

$$A = \mathrm{im}\left(\frac{\bar{x}\mathrm{d}x}{1+|x|^2}\right) \quad \Rightarrow \quad F = \mathrm{im}\left(\frac{\mathrm{d}\bar{x}\wedge\mathrm{d}x}{(1+|x|^2)^2}\right)$$

SU(2)-Anti-Instanton:

$$A = \mathrm{im}\left(\frac{x\mathrm{d}\bar{x}}{1+|x|^2}\right) \quad \Rightarrow \quad F = \mathrm{im}\left(\frac{\mathrm{d}x\wedge\mathrm{d}\bar{x}}{(1+|x|^2)^2}\right)$$

Belavin et al. 1975, Atiyah 1979

Elementary Solution: The Higher Instanton

The quaternionic form of the BPST instanton solution translates perfectly.

Solution to the higher instanton equations $H = \star H$, $F = \mathsf{t}(B)$:

- Same inner derivation 2-crossed module as for ABJM
- Recall BPST instanton:

$$A = \mathrm{im}\left(\frac{\bar{x}\mathrm{d}x}{1+|x|^2}\right) \quad \Rightarrow \quad F = \mathrm{im}\left(\frac{\mathrm{d}\bar{x}\wedge\mathrm{d}x}{(1+|x|^2)^2}\right)$$

- Solution in coordinates $x = x^M\sigma_M$, $\hat{x} = x^M\bar{\sigma}_M$

$$A = \begin{pmatrix} \frac{\hat{x}\,\mathrm{d}x}{1+|x|^2} & 0 \\ 0 & \frac{\mathrm{d}x\,\hat{x}}{1+|x|^2} \end{pmatrix} \quad B = F + \begin{pmatrix} 0 & \frac{\hat{x}\,\mathrm{d}x\wedge\mathrm{d}\hat{x}}{(1+|x|^2)^2} \\ 0 & 0 \end{pmatrix}$$

$$F := \mathrm{d}A + A\wedge A = \begin{pmatrix} \frac{\mathrm{d}\hat{x}\wedge\mathrm{d}x}{(1+|x|^2)^2} + \frac{2\,\mathrm{d}\hat{x}\,x\wedge\mathrm{d}\hat{x}\,x}{(1+|x|^2)^2} & 0 \\ 0 & -\frac{\mathrm{d}x\wedge\mathrm{d}\hat{x}}{(1+|x|^2)^2} \end{pmatrix}$$

$$H := \mathrm{d}B + A \rhd B = \begin{pmatrix} 0 & \frac{\mathrm{d}\hat{x}\wedge\mathrm{d}x\wedge\mathrm{d}\hat{x}}{(1+|x|^2)^3} \\ 0 & 0 \end{pmatrix}$$

Review: The 't Hooft-Polyakov Monopole

The 't Hooft-Polyakov Monopole is a non-singular solution with charge 1.

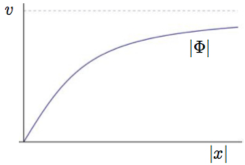
Recall 't Hooft-Polyakov monopole (e_i generate $\mathfrak{su}(2)$, $\xi = v|x|$):

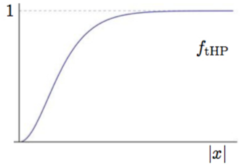
$$\Phi = \frac{e_i x^i}{|x|^2}(\xi \coth(\xi) - 1)\ , \quad A = \varepsilon_{ijk}\frac{e_i x^j}{|x|^2}\left(1 - \frac{\xi}{\sinh(\xi)}\right)\ \mathrm{d}x^k$$

- At S_2^∞: $\Phi \sim g(\theta) e_3 g(\theta)^1$.
  $g(\theta) : S^2_\infty \to \mathsf{SU}(2)/\mathsf{U}(1)$: winding 1
- Charge $q = 1$ with

$$2\pi q = \tfrac{1}{2}\int_{S^2_\infty} \frac{\mathrm{tr}\,(F^\dagger \Phi)}{||\Phi||} \quad \text{with} \quad ||\Phi|| := \sqrt{\tfrac{1}{2}\,\mathrm{tr}\,(\Phi^\dagger\Phi)}$$

- Higgs field non-singular:





We can write down a non-abelian self-dual string with winding number 1.

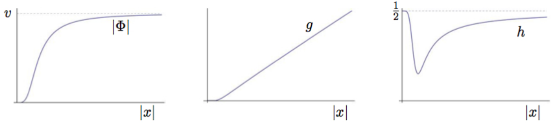
Self-Dual String (e_μ generate DCM $\mathfrak{su}(2)\times\mathfrak{su}(2)\xrightarrow{\mathsf{t}}\mathbb{R}^4$, $\xi = v|x|^2$):

$$\Phi = \frac{e_\mu x^\mu}{|x|^3}\, f(\xi)\ , \quad B_{\mu\nu} = \varepsilon_{\mu\nu\kappa\lambda}\frac{e_\kappa x^\lambda}{|x|^3}\, g(\xi)\ , \quad A_\mu = \varepsilon_{\mu\nu\kappa\lambda} D(e_\nu, e_\kappa)\,\frac{x^\lambda}{|x|^2}\, h(\xi)$$

- At S_3^∞: $\Phi \sim g(\theta) \rhd e_4$. $g(\theta) : S^3_\infty \to \mathsf{SU}(2)$ has winding 1.
- Charge $q = 1$:

$$(2\pi)^3 q = \tfrac{1}{2}\int_{S^3_\infty} \frac{(H,\Phi)}{||\Phi||} \quad \text{with} \quad ||\Phi|| := \sqrt{\tfrac{1}{2}(\Phi,\Phi)}\ ,$$

- Higgs field non-singular:



What I didn't have time to talk about...

There is much more evidence for using higher structures in M-theory.

- 6d (1,0) models from tensor hierarchies
  Samtleben et al., 1108.4060, also 1108.5131
  - (1,0) tensor + vector multiplets with new gauge structure
  - These are higher gauge theories.
  - New gauge structure: symplectic Lie n-algebroids
    S Palmer&CS 1308.2622, Samtleben et al. 1403.7114
- HGT a very nice playground, particularly for PhD students:
  - Higher Magnetic Bags D Harland, S Palmer&CS 1204.6685
  - Proof of Higher Poincaré Lemma G Demessie&CS 1406.5342

Summary:

- ✓ Clear physical and mathematical motivation to study HGT
- ✓ ABJM model is a HGT
- ✓ Various twistor constructions with non-abelian gerbes
- ✓ 6d superconformal tensor multiplet equations
- ✓ Explicit higher monopole and instanton solutions

Future directions:

- ▷ Allow for more general higher groups
- ▷ Continue translation of higher ADHM-constructions
- ▷ Geometric Quant. with higher Hilbert spaces
- ▷ Study categorified IKKT model

Higher Gauge Theory and M-theory

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