Quantization of 2-plectic Manifolds

Christian Sämann

School of Mathematical and Computer Sciences
Heriot Watt University, Edinburgh

Bayrischzell Workshop, 22.5.2011

Based on:

- Josh DeBellis, CS and Richard Szabo
  arXiv:1001.3275 (JMP), arXiv:1012.2236 (JHEP)
- CS and Richard Szabo, in preparation

Two-plectic Manifolds

Multisymplectic manifolds are a natural generalization of symplectic manifolds.

Symplectic manifolds

Manifold M with closed 2-form ω such that $\iota_v\omega = 0 \Leftrightarrow v = 0$.

- Poisson structure $\rightarrow$ Phase spaces in Hamiltonian dynamics.
- Starting point for quantization.

p-plectic manifolds

Manifold M with closed $p+1$-form ω such that $\iota_v\omega = 0 \Leftrightarrow v = 0$.

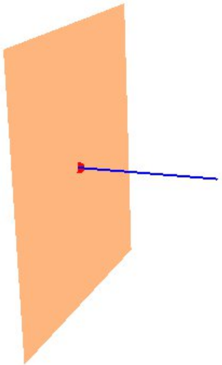
- 1-plectic: symplectic, 2-plectic: 3-form ω
- (Often) Nambu-Poisson structure $\rightarrow$ multiphase spaces in Nambu mechanics.
- Starting point for higher quantization (?)

- Why should we be interested in such manifolds?
- Why should we quantize them?

D1-D3-Branes and the Nahm Equation

D1-branes ending on D3-branes can be described by the Nahm equation.

dim	0	1	2	3	...	6
$D1$	$\times$					$\times$
$D3$	$\times$	$\times$	$\times$	$\times$		



D1-branes ending on D3-branes:

A Monopole appears.

$X^i \in \mathfrak{u}(N)$: transverse fluctuations

Nahm equation: $(s = x^6)$

$$\frac{\mathrm{d}}{\mathrm{d}s}X^i + \varepsilon^{ijk}[X^j, X^k] = 0$$

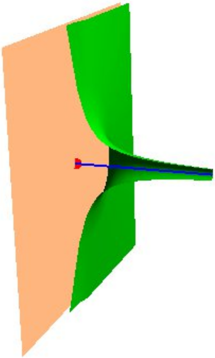
Note $\mathsf{SO}(3)$-invariance.

Solution: $X^i = r(s)G^i$ with

$$r(s) = \frac{1}{s}\ , \quad G^i = \varepsilon^{ijk}[G^j, G^k]$$

Nahm, Diaconescu, Tsimpis

dim	0	1	2	3	...	6
$D1$	×					×
$D3$	×	×	×	×		



Solution: $X^i = r(s)G^i$

$$r(s) = \frac{1}{s}\ , \quad G^i = \varepsilon^{ijk}[G^j, G^k]$$

Matches profile from SUGRA analysis

The D1-branes form a fuzzy funnel:

G^i form irrep of $\mathfrak{su}(2)$:
coordinates on fuzzy sphere S^2_F

D1-worldvolume polarizes: $2d \rightarrow 4d$

Myers

Lifting D1-D3-Branes to M2-M5-Branes

The lift to M-theory is performed by a T-duality and an M-theory lift

IIB	0	1	2	3	4	5	6
$D1$	×						×
$D3$	×	×	×	×			

T-dualize along x^5:

IIA	0	1	2	3	4	5	6
$D2$	×					×	×
$D4$	×	×	×	×		×	

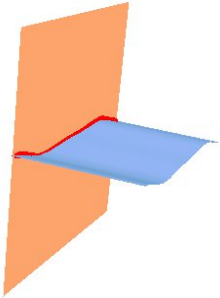
Interpret x^4 as M-theory direction:

M	0	1	2	3	4	5	6
$M2$	×					×	×
$M5$	×	×	×	×	×	×	

The Basu-Harvey lift of the Nahm Equation

M2-branes ending on M5-branes yield a Nahm equation with a cubic term.

M	0	1	2	3	4	5	6
$M2$	$\times$					$\times$	$\times$
$M5$	$\times$	$\times$	$\times$	$\times$	$\times$	$\times$	



Basu, Harvey, hep-th/0412310

A Self-Dual String appears.

Substitute SO(3)-inv. Nahm eqn.

$$\frac{\mathrm{d}}{\mathrm{d}s}X^i + \varepsilon^{ijk}[X^j, X^k] = 0$$

by the SO(4)-invariant equation

$$\frac{\mathrm{d}}{\mathrm{d}s}X^\mu + \varepsilon^{\mu\nu\rho\sigma}[X^\nu, X^\rho, X^\sigma] = 0$$

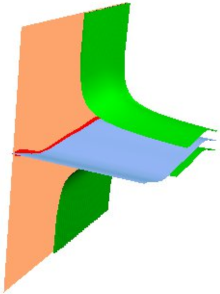
Solution: $X^\mu = r(s)G^\mu$ with

$$r(s) = \frac{1}{\sqrt{s}}\ , \quad G^\mu = \varepsilon^{\mu\nu\rho\sigma}[G^\nu, G^\rho, G^\sigma]$$

The Basu-Harvey lift of the Nahm Equation

Also M2-branes end on M5-branes by forming a fuzzy funnel.

M	0	1	2	3	4	5	6
$M2$	$\times$					$\times$	$\times$
$M5$	$\times$	$\times$	$\times$	$\times$	$\times$	$\times$	



Solution: $X^\mu = r(s) G^\mu$

$$r(s) = \frac{1}{\sqrt{s}}\ ,\quad G^\mu = \varepsilon^{\mu\nu\rho\sigma}[G^\nu, G^\rho, G^\sigma]$$

Matches profile from SUGRA analysis

The M2-branes form a fuzzy funnel:

G^μ form a rep of $\mathfrak{so}(4)$:
coordinates on fuzzy sphere S^3_F

M2-worldvolume polarizes: $3d \to 6d$

- 3-form structure appears
- quantization of S^3 required.

Further Motivation

There are more appearances of 2-plectic manifolds in string-/M-theory.

- M5-brane perspective: Turning on 3-form background,

$$C = \theta \mathrm{d}x^0 \wedge \mathrm{d}x^1 \wedge \mathrm{d}x^2 + \theta' \mathrm{d}x^0 \wedge \mathrm{d}x^1 \wedge \mathrm{d}x^2 ~,$$

the self-dual strings move in a space with

$$[x^0, x^1, x^2] = \theta \quad \text{and} \quad [x^3, x^4, x^5] = \theta' ~.$$

Chu, Smith, 0901.1847

- Baez et al.: The phase space of the bosonic string is believed to be 2-plectic
- Using T-duality, Lüst, 1010.1361, conjectured that closed strings on T^3 in 3-form background lead to:

$$[x^i, x^j] \sim \varepsilon^{ijk} p_k ~, \quad [x^i, p_j] \sim \delta^i_j ~, \quad [p_i, p_j] \sim \varepsilon^{ijk} p_k ~.$$

Note that here, the Jacobi identity is not satisfied.

Outline

- Quantization axioms for 2-plectic manifolds
- Quantized 2-plectic manifolds as vacua of 3-algebra models
- Another approach: Quantization via Groupoids

Axioms of Quantization

Quantization is nontrivial and far from being fully understood.

Classical level: states are points on a Poisson manifold $\mathcal{M}$.
observables are functions on $\mathcal{M}$.

Quantum level: states are rays in a complex Hilbert space $\mathscr{H}$.
observables are hermitian operators on $\mathscr{H}$.

Full Quantization

A full quantization is a map $\hat{-} : \mathcal{C}^\infty(\mathcal{M}) \to \mathsf{End}\,(\mathscr{H})$ satisfying

1. $f \mapsto \hat{f}$ is linear over $\mathbb{C}$, $f = f^* \Rightarrow \hat{f} = \hat{f}^\dagger$.
2. the constant function $f = 1$ is mapped to the identity on $\mathscr{H}$.
3. Correspondence principle: $\{f_1, f_2\} = g \Rightarrow [\hat{f}_1, \hat{f}_2] = \hat{g}$.
4. The quantized coordinate functions act irreducibly on $\mathscr{H}$.

Problem:

Groenewold-van Howe: no full quant. for $T^*\mathbb{R}^n$ or S^2 (T^2 OK)

Three approaches to weaken the axioms of a full quantization:

- Drop irreducibility condition
- Quantize a subset of $\mathcal{C}^\infty(\mathcal{M})$
- Correspondence principle applies only to $\mathcal{O}(\hbar)$

The first two yield prequantization and geometric quantization.
The last approach leads eventually to deformation quantization.

My favorite method for this talk:
Berezin quantization (or physicists' fuzzy geometry),
a hybrid of geometric and deformation quantization.

Hilbert space

$\mathscr{H}$ is the space of global holomorphic sections of a certain line bundle: $\mathscr{H} = H^0(\mathcal{M}, L)$. For $\mathcal{M} = \mathbb{C}P^1$: $L := \mathcal{O}(k)$.

$$\mathscr{H}_k \cong \mathrm{span}(z_{\alpha_1} \ldots z_{\alpha_k}) \cong \mathrm{span}(\hat{a}^\dagger_{\alpha_1} \ldots \hat{a}^\dagger_{\alpha_k}|0\rangle)$$

Coherent states

For any $z \in \mathcal{M}$: coherent st. $|z\rangle \in \mathscr{H}$. Here: $|z\rangle = \frac{1}{k!}(\bar{z}_\alpha \hat{a}^\dagger_\alpha)^k|0\rangle$.

Quantization

Quantization is the inverse map on the image $\Sigma = \sigma(\mathcal{C}^\infty(\mathcal{M}))$ of

$$f(z) = \sigma(\hat{f}) = \mathrm{tr}\left(\frac{|z\rangle\langle z|}{\langle z|z\rangle}\hat{f}\right) , \quad \text{Bridge: } \mathcal{P} := \frac{|z\rangle\langle z|}{\langle z|z\rangle}$$

Axioms of Generalized Quantization

We propose a generalization of the quantization axioms to p-plectic manifolds.

Problem is notoriously difficult, and many people tried to extend geometric quantization. Berezin quantization should be easier.
Keep: a complex Hilbert space $\mathscr{H}$ and $\mathsf{End}\,(\mathscr{H})$ as observables.

Generalized quantization axioms

A full quantization is a map $\hat{\ } : \Sigma \to \mathsf{End}\,(\mathscr{H}),\ \Sigma \subset \mathcal{C}^\infty(M)$ satisfying

1. $f \mapsto \hat{f}$ is linear over $\mathbb{C}$, $f = f^* \Rightarrow \hat{f} = \hat{f}^\dagger$.
2. the constant function $f = 1$ is mapped to the identity on $\mathscr{H}$.
3. Correspondence principle:

$$\lim_{\hbar \to 0} \left\| \frac{\mathrm{i}}{\hbar} \sigma\big([\hat{f}_1, \ldots, \hat{f}_n]\big) - \{f_1, \ldots, f_n\} \right\|_{L^2} = 0$$

If $\mathcal{M}$ is a Poisson manifold, this holds for Berezin quantization.

Quantization of $\mathbb{R}^3$

The quantized Nambu-Heisenberg algebra corresponds to the space $\mathbb{R}^3_\lambda$.

We start from $\mathbb{R}^3$ with $\omega = \varepsilon_{ijk} \mathrm{d}x^i \wedge \mathrm{d}x^j \wedge \mathrm{d}x^k$.

We find $\{f, g, h\} = \varepsilon^{ijk} \frac{\partial}{\partial x^i} f \frac{\partial}{\partial x^j} g \frac{\partial}{\partial x^k} h$.

What is the geometry of $[\hat{x}, \hat{y}, \hat{z}] = -\mathrm{i}\,\hbar\,\mathbb{1}$?

One possible interpretation as $\mathbb{R}^3_\lambda$:
Take a fuzzy sphere with Hilbert space $H^0(\mathbb{C}P^1, \mathcal{O}(k))$. Define:

$$[\hat{x}^1, \hat{x}^2, \hat{x}^3] = \sum_{i,j,k} \varepsilon^{ijk} \hat{x}^i \hat{x}^j \hat{x}^k = -\mathrm{i}\,\frac{6R^3}{k}\,\mathbb{1}_{\mathscr{H}_k}$$

Radius of this fuzzy sphere: $R_{F,k} = \sqrt{1 + \frac{2}{k}}\ \sqrt[3]{\frac{\hbar\, k}{6}}$.
Now "discretely foliate" $\mathbb{R}^3$ by fuzzy spheres. $\Rightarrow \mathbb{R}^3_\lambda$.

Brief Review: The IKKT model

Noncommutative geometries arise as stable solutions of the IKKT matrix model.

Fully dimensionally reduce maximally SUSY Yang-Mills theory:

$$S = \operatorname{tr}\left([X^I, X^J][X_I, X_J] + \mu_I (X^I)^2 + C_{IJK} X^I [X^J, X^K] + \text{fermions}\right)$$

where $X^I \in \mathfrak{u}(N)$, $I = 0, \ldots, 9$. Stable classical solutions:

Moyal spaces	$\mu_i = C_{ijk} = 0$	$[X^i, X^j] \sim \theta^{ij}\,\mathbb{1}$
Fuzzy sphere	$C_{123} = 1$	$[X^i, X^j] \sim \varepsilon^{ijk} X_k$
NC Hpp-waves	$C_{123} = 1,\ \mu_1 = \mu_2 = \mu$	$[X^1, X^2] \sim \theta^{12}\,\mathbb{1}$ $[X^3, X^i] \sim \theta^{ij} X^j$

First two: BPS. The third: Nappi-Witten algebra.

Hpp-waves: 4d Cahen-Wallach symm. space in Brinkman coords.:

$$\mathrm{d}s_4^2 = 2\alpha\,\beta\,\mathrm{d}x^+\,\mathrm{d}x^- + \gamma^2\,|\mathrm{d}z|^2 - \tfrac{1}{4}\,\beta^2\left(\gamma^2\,|z|^2 - b\right)(\mathrm{d}x^+)^2$$

Identification:

$$X^1 + \mathrm{i}X^2 \sim z\ , \quad X^3 \sim J\ , \quad X^4 \sim \mathbb{1}\ .$$

The Dimensionally Reduced BLG Model

The theory corresponding to SYM is the BLG model, which we dimensionally reduce.

Basu-Harvey $\rightarrow$ Bagger, Lambert and Gustavsson developed a Chern-Simons matter theory for M2-branes. Reduced form:

$$
\begin{aligned}
S = &- \tfrac{1}{2}\left(A_\mu X^I, A^\mu X^I\right) + \tfrac{\mathrm{i}}{2}\left(\bar{\Psi}, \Gamma^\mu A_\mu \Psi\right) - \tfrac{1}{2}\sum_{I=1}^{8} \mu_{1,I}^2 \left(X^I, X^I\right) \\
&+ \tfrac{\mathrm{i}}{2}\,\mu_2\left(\bar{\Psi}, \Gamma_{3456}\Psi\right) + C^{IJKL}\left([X^I, X^J, X^K], X^L\right) \\
&+ \tfrac{\mathrm{i}}{4}\left(\bar{\Psi}, \Gamma_{IJ}[X^I, X^J, \Psi]\right) - \tfrac{1}{12}\left([X^I, X^J, X^K], [X^I, X^J, X^K]\right) \\
&+ \tfrac{1}{6}\,\epsilon^{\mu\nu\lambda}\left(\!\left(A_\mu, [A_\nu, A_\lambda]\right)\!\right) + \frac{1}{4\gamma^2}\left(\!\left([A_\mu, A_\nu], [A^\mu, A^\nu]\right)\!\right) .
\end{aligned}
$$

Matter fields X^I, Ψ in vector space forming an orth. rep. of the gauge algebra in which A_μ lives. ("3-Lie algebra")

This model should take over the role of the IKKT model!

Solutions in the 3-Lie Algebra Model

Stable solutions of the IKKT model have a 3-Lie algebra analogue.

Stable classical solutions:

$\mathbb{R}^3_\lambda$	$\mu_I = C_{IJKL} = 0$	$[X^i, X^j, X^k] \sim \varepsilon^{ijk}\mathbb{1}$
Fuzzy S^3	$C_{1234} = 1$	$[X^i, X^j, X^k] \sim \varepsilon^{ijkl} X^l$
NC Hpp-waves	$C_{1234} = 1,\ \mu_1 = \mu_2 = \mu$	$[X^1, X^2, X^3] \sim \theta^{123}\mathbb{1}$ $[X^4, X^i, X^j] \sim \theta^{ijk} X^k$

First two solutions again BPS. The third: 5d Hpp-wave backgrnd.:

$$\mathrm{d}s_4^2 = 2\alpha\,\beta\,\mathrm{d}x^+\,\mathrm{d}x^- + \gamma^2\,|\mathrm{d}\vec{z}|^2 - \tfrac{1}{4}\,\beta^2\left(\gamma^2\,|\vec{z}|^2 - b\right)(\mathrm{d}x^+)^2$$

Identification:

$$X^1 + \mathrm{i}X^2 \sim z\ ,\quad X^3 \sim z^3\ ,\quad X^4 \sim J\ ,\quad X^5 \sim \mathbb{1}\ .$$

Further Features of the 3-Algebra Matrix Model

The 3-algebra model has more interesting features.

- Recall: IKKT expanded around sol.: NC YM theory on sol.
  Here: 3-algebra model exp. around sol.: expected part + ...
- Close connection to the cubic $\mathfrak{osp}(1|32)$-cubic matrix model:

$$S_{\mathrm{CSM}} = \mathrm{str}\big(M^3\big) \ ,$$

The Groupoid Approach to Quantization

The elements of geometric quantization can be written in a groupoid language.

Groupoids: Objects + composable, invertible arrows between them.

Why groupoids?

- Quantization of the dual of a Lie algebra:
  Twisted convolution algebra of integrating group
- Poisson struct. $\rightarrow$ Lie algebroid $\rightarrow$ Conv. alg. on Lie groupoid
- Construction avoids Hilbert spaces, useful for 2-plectic case

Procedure (Eli Hawkins, math/0612363, Weinstein, Renault, ...)

1. Integrating groupoid $s, t : \Sigma \rightrightarrows M$, ω, $\partial^* \omega = 0$, t Poisson
2. Construct a prequantization of Σ with data (L, ∇)
3. Endow Σ with a groupoid polarization
4. Construct a twist element
5. Obtain twisted polarized convolution algebra of Σ.

Example: Groupoid quantization of $\mathbb{R}^2$

The Moyal plane is conveniently reproduced in groupoid language.

Starting point: $M = \mathbb{R}^2$, Poisson structure π^{ij}, $i, j = 1, 2$.

1. Lie groupoid: $\Sigma = M \times M^*$, coords. (x^i, y_i), $\omega = \mathrm{d}x^i \wedge \mathrm{d}y_i$

   $$s(x^i, y_i) = (x^i + \tfrac{1}{2}\pi^{ij} y_j) \quad \text{and} \quad t(x^i, y_i) = (x^i - \tfrac{1}{2}\pi^{ij} y_j)$$

   Note: t is indeed a Poisson map: $\{t^* f, t^* g\}_\omega = t^* \{f, g\}_\pi$

   $$x^i + \pi^{ij}(y_j + y'_j) \longrightarrow x^i + \pi^{ij}(y_j - y'_j) \longrightarrow x^i - \pi^{ij}(y_j + y'_j)$$

   From this: pr_1, pr_2 and m. $\partial^* \omega = p_1^* \omega - m^* \omega + p_2^* \omega = 0$
2. Prequantization: L trivial line bundle over Σ, $F = -\mathrm{i}2\pi\omega$
3. Polarization: Induced by symplectic prepotential $\theta = -x^i \mathrm{d}y_i$
4. Twist element: $\partial^* \theta = \sigma_0^{-1} \mathrm{d}\sigma_0 = \mathrm{d}(-\tfrac{1}{2}\pi^{ij} y_i y'_j)$
5. Twisted polarized convolution algebra: Moyal product on M

Hawkins

How do we extend this to $\mathbb{R}^3$?

1. Strict approach: Look at 2-Groupoids (2-Hilbert spaces) etc. (work in progress, cf. Freed, Baez, Rogers ...)
2. Trick: Map the 2-plectic forms back to symplectic ones.

Consider the following double fibration:

$$\begin{array}{ccc} & \mathcal{L}M \times S^1 & \\ {}^{ev}\swarrow & & \searrow^{pr} \\ M & & \mathcal{L}M \end{array}$$

Transgression

$$\mathcal{T} : \Omega^{k+1}(M) \to \Omega^k(\mathcal{L}M)~, \quad \mathcal{T} = pr_! \circ ev^*$$

$$(\mathcal{T}\omega)_x(v_1(\tau), \ldots, v_k(\tau)) := \int_{S^1} \mathrm{d}\tau\, \omega(v_1(\tau), \ldots, v_k(\tau), \dot{x}(\tau))$$

Previous application: Lift ADHMN construction to M-theory

CS, Papageorgakis&CS, Palmer&CS

Explicitly, this works as follows:

We start from $\mathbb{R}^3$ with 2-plectic form $\varpi = \varepsilon_{ijk}\mathrm{d}x^i \wedge \mathrm{d}x^j \wedge \mathrm{d}x^k$.

Transgression yields a symplectic form on loop space $\mathcal{L}\mathbb{R}^3$:

$$\omega = \oint \mathrm{d}\tau \oint \mathrm{d}\sigma \; \varepsilon_{ijk}\dot{x}^k(\tau)\delta(\tau-\sigma) \; \delta x^i(\tau) \wedge \delta x^j(\sigma)$$

This gives a Poisson bracket

$$\{x^i(\tau), x^j(\sigma)\} := \varepsilon^{ijk}\dot{x}_k(\tau)\delta(\tau-\sigma)$$

Corresponding commutator relation previously derived in M-theory.
Kawamoto&Sasakura
Bergshoeff&Berman&van der Schaar&Sundell

Groupoid Quantization of $\mathbb{R}^3$

The groupoid approach can be extended to higher structures using loop space.

Starting point: $M = \mathbb{R}^3$, Poisson structure $\pi^{ijk}\dot{x}_k$, $i, j = 1, 2$.

1. Lie groupoid: $\Sigma = LM \times LM^*$, coords. (x^i, y_i), $\omega = \mathrm{d}x^i \wedge y_i$

   $$s(x^i, y_i) = (x^i + \tfrac{1}{2}\pi^{ijk} y_j \dot{x}_k) \quad \text{and} \quad t(x^i, y_i) = (x^i - \tfrac{1}{2}\pi^{ijk} y_j \dot{x}_k)$$

   Note: t is indeed a Poisson map: $\{t^* f, t^* g\}_\omega = t^*\{f, g\}_\pi$

   $$x^i + \pi^{ij}(y_j + y_j')\dot{x}_k \to x^i + \pi^{ijk}(y_j - y_j')\dot{x}_k \to x^i - \pi^{ijk}(y_j + y_j')\dot{x}_k$$

   From this: pr_1, pr_2 and m. $\partial^*\omega = p_1^*\omega - m^*\omega + p_2^*\omega = 0$
2. Prequantization: L trivial line bundle over Σ, $F = -\mathrm{i}2\pi\omega$
3. Polarization: Induced by symplectic prepotential $\theta = -x^i \mathrm{d}y_i$
4. Twist element: $\partial^*\theta = \sigma_0^{-1}\mathrm{d}\sigma_0 = \mathrm{d}(-\tfrac{1}{2}\pi^{ijk} y_i y_j' \dot{x}_k)$
5. Twisted polar. conv. alg.: $[x^i(\tau), x^j(\sigma)] = \delta(\tau - \sigma)\pi^{ijk}\dot{x}_k(\tau)$

Summary:

- ✓ Naive approach to quantizing 2-plectic manifolds works OK
- ✓ IKKT-like model can be written down, has expected features.
- ✓ Groupoids offer a more general approach to quantization

Future directions:

- ▷ Extend quantization of $\mathbb{R}^3$ to 2-groupoid.
- ▷ Extend groupoid constructions to other spaces (S^3).
- ▷ Unify picture: Higher Poisson structures? Courant algebroids?
- ▷ Rewrite BLG model using the new function algebras.

Quantization of 2-plectic Manifolds

Christian Sämann

School of Mathematical and Computer Sciences
Heriot Watt University, Edinburgh

Bayrischzell Workshop, 22.5.2011