The elementary non-abelian self-dual string and the (2,0)-theory

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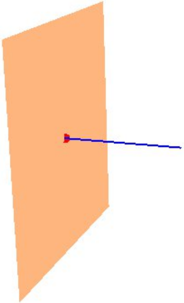
Based on:

- CS & L Schmidt, arXiv:1705.????

Motivation: The Dynamics of Multiple M5-Branes

To understand M-theory, an effective description of M5-branes would be very useful.

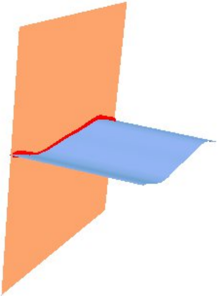


D-branes

- D-branes interact via strings.
- Effective description: theory of endpoints
- Parallel transport of these: Gauge theory
- Study string theory via gauge theory

M5-branes

- M5-branes interact via M2-branes.
- Eff. description: theory of self-dual strings
- Parallel transport: Higher gauge theory
- Long sought $(2,0)$-theory a HGT?



What we know about the (2,0)-theory

Multiple M5-branes are described by a $\mathcal{N} = (2,0)$ superconformal field theory.

What we know about 6d $\mathcal{N} = (2,0)$ SCFT:

- String theory considerations: conformal fixed point in 6d
  Witten, Strominger 1995
- Field content: $\mathcal{N} = (2,0)$ supermultiplet in 6d:
  - a self-dual 3-form field strength
  - five (Goldstone) scalars
  - fermionic partners
- A theory of essentially tensionless light strings
- Supergravity decouples, so study string dynamics separately
- Observables: Wilson surfaces, i.e. parallel transport of strings
- No Lagrangian description known
- As important as $\mathcal{N} = 4$ super Yang-Mills for string theory
- Huge interest in string theory: AGT, AdS$_7$-CFT$_6$, S-duality, ...
- Mathematics: Geom. Langlands, Khovanov Homology, ...

Parallel Transport of Strings is Problematic

Parallel transport of particles in representation of gauge group G:

- holonomy functor hol : path $\gamma \mapsto \mathrm{hol}(\gamma) \in \mathsf{G}$
- $\mathrm{hol}(\gamma) = P\exp(\int_\gamma A)$, P: path ordering, trivial for $\mathsf{U}(1)$.

Parallel transport of strings with gauge group $\mathsf{U}(1)$:

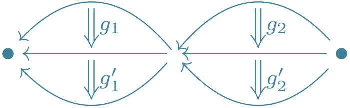
- map hol : surface $\sigma \mapsto \mathrm{hol}(\sigma) \in \mathsf{U}(1)$
- $\mathrm{hol}(\sigma) = \exp(\int_\sigma B)$, B: connective structure on gerbe.

Nonabelian case:

- much more involved!
- no straightforward definition of surface ordering

Naïve no-go theorem

Imagine parallel transport of string with gauge degrees in Lie(G):



Consistency of parallel transport requires:

$$(g_1'g_2')(g_1g_2) = (g_1'g_1)(g_2'g_2)$$

This renders group G abelian.

Eckmann and Hilton, 1962
Physicists 80'ies and 90'ies

Way out: 2-categories, Higher Gauge Theory.

Two operations $\circ$ and $\otimes$ satisfying Interchange Law:

$$(g_1' \otimes g_2') \circ (g_1 \otimes g_2) = (g_1' \circ g_1) \otimes (g_2' \circ g_2)\ .$$

Objection to a classical (2,0)-theory

Standard objection beyond the previous no-go theorem:

- theory at conformal fixed points ⇒ no dimensionful parameter
- fixed points are isolated ⇒ no dimensionless parameter
- "No parameters ⇒ no classical limit ⇒ no Lagrangian."

Answers:

- Same arguments for M2-brane Schwarz, 2004
- There, integer parameters arose from orbifold $\mathbb{R}^8/\mathbb{Z}_k$
- Same should happen for M5-branes
- Even if no Lagrangian, BPS-states may exist classically ⇒ "self-dual strings"
- Even if not, study quantum features of related theories.

To formulate M5-brane theory: Need category theory

Some quotes:

- “We will need to use some very simple notions of category theory, an esoteric subject noted for its difficulty and irrelevance.”

  G. Moore and N. Seiberg, 1989
- “We’ll only use as much category theory as is necessary. Famous last words...”

  Roman Abramovich
- “Category theory is the subject where you can leave the definitions as exercises.”

  John Baez

Reminder: Category Theory

Example:

$$\begin{tikzcd}
0 \arrow[loop above, "\mathrm{id}_0"] \arrow[r] \arrow[rr, bend right] & 1 \arrow[loop above, "\mathrm{id}_1"] \arrow[r] & 2 \arrow[loop above, "\mathrm{id}_2"]
\end{tikzcd}$$

Category $\mathscr{C}$:

- Classes of Objects $\mathscr{C}_0 = \{0, 1, 2\}$ and Arrows $\mathscr{C}_1$: $\mathscr{C}_1 \rightrightarrows \mathscr{C}_0$
- Associative composition always possible: $$\begin{tikzcd}
a & b \arrow[l, bend right, "f"'] & c \arrow[l, bend right, "g"'] \arrow[ll, bend left, "h=f\circ g"]
\end{tikzcd}$$
- Each object has identity morphism: $$\begin{tikzcd}
a \arrow[loop above, "\mathrm{id}_\mathrm{a}"]
\end{tikzcd}, \quad \begin{array}{l} f \circ \mathrm{id}_b = f \\ \mathrm{id}_b \circ g = g \end{array}$$

Examples:

- set S: $S \rightrightarrows S$
- Set: Set_0: sets, Set_1: functions between sets
- Vect: Vect_0: vector spaces, Vect_1: linear maps between them
- Mfd^∞: Mfd^∞_0: smooth manifolds, Mfd^∞_1: smooth maps

Reminder: Categorification via Internalization

Internalization in a category $\mathscr{C}$

- Math. notion as objects, arrows, and commutative diagrams.
- Restrict objects/arrows to those of category $\mathscr{C}$.

Group G as morphisms:

$$\mathbb{1} : * \longrightarrow \mathsf{G}\ , \quad (-)^{-1} : \mathsf{G} \longrightarrow \mathsf{G}\ , \quad \mathsf{m} : \mathsf{G} \times \mathsf{G} \longrightarrow \mathsf{G}$$

such that identities hold:

$$\begin{tikzcd}
\mathsf{G} \arrow[r, "(\mathbb{1},\mathrm{id})"] \arrow[d, "(\mathrm{id},\mathbb{1})"'] \arrow[dr, "\mathrm{id}"] & \mathsf{G} \times \mathsf{G} \arrow[d, "\mathsf{m}"] \\
\mathsf{G} \times \mathsf{G} \arrow[r, "\mathsf{m}"'] & \mathsf{G}
\end{tikzcd}$$

, etc.

Examples:

- Group (object) internal to Top: topological group
- Group (object) internal to Mfd^∞: Lie group
- Group (object) internal to Cat: strict 2-group
- Category internal to Mfd^∞: 2-space
- Category internal to Vect: 2-vector space

Categorification

Categorification not unique, but can categorify almost anything:

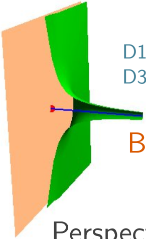
- Categorified groups or 2-groups
- Categorified Lie algebras or Lie 2-algebras
- Categorified principal bundles or principal 2-bundles
- Categorified connections

Issues:

- Very complicated structures packaged in very compact form
- Unpacking of definitions is complicated, requires much work
- Lots and lots of equivalences/isomorphisms
- Usually: Few available examples

self-dual strings (BPS in (2,0)-theory): important examples.

The non-abelian self-dual string



	0	1	2	3	4	5	6
D1	×						×
D3	×	×	×	×			

BPS configuration

Perspective of D3:

Bogomolny monopole eqn.

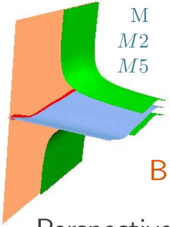
$$F = \nabla^2 = *\nabla\Phi \text{ on } \mathbb{R}^3$$

$\updownarrow$ Nahm transform $\updownarrow$

Perspective of D1:

Nahm eqn.

$$\frac{\mathrm{d}}{\mathrm{d}x^6} X^i + \varepsilon^{ijk}[X^j, X^k] = 0$$



M	0	1	2	3	4	5	6
$M2$	×					×	×
$M5$	×	×	×	×	×	×	

BPS configuration

Perspective of M5:

Abelian Self-dual string eqn.

$$H := \mathrm{d}B = *\mathrm{d}\Phi \text{ on } \mathbb{R}^4$$

$\updownarrow$ genlzd. Nahm transform (?) $\updownarrow$

Perspective of M2:

Hoppe-Basu-Harvey eqn. (??)

$$\frac{\mathrm{d}}{\mathrm{d}x^6} X^\mu + \varepsilon^{\mu\nu\rho\sigma}[X^\nu, X^\rho, X^\sigma] = 0$$

Problem: What is a non-abelian self-dual string?

Recall:

- Abelian Dirac Monopole: singular on $\mathbb{R}^3$
- Non-abelian 't Hooft–Polyakov Monopole: non-singular on $\mathbb{R}^3$
- Abelian Self-Dual String: singular on $\mathbb{R}^4$

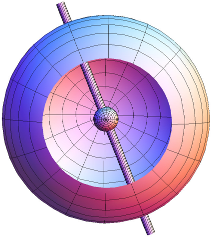
Goal:

- Non-abelian self-dual string with non-singular solution on $\mathbb{R}^4$

Steps:

- Identity gauge structure
- Identify equations of motion
- Find at least elementary (charge 1) solution

Identifying gauge structure: Monopoles



Monopoles

- Solution to Bogomolny equation $F = *\nabla\phi$
- Abelian: singular on $\mathbb{R}^3$, Dirac strings
- Principal bundle over S^2
- Non-Abelian: non-singular on $\mathbb{R}^3$

$$\begin{tikzcd}
\mathsf{U}(1) \arrow[r, hook] \arrow[rrr, bend left, dashed, "\rho"] & \mathsf{SU}(2) \cong S^3 \arrow[d, "\pi"] \arrow[rrr, bend left, dashed, "\pi\times\mathrm{id}"] & & \mathsf{SU}(2) \arrow[r, hook] & S^2 \times \mathsf{SU}(2) \arrow[d, "\mathrm{pr}"] \\
& S^2 \arrow[rrr, bend left, dashed, "\mathrm{id}"] & & & S^2
\end{tikzcd}$$

abelian, Dirac

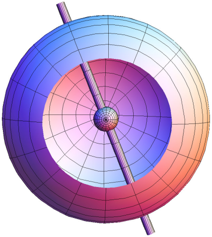
non-Abelian, 't Hooft-Polyakov

$\Rightarrow$ Choose $\mathsf{SU}(2)$, as trivialization possible.

Identifying gauge structure: Self-Dual Strings

Self-Dual Strings

- Abelian: singular on $\mathbb{R}^4$, Dirac strings
- Solution to $H = *\nabla\phi$
- Gerbe over S^3
- Non-Abelian: ?



$$\begin{tikzcd}
\mathrm{BU}(1) \arrow[r, hook] \arrow[rr, dashed, bend left, "\rho"] & \mathscr{G}_{\mathrm{F}} \arrow[d, "\pi"] \arrow[rr, dashed, bend left, "\pi\times\mathrm{id}"] & \mathscr{G}_{\mathrm{F}} \arrow[r, hook] & (S^3 \rightrightarrows S^3)\times\mathscr{G}_{\mathrm{F}} \arrow[d, "\mathrm{pr}"] \\
& (S^3 \rightrightarrows S^3) \arrow[rr, dashed, bend left, "\mathrm{id}"] & & (S^3 \rightrightarrows S^3)
\end{tikzcd}$$

abelian

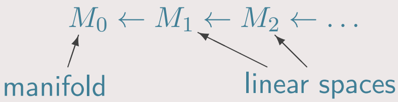
non-Abelian ?

$\Rightarrow$ Choose $\mathscr{G}_F$, with 2-group structure: String 2-group
(many other reasons for this)

NQ-Manifolds

N-manifolds, NQ-manifold

- $\mathbb{N}_0$-graded manifold with coordinates of degree $0, 1, 2, \ldots$



$$M_0 \leftarrow M_1 \leftarrow M_2 \leftarrow \ldots$$

- NQ-manifold: vector field Q of degree 1, $Q^2 = 0$
- Physicists: think ghost numbers, BRST charge, SFT

Examples:

- Tangent algebroid $T[1]M$, $\mathcal{C}^\infty(T[1]M) \cong \Omega^\bullet(M)$, $Q = \mathrm{d}$
- Lie algebra $\mathfrak{g}[1]$, coordinates ξ^a of degree 1:

$$Q = -\tfrac{1}{2} f^c_{ab} \xi^a \xi^b \frac{\partial}{\partial \xi^c} \quad , \qquad \text{Jacobi identity} \Leftrightarrow \; Q^2 = 0$$

cf. most talks yesterday

L_∞-Algebras, Lie 2-Algebras

Lie n-algebroid or n-term L_∞-algebroid:

$$M_0 \leftarrow M_1 \leftarrow M_2 \leftarrow \ldots \leftarrow M_n \leftarrow * \leftarrow * \leftarrow \ldots$$

Lie n-algebra or n-term L_∞-algebra:

$$* \leftarrow M_1 \leftarrow M_2 \leftarrow \ldots \leftarrow M_n \leftarrow * \leftarrow * \leftarrow \ldots$$

Important example: Lie 2-algebra

- Graded vector space: $W[1] \leftarrow V[2]$, Category: $V \oplus W \rightrightarrows W$
- Coordinates: w^a of degree 1 on $W[1]$, v^i of degree 2 on $V[2]$
- Most general vector field Q of degree 1:

$$Q = -m_i^a v^i \frac{\partial}{\partial w^a} - \tfrac{1}{2} m_{ab}^c w^a w^b \frac{\partial}{\partial w^c} - m_{ai}^j w^a v^i \frac{\partial}{\partial v^j} - \tfrac{1}{3!} m_{abc}^i w^a w^b w^c \frac{\partial}{\partial v^i}$$

- Induces "brackets"/"higher products":

$$\mu_1(\tau_i) = m_i^a \tau_a \ , \quad \mu_2(\tau_a, \tau_b) = m_{ab}^c \tau_c \ , \quad \ldots \ , \quad \mu_3(\tau_a, \tau_b, \tau_c) = m_{abc}^i \tau_i$$

- $Q^2 = 0 \Leftrightarrow$ Homotopy Jacobi identities, e.g. $\mu_1(\mu_1(-)) = 0$
- Failure of Jacobi identity: $\mu_2(x, \mu_2(y, z)) + \ldots = \mu_1(\mu_3(x, y, z))$

String Lie 2-algebra

- Recall: $\mathscr{G}_F$ can be extended to String 2-group model
- Lie differentiate (e.g. Demessie, CS, cf. Jan's talk)
- Result: String Lie 2-algebra $\mathfrak{string}(3) = \mathbb{R}[1] \rightarrow \mathfrak{su}(2)$ with

$$Q\xi^\alpha = -\tfrac{1}{2} f^\alpha_{\beta\gamma}\xi^\beta\xi^\gamma~, \quad Qb = -\tfrac{1}{3!} f_{\alpha\beta\gamma}\xi^\alpha\xi^\beta\xi^\gamma~.$$

or

$$\mu_2(x_1,x_2) = [x_1,x_2]~, \quad \mu_3(x_1,x_2,x_3) = (x_1,[x_2,x_3])$$

Gauge Theory from NQ-manifolds

- Ideas: Atiyah, Strobl et al., Sati, Schreiber, Stasheff
- Recall: Chevalley-Eilenberg algebra of Lie algebra $\mathfrak{g}$:

$$\mathrm{CE}(\mathfrak{g}) = \mathcal{C}^\infty(\mathfrak{g}[1])\ , \quad Q\xi^\alpha = -\tfrac{1}{2} f^\alpha_{\beta\gamma}\xi^\beta\xi^\gamma$$

- Double to Weil algebra

$$\mathrm{W}(\mathfrak{g}) := \mathcal{C}^\infty(T[1]\mathfrak{g}[1])\ , \quad Q = Q_{\mathrm{CE}} + \sigma\ , \quad \sigma Q_{\mathrm{CE}} = -Q_{\mathrm{CE}}\sigma$$

- Potentials/curvatures/Bianchi identities from dga-morphisms

$$\begin{gathered}(A,F) : \mathrm{W}(\mathfrak{g}) \to \Omega^\bullet(M) = W(M)\\ \xi^\alpha \mapsto A^\alpha\\ (\sigma\xi^\alpha) = Q\xi^\alpha + \tfrac{1}{2} f^\alpha_{\beta\gamma}\xi^\beta\xi^\gamma \mapsto F^\alpha = (\mathrm{d}A + \tfrac{1}{2}[A,A])^\alpha\\ Q(\sigma\xi^\alpha) = -f^\alpha_{\beta\gamma}(\sigma\xi^\alpha)\xi^\beta \mapsto (\nabla F)^\alpha = 0\end{gathered}$$

- Gauge transformations: homotopies between dga-morphisms
- Topological invariants: invariant polynomials in $\mathrm{W}(\mathfrak{g})$

Derive equations of motion

Field Content with values in $\mathfrak{string}(3)$:

$$A \in \Omega^1(\mathbb{R}^4) \otimes \mathfrak{su}(2)\ , \quad B \in \Omega^2(\mathbb{R}^4) \otimes \mathbb{R}\ ,$$
$$\phi \in \Omega^0(\mathbb{R}^4) \otimes \mathbb{R}$$

Equations of motion: CS, Schmidt (2017)

- Straightforward category theoretical discussions yields:

$$H = \mathrm{d}B + \tfrac{1}{3!}(A, [A, A]) = *\nabla\phi\ , \quad F = \mathrm{d}A + \tfrac{1}{2}[A, A] = 0\ .$$

- Only abelian solutions, gauge invariance? ⇒ Drop $F = 0$
- "Bianchi identity:"

$$\mathrm{d}H = \tfrac{1}{3!}\mathrm{d}(A, [A, A]) = *\Box\phi\ .$$

- Anomaly cancellation in string theory suggests:

$$\mathrm{d}H = F \wedge F$$

- Need to modify equations of motion

Derive equations of motion

Field Content with values in $\mathfrak{string}(3)$:

$$A \in \Omega^1(\mathbb{R}^4) \otimes \mathfrak{su}(2)\ , \quad B \in \Omega^2(\mathbb{R}^4) \otimes \mathbb{R}\ ,$$

$$\phi \in \Omega^0(\mathbb{R}^4) \otimes \mathbb{R}$$

Use categorical equivalences:

- SU(N)-bundle: U(N)-bundle + trivialization of U(1)-factor
- String(3)-bundle as Spin(3)-bundle + BBU(1)-factor

  Sati, Schreiber, Stasheff (2009)
- This yields (cf. Jan's talk)

$$H = \mathrm{d}B + \tfrac{1}{2}(A, \mathrm{d}A) + \tfrac{1}{3!}(A, [A, A]) = *\mathrm{d}\phi \ \Rightarrow \ dH = (F, F) = *\Box\phi$$

- ϕ should "know" all about configuration, thus demand also

$$F = \mathrm{d}A + \tfrac{1}{2}[A, A] = *F$$

- Note: Nice reduction of second equation to monopoles on $\mathbb{R}^3$
- First equation requires only little more work, also reduces
- Equations are BPS equations for (1,0)-model (more later)

Bogomolny trick

Bogomolny's trick familiar from instanton/monopoles works, too:

$$S = \int_{\mathbb{R}^4} H \wedge *H + (\mathrm{d}\varphi, *\mathrm{d}\varphi) + (F, *F)$$

cf. Jan's talk

Rewrite:

$$S = \int_{\mathbb{R}^4} (H - *\mathrm{d}\varphi) \wedge *(H - *\mathrm{d}\varphi) + 2H \wedge \mathrm{d}\varphi + \frac{1}{2}\big((F - *F), *(F - *F)\big) + \big(F, F\big)$$

Minimum/Bogomolny bound:

$$H = *\mathrm{d}\varphi \ , \quad F = *F$$

Topological invariants

$$S = 2\int_{\mathbb{R}^4} H \wedge \mathrm{d}\varphi + \int_{\mathbb{R}^4} \big(F, F\big) = 2\int_{S^3_\infty} H \wedge \varphi$$

Solution

Equations of motion:

$$F = \mathrm{d}A + \tfrac{1}{2}[A, A] = *F \quad \text{and} \quad H = \mathrm{d}B + \mathrm{CS}(A) = *\mathrm{d}\phi$$

Solution:

$$A_\mu(x) = \frac{2}{\mathrm{i}} \frac{\eta^i_{\mu\nu}\,\sigma_i\,(x - x_0)^\nu}{\rho^2 + (x - x_0)^2}\;, \quad B = 0\;, \quad \varphi = \frac{2((x - x_0)^2 - \rho^2}{3\big((x - x_0)^2 + \rho^2\big)^2}$$

cf. also Akyol, Papadopoulos 2012

Consistency checks:

- Globally non-singular on $\mathbb{R}^4$
- Approaches abelian solution $\frac{1}{x^2}$ as $x \to \infty$
- Moduli: same as instanton:
  - Position
  - Size (unexpected, conformal symmetry of $F = *F$)
  - Residual global SU(2) gauge symmetry

Extension from BPS to (1,0)-theory

BPS states work, what about classical (2,0)-theory?

Recall:

- 6d (1,0)-model derived from tensor hierarchies
  Samtleben, Sezgin, Wimmer (2011)
- Issue 1: Choice of gauge structure unclear
- Issue 2: cubic interactions
- Issue 3: scalar fields with wrong sign kinetic term

Previous observation:

- Gauge structure has underlying Lie n-algebra
  Palmer, CS (2013), Samtleben et al. (2014)
  Schmidt, CS (2017)

New:

- $\mathfrak{string}(3)$ can be "doubled" to carry suitable inner product:

$$\mathfrak{su}(2) \leftarrow \mathbb{R} \quad \rightarrow \quad \mathfrak{su}(2) \oplus \mathbb{R} \leftarrow \mathbb{R}^2 \leftarrow \mathbb{R} \oplus \mathfrak{su}(2)^*$$

- Suitable gauge structure for (1,0)-model
- Solves Issues 1 and 2.

Properties of resulting (1,0)-theory

Field content:

- (1,0) tensor multiplet (ϕ, χ^i, B), values in $\mathbb{R}^2$, $\phi = \phi_s + \phi_r$, …
- (1,0) vector multiplet (A, λ^i, Y^{ij}), values in $\mathfrak{su}(2) \oplus \mathbb{R}$
- C-field, values in $\mathbb{R} \oplus \mathfrak{su}(2)^*$

Action (schematically):

$$S = \int_{\mathbb{R}^{1,5}} \Big(\mathcal{H} \wedge *\mathcal{H} + \mathrm{d}\phi \wedge *\mathrm{d}\phi - *\langle \bar{\chi}, \not{\partial}\chi \rangle + \mathcal{H}_s \wedge *\langle \bar{\lambda}, \gamma_{(3)}\lambda \rangle + *\langle Y, \bar{\lambda} \rangle \chi_s \\ + \phi_s \big((\mathcal{F}, *\mathcal{F}) - *(Y, Y) + *(\bar{\lambda}, \not{\nabla}\lambda) \big) + \langle \bar{\lambda}, \mathcal{F} \rangle \wedge *\gamma_{(2)}\chi_s \\ + \mu_1(C) \wedge \mathcal{H}_s + B_s \wedge (\mathcal{F}, \mathcal{F}) + B_s \wedge ([A, A], [A, A]) \Big)$$

Reduction to 4d SYM (Crucial test):

- Compactify along x^{10}, x^9 interpret as $\langle \phi_s \rangle = \frac{1}{\pi^2 R_{10} R_9} = \frac{1}{\pi^2 R_{10}^2}$
- Strong coupling expansion around $\langle \phi_s \rangle$ (cf. M2 → D2)
- This yields 4d $\mathcal{N} = 2$ SYM with gauge group $\mathsf{SU}(2)$!

Conclusions

Summary and Outlook.

Summary:

- Categorified structures appear naturally in M-theory
- Higher analogue of $\mathsf{SU}(2)$ is $\mathsf{String}(3)$
- There is a non-abelian self-dual string
- Better understanding of (1,0)-theory
- Dimensional reduction to $\mathcal{N} = 4$ SYM works!

Soon to come:

- ▷ Interpretation of fuzzy S^3
- ▷ Categorified Integrability
- ▷ Access new superconformal field theories
- ▷ Better understanding of M-theory

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