EL_∞-algebras, Generalized Geometry, and Tensor Hierarchies

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Based on:

- arXiv:2106.00108 with Leron Borsten and Hyungrok Kim
- arXiv:1908.08086 with Lennart Schmidt

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5) What is a small cofibrant replacement for the operad $\mathcal{L}ie$?

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- Is there more to it?

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In order to further understand the above:
understand symplectic L_∞-algebroids!

Third question

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- Where does $(-,-)$ come from?

Relevance of third question

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Mathematics:

- Higher geometry would be much less beautiful.

Summary

All these questions:

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4) Cofibrant replacement of $\mathcal{L}ie$?
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EL_∞-algebras

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Conclusions

We are looking for a weak generalization of L_∞-algebras, generalizing the 2- and 3-term EL_∞-algebras in the literature.

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- "Homotopy $\mathcal{O}$-algebras or $\mathcal{O}_\infty$-algebra is an algebra over the Koszul resolution of the Koszul-dual cooperad."

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- For $\mathfrak{g} = \oplus_{i \leq 0} \mathfrak{g}_i$: categorified Lie algebras

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- Question: which operad produces weak L_∞-/EL_∞-algebras?

Hemistrict Lie 2-algebras: differential graded algebras $\mathfrak{L}$ with

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such that

$$\varepsilon_1(\varepsilon_1(x_1)) = 0 ,$$

Generalizes hemistrict Lie 2-algs and specializes dg-Leibniz algs.

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Generalize, preserving differential compatibility: $h\mathcal{L}ie$-algebras

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Graded vector space $\mathfrak{L}$ with

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$$h\mathcal{L}ie \xrightarrow{\text{Koszul duality}} \mathcal{E}ilh$$

$$hLie \xrightarrow{\text{Koszul duality}} \mathcal{E}ilh$$

$\mathcal{E}ilh$-algebras

Graded vector space V, tensor products $\oslash_i$, $|\oslash_i| = i$, $i \in \mathbb{N}$,

$$a \oslash_i (b \oslash_j c) = \sum (\dots \oslash_k \dots) \oslash_l \dots \ ,$$

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Differential:

$$Q(a \oslash_i b) = (-1)^i \big((Qa) \oslash_i b + (-1)^{|a|} a \oslash_i Qb\big) + (-1)^i (a \oslash_{i+1} b) - (-1)^{|a|\,|b|} (b \oslash_{i+1} a) \ .$$

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Duality explicitly:

- $hLie$-algebra:

$$\varepsilon_1(\tau_\alpha) = m_\alpha^\beta \tau_\beta \ , \quad \varepsilon_2^i(\tau_\alpha, \tau_\beta) = m_{\alpha\beta}^{i,\gamma} \tau_\gamma$$

- $\mathcal{E}ilh$-algebra:

$$Qt^\alpha = \pm m_\beta^\alpha t^\beta \pm m_{\beta\gamma}^{i,\alpha} t^\beta \oslash_i t^\gamma$$

EL_∞-algebras

EL_∞-algebra are homotopy $h\mathcal{L}ie$-algebras.

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amounting to $Q^2 = 0$ in the corresponding dual $\mathcal{E}ilh$-algebra.

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Note: if $\varepsilon_k^I = 0$ for $I \neq (0,0,\dots,0)$, then this is L_∞-algebra.

Properties of EL_∞-algebras

They generalize:

- (dg) Lie algebras

They specialize:

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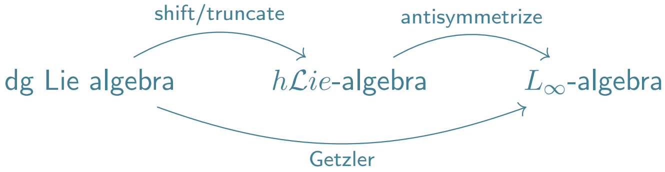
- $\Rightarrow$ They are weak Lie ∞-algebras

Answer to question 1:

What is the algebraic structure underlying Courant algebroids?



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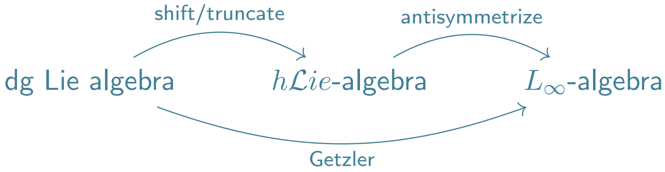

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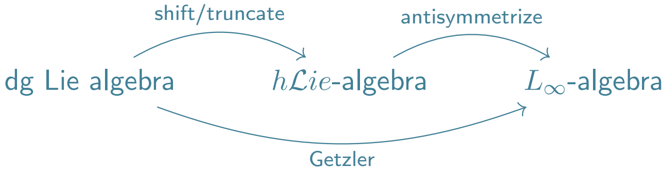
Derived bracket constructions



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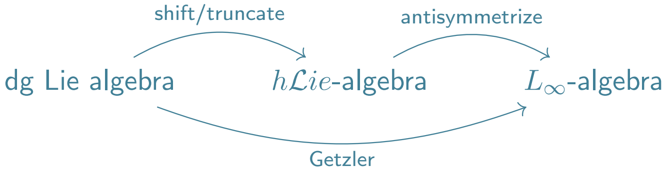

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- Differential graded Lie algebras truncate to $h\mathcal{L}ie$-algebras
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- Note: $h\mathcal{L}ie$-algebras are much easier to handle!

Differential graded Lie algebra Roytenberg (2002):

- Graded manifold $\mathcal{M} := T^*[2]T[1]M$, $\overset{0}{x^\mu}, \overset{1}{\xi^\mu}, \overset{1}{\zeta_\mu}, \overset{2}{p_\mu}$
- $\mathfrak{g} := C^\infty(T^*[2]T[1]M)$, degree is coordinate degree
- Lie bracket: Poisson bracket of $\omega = \mathrm{d}x^\mu \wedge \mathrm{d}p_\mu + \mathrm{d}\xi^\mu \wedge \mathrm{d}\zeta^\mu$
- differential: $Q = \{S, -\}$, $S = \xi^\mu p_\mu + \frac{1}{3!}\varpi_{\mu\nu\kappa}\xi^\mu\xi^\nu\xi^\kappa$

Example: Courant algebroid

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$h\mathcal{L}ie$-algebra:

- $\mathfrak{E} = \mathfrak{E}_{-1} \xrightarrow{\varepsilon_1} \mathfrak{E}_0 = C^\infty_0(\mathcal{M}) \xrightarrow{Q} C^\infty_1(\mathcal{M})$
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Generalizes to all generalized tangent bundles

Answer to question 3

What are "good" curvatures for non-abelian gauge potentials?

All direct categorifications of gauge theory yield the following:

- Higher gauge Lie algebra
  - Two gauge Lie algebras: $\mathfrak{g}$ and $\mathfrak{h}$
  - Morphism $\mu_1 : \mathfrak{h} \to \mathfrak{g}$.
  - Action $\mu_2 : \mathfrak{g} \curvearrowright \mathfrak{h}$

Naive higher curvatures

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- BRST complex is not closed off-shell

  $\Rightarrow$ fake curvature 2-form needs to vanish

  $\Rightarrow$ but then everything becomes locally abelian...

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- Principal bundles + ∇ are not trivially higher bundles

Naive higher curvatures

All direct categorifications of gauge theory yield the following:

- Higher gauge Lie algebra
  - Two gauge Lie algebras: $\mathfrak{g}$ and $\mathfrak{h}$
  - Morphism $\mu_1 : \mathfrak{h} \to \mathfrak{g}$.
  - Action $\mu_2 : \mathfrak{g} \curvearrowright \mathfrak{h}$
- Higher non-abelian gauge potentials
  - $A \in \Omega^1(U, \mathfrak{g})$
  - $B \in \Omega^2(U, \mathfrak{h})$.
- Higher non-abelian curvature forms
  - Fake curvature $\mathrm{d}A + \frac{1}{2}\mu_2(A, A) + \mu_1(B)$
  - 3-form curvature $\mathrm{d}B + \mu_2(A, B)$

Many, many problems with this:

- BRST complex is not closed off-shell

  $\Rightarrow$ fake curvature 2-form needs to vanish

  $\Rightarrow$ but then everything becomes locally abelian...
- Principal bundles + ∇ are not trivially higher bundles
- Does not match mathematical or physical expectations

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Curvatures:

$$F := \mathrm{d}A + \tfrac{1}{2}[A, A]$$

$$H := \mathrm{d}B - \tfrac{1}{3!}\mu_3(A, A, A) + (A, F)$$

$$= \mathrm{d}B + \underbrace{(A, \mathrm{d}A) + \tfrac{1}{3}(A, [A, A])}_{\mathrm{cs}(A)}$$

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$$\mathrm{d}F + [A, F] = 0 \ , \quad \mathrm{d}H - (F, F) = 0$$

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Gauge transformations:

$$\delta A = \mathrm{d}\Lambda_0 + \mu_2(A, \Lambda_0) \qquad \delta F = -\mu_2(F, \Lambda_0)$$

$$\delta B = \mathrm{d}\Lambda_1 + (\Lambda_0, F) - \tfrac{1}{2}\mu_3(A, A, \Lambda_0) \qquad \delta H = 0$$

Observations

Evident question:

Where do the structure constants for adjustment come from?

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$$\varepsilon_2(x_1, x_2) = [x_1, x_2] \ , \quad \varepsilon_2(x_1, r) = 0 \ ,$$
$$\varepsilon_3(x_1, x_2, x_3) = (1-\alpha)(x_1, [x_2, x_3]) \ ,$$
$$\mathsf{alt}(x_1, x_2) = -2\alpha(x_1, x_2)$$

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Conjecture:

Adjustment data from alternators in weak Lie n-algebras

Theorem

Any EL_∞-algebra obtained by shift/truncation from a differential graded Lie algebra admits a mathematically natural adjustment of the definition of the resulting curvatures.

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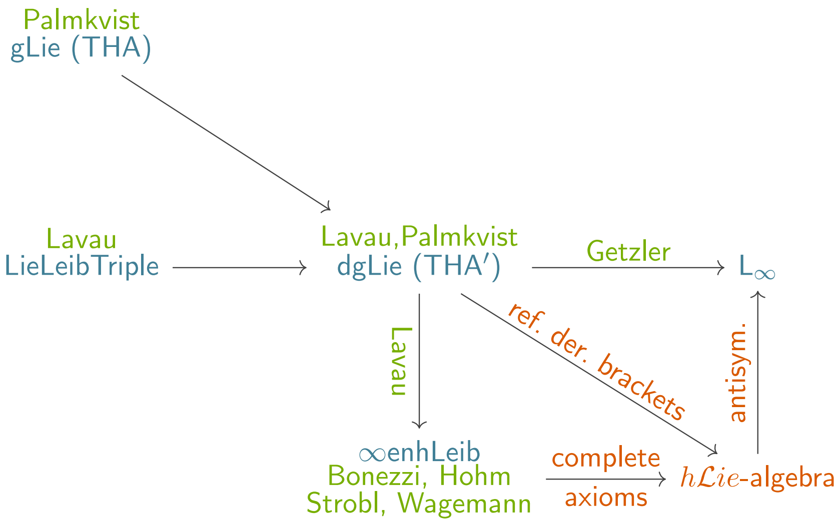
- Tensor hierarchies from dgLA $\Rightarrow$ adjusted higher gauge theory

Example: Tensor hierarchies



graded Lie algebra/tensor hierarchy algebras (reps. of $\mathfrak{e}_{6(6)}$)

$V_{\mathfrak{e}_{6(6)}}$	$=$	V_{-5}	$\oplus$	V_{-4}	$\oplus$	V_{-3}	$\oplus$	V_{-2}	$\oplus$	V_{-1}	$\oplus$	V_0	$\oplus$	V_1
$\rho_{(k)}$		$\mathbf{27}\oplus\mathbf{1728}$		$\mathbf{351}_c$		$\mathbf{78}$		$\mathbf{27}$		$\mathbf{27}_c$		$\mathbf{78}$		$\mathbf{351}$

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- L_∞-algebra + extra structure, cf. Palmer, CS (2013)

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- Lift above constructions to true/non-strict EL_∞-algebras?

Thank You!