Strong Homotopy Lie Algebras and Field Theories

Christian Saemann

School of Mathematical and Computer Sciences
Heriot-Watt University, Edinburgh

Corfu2019 Workshop, 20.9.2019

Relevant papers (besides vast literature on BV/BRST):

- arXiv:1809.09899 with B Jurco, L Raspollini and M Wolf
- arXiv:1903.05713 with T Macrelli and M Wolf
- arXiv:1910.xxxxx with B Jurco, T Macrelli and M Wolf

Strong homotopy Lie algebras from Physics

Homotopy Maurer–Cartan Theory

BRST/BV-Formalism

Scattering amplitudes

Strong homotopy Lie algebras from Physics

"... and there is no new thing under the sun."

Ecclesiastes

"It's like déjà vu all over again."

Yogi Berra

Berends–Giele recursion relation

Nuclear Physics B306 (1988) 759–808
North-Holland, Amsterdam

RECURSIVE CALCULATIONS FOR PROCESSES WITH *n* GLUONS

F.A. BERENDS and W.T. GIELE*

Instituut-Lorentz, University of Leiden, P.O.B. 9506, 2300 RA Leiden, The Netherlands

Received 30 December 1987

- Recursion relation for currents in Yang–Mills theory
- Directly translate to relations for amplitudes
- These were used to prove Parke–Taylor (MHV) formula
- Relations have deep algebraic meaning: Quasi-Isomorphisms!

Nuclear Physics B390 (1993) 33–152
North-Holland

NUCLEAR
PHYSICS B

Closed string field theory: Quantum action and the Batalin–Vilkovisky master equation

Barton Zwiebach [1]

School of Natural Sciences, Institute for Advanced Study, Olden Lane, Princeton NJ 08540, USA

Received 30 June 1992
Accepted for publication 18 September 1992

This paper has all the important ingredients:

- BV-formalism
- L_∞-algebras
- quantum L_∞-algebras

Why should I care?

Strong homotopy Lie algebras = L_∞-algebras.

They are everywhere:

- L_∞-algebras from BV (this talk) $\rightarrow$ perturbative QFT
- Basis for string field theory (because of above)
- in higher gauge theories (SUGRA, string/M-theory,...)
- L_∞-algebroids in AKSZ formalism
- L_∞-algebroids in Generalized Geometry/T-duality
- L_∞-algebroids in nonassociative geometry
- Deformation theory
- Beautiful mathematics

"Before functoriality, people lived in caves."

Brian Conrad

Two formulations dual to each other:

- Formulation as differential graded algebras (dga)
- Formulation as "higher brackets" (from codifferential)
- Both are important and helpful!
- Signs are messy, but can usually be reconstructed

L_∞-algebras as differential graded algebras

- Graded vector space $E = \cdots \oplus E_{-1} \oplus E_0 \oplus E_1 \oplus \dots$
- Vector field Q on E, $|Q| = 1$, $Q^2 = 0$

Note: E vector bundle $\rightarrow$ L_∞-algebroid

Example: Lie algebras

$E = \mathfrak{g}[1]$, coordinate functions ξ^α of degree 1:

$$Q = -\tfrac{1}{2} f^\alpha_{\beta\gamma} \xi^\beta \xi^\gamma \frac{\partial}{\partial \xi^\alpha} \;, \quad Q^2 = 0 \Leftrightarrow \text{Jacobi identity}$$

Example: BRST complex

$E =$ ghosts$[1] \oplus$ fields, coords.: c, $|c| = 1$ and A, $|A| = 0$:

$$Qc = -\tfrac{1}{2}[c,c] \;, \quad QA = \mathrm{d}c + [A,c]$$

L_∞-algebras as differential graded algebras

- Graded vector space $E = \cdots \oplus E_{-1} \oplus E_0 \oplus E_1 \oplus \ldots$
- Vector field Q on E, $|Q| = 1$, $Q^2 = 0$

- Q encoded in "structure constants":

$$Q = \pm \sum_{i\geq 0} \frac{1}{i!} m^\beta_{\alpha_1\ldots\alpha_i} \xi^{\alpha_1} \ldots \xi^{\alpha_i} \frac{\partial}{\partial \xi^\beta}$$

- These encode higher brackets on basis τ_α of $\mathsf{L} = E[-1]$:

$$\mu_i(\tau_{\alpha_1}, \ldots \tau_{\alpha_i}) = m^\beta_{\alpha_1\ldots\alpha_i} \tau_\beta \ .$$

L_∞-algebras as higher brackets

- Graded vector space $\mathsf{L} = \cdots \oplus \mathsf{L}_{-1} \oplus \mathsf{L}_0 \oplus \mathsf{L}_1 \oplus \ldots$
- Higher brackets/products $\mu_i : \mathsf{L}^{\wedge i} \to \mathsf{L}$, $|\mu_i| = 2 - i$
- Higher Jacobi identities: $(\Leftrightarrow Q^2 = 0)$

$$\sum_{i+j=n} \sum_{\sigma\in \mathrm{Sh}(i,n-i)} \pm \mu_{i+1}(\mu_j(\ell_{\sigma(1)}, \ldots, \ell_{\sigma(j)}), \ell_{\sigma(j+1)}, \ldots, \ell_{\sigma(n)}) = 0$$

- Graded vector space: $* \leftarrow W[1] \leftarrow V[2] \leftarrow * \leftarrow \dots$
- Coords: w^a of degree 1 on $W[1]$, v^i of degree 2 on $V[2]$
- Most general vector field Q of degree 1:

$$Q = -m_i^a v^i \frac{\partial}{\partial w^a} - \tfrac{1}{2} m_{ab}^c w^a w^b \frac{\partial}{\partial w^c} - m_{ai}^j w^a v^i \frac{\partial}{\partial v^j} - \tfrac{1}{3!} m_{abc}^i w^a w^b w^c \frac{\partial}{\partial v^i}$$

- Induces "brackets"/"higher products":
  - $\mu_1(\tau_i) = m_i^a \tau_a$
  - $\mu_2(\tau_a, \tau_b) = m_{ab}^c \tau_c \;, \quad \mu_2(\tau_a, \tau_i) = m_{ai}^j \tau_j$
  - $\mu_3(\tau_a, \tau_b, \tau_c) = m_{abc}^i \tau_i$
- $Q^2 = 0 \Leftrightarrow$ Homotopy Jacobi identities, e.g.
  - $\mu_1(\mu_1(-)) = 0$: μ_1 is a differential
  - $\mu_1(\mu_2(x,y)) = \mu_2(\mu_1(x), y) \pm \mu_2(x, \mu_1(y))$: compatible w. μ_2,
  - $\mu_2(x, \mu_2(y,z)) + \text{cycl.} = \mu_1(\mu_3(x,y,z))$: Jacobiator
- Analogously: Lie 3-, 4-, ...-algebras

L_∞-algebras are generalizations of dg Lie algebras.

L_∞-algebras: Inner products

Inner product on Lie algebra $\mathfrak{g}$: $\langle -, - \rangle : \mathfrak{g} \times \mathfrak{g} \to \mathbb{R}$

- positive definite/non-degenerate
- symmetric
- bilinear
- satisfying cyclic relation:

$$\langle \ell_1, [\ell_2, \ell_3] \rangle = \langle \ell_2, [\ell_3, \ell_1] \rangle$$

generalized naturally (more later) to

Cyclic structure on L_∞-algebra L: $\langle -, - \rangle : \mathsf{L} \times \mathsf{L} \to \mathbb{R}$

- non-degenerate
- graded symmetric
- bilinear
- satisfying cyclic relation:

$$\langle \ell_1, \mu_i(\ell_2, \dots, \ell_{1+i}) \rangle = \pm \langle \ell_2, \mu_i(\ell_3, \dots, \ell_{1+i}, \ell_1) \rangle$$

"dg commutative algebra $\otimes$ L_∞-algebra yields L_∞-algebra"

Example: $\Omega^\bullet(M,\mathsf{L}) := \Omega^\bullet(M)\otimes\mathsf{L} = \bigoplus_{k\in\mathbb{Z}}\Omega^\bullet_k(M,\mathsf{L})$:

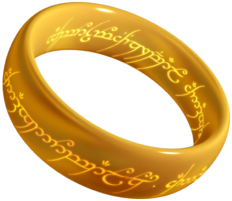
- $\Omega^\bullet_k(M,\mathsf{L}) := \Omega^0(M)\otimes\mathsf{L}_k \oplus \Omega^1(M)\otimes\mathsf{L}_{k-1} \oplus \cdots \oplus \Omega^d(M)\otimes\mathsf{L}_{k-d}$
- Higher products:

$$\begin{aligned}\hat\mu_1(\alpha_1\otimes\ell_1) &:= \mathrm{d}\alpha_1\otimes\ell_1 \pm \alpha_1\otimes\mu_1(\ell_1)\\ \hat\mu_i(\alpha_1\otimes\ell_1,\ldots,\alpha_i\otimes\ell_i) &:= \pm(\alpha_1\wedge\ldots\wedge\alpha_i)\otimes\mu_i(\ell_1,\ldots,\ell_i)\end{aligned}$$

- Cyclic structure for compact manifolds and cyclic L:

$$\langle\alpha_1\otimes\ell_1,\alpha_2\otimes\ell_2\rangle_{\Omega^\bullet(M,\mathsf{L})} := \pm\int_M \alpha_1\wedge\alpha_2\ \langle\ell_1,\ell_2\rangle_{\mathsf{L}}$$

Homotopy Maurer–Cartan Theory



"One ring to rule them all ..."

Maurer–Cartan equation for differential graded Lie algebra, $(\mathfrak{g}, \mathrm{d})$:

$$\mathrm{d}a + \tfrac{1}{2}[a,a] = 0\ , \quad a \in \mathfrak{g}\ .$$

L_∞-algebras are generalizations of dg Lie algebras.

Homotopy Maurer–Cartan equation:

$$f := \mu_1(a) + \tfrac{1}{2}\mu_2(a,a) + \tfrac{1}{3!}\mu_3(a,a,a) + \cdots = 0\ , \quad a \in \mathsf{L}_1$$

Nomenclature: a: gauge potential f: curvature

Bianchi identity:

$$\mu_1(f) - \mu_2(f,a) + \tfrac{1}{2}\mu_3(f,a,a) - \tfrac{1}{3!}\mu_4(f,a,a,a) + \cdots = 0\ .$$

Homotopy Maurer–Cartan Action:

$$S_{\mathrm{MC}}[a] := \sum_{i\geq 1} \frac{1}{(i+1)!}\langle a, \mu_i(a,\ldots,a)\rangle_{\mathsf{L}}\ .$$

Also: this is the structure underlying closed string field theory.

Example: Chern–Simons Theory

Tensor product L_∞-algebra $\hat{\mathsf{L}} = \Omega^\bullet(M) \otimes \mathfrak{g}$ with $\mathfrak{g}$ Lie algebra:

- gauge potential

$$A \in \hat{\mathsf{L}}_1 = \Omega^1(M) \otimes \mathfrak{g}$$

- higher products:

$$\hat{\mu}_1 = \mathrm{d} \quad \text{and} \quad \mu_2 = [-,-]$$

- Homotopy Maurer–Cartan equation:

$$F := \mathrm{d}A + \tfrac{1}{2}[A, A] = 0$$

- Homotopy Maurer–Cartan action:

$$S_{\mathrm{MC}}[A] := \int_M \left\langle \tfrac{1}{2}(A, \mathrm{d}A) + \tfrac{1}{3!}(A, [A, A]) \right\rangle \, .$$

For $d = 4$, need cyclic "Lie 2-algebra:" $\mathsf{L} = \mathsf{L}_{-1} \oplus \mathsf{L}_0$.

Tensor product L_∞-algebra $\hat{\mathsf{L}} = \Omega^\bullet(M) \otimes \mathsf{L}$:

- gauge potential

$$A + B \in \hat{\mathsf{L}}_1 = \Omega^1(M) \otimes \mathsf{L}_0 \oplus \Omega^2(M) \otimes \mathsf{L}_{-1}$$

- higher products are $\hat{\mu}_1 = \mathrm{d} + \mu_1$, μ_2, μ_3
- Homotopy Maurer–Cartan equation:

$$F = \mathrm{d}A + \tfrac{1}{2}\mu_2(A, A) + \mu_1(B)$$
$$H = \mathrm{d}B + \mu_2(A, B) - \tfrac{1}{3!}\mu_3(A, A, A)$$

- Homotopy Maurer–Cartan action:

$$S_{\mathrm{MC}} = \int_M \Big\{ \langle B, \mathrm{d}A + \tfrac{1}{2}\mu_2(A, A) + \tfrac{1}{2}\mu_1(B) \rangle_{\mathsf{L}} + \\ + \tfrac{1}{4!}\langle \mu_3(A, A, A), A \rangle_{\mathsf{L}} \Big\} ,$$

Generalizes to arbitrary dimensions $d \geq 3$!

BRST/BV-Formalism

Classical space of observables:

$$\begin{array}{ccc} & \text{Functionals on fields } \mathfrak{F} & \\ \diagdown & & \diagup \\ \text{ideal } \mathfrak{I} := \langle\text{solutions to eom}\rangle & & \text{gauge symmetry } \mathfrak{G} \end{array}$$

Observation:

- Orbit spaces are often not nice
- Better: derived quotient
  - Consider action groupoid
  - quotient space in cohomology.

Action Lie groupoid ("derived quotient")

$$(\text{symmetry group} \ltimes \text{field space}) \rightrightarrows \text{field space}$$

$$\Phi \xrightarrow{(g,\Phi)} g \rhd \Phi$$

This differentiates to the action Lie algebroid

$$\mathfrak{F}_{\rm BRST} := \big(\text{Lie}(\text{symmetry group}) \ltimes \text{field space} \to \text{field space}\big)$$

BRST complex is the dga-description of this Lie algebroid.

Chevalley–Eilenberg resolution:

$$0 \longrightarrow \mathscr{C}^\infty(\mathfrak{F}/\mathfrak{G}) \cong H^0(\mathfrak{F}/\mathfrak{G}) \hookrightarrow \mathscr{C}^\infty_0(\mathfrak{F}_{\rm BRST}) \xrightarrow{Q} \mathscr{C}^\infty_1(\mathfrak{F}_{\rm BRST}) \xrightarrow{Q} \cdots$$

Classical observables:

field configurations modulo symmetries satisfying eom

- Field space $\mathfrak{F}$
- Enlarged: $\mathfrak{F}_{\rm BV} := T^*[-1]\mathfrak{F}$ coords. fields Φ^A, "antifields" Φ^+_A
- $S_{\rm BV}$ defines $Q_{\rm BV} = \{S_{\rm BV}, -\}$ with $Q^2_{\rm BV} = 0$
- Note: $Q_{\rm BV}\Phi^+_A = \{S_{\rm BV}, \Phi^+_A\} = \delta_{\Phi^A} S$, classical eoms.
- Note: $Q_{\rm BV}(\mathscr{C}^\infty_{-1}(T^*[-1]\mathfrak{F})) \; = \; \mathfrak{I}$, ideal vanishing on solutions

Koszul–Tate resolution:

$$\cdots \xrightarrow{Q} \mathscr{C}^\infty_{-1}(T^*[-1]\mathfrak{F}) \xrightarrow{Q} \mathscr{C}^\infty_0(T^*[-1]\mathfrak{F}) \longrightarrow H^0(T^*[-1]\mathfrak{F}) = \mathscr{C}^\infty(\mathfrak{F})/\mathfrak{I} \longrightarrow 0$$

Classical BRST-BV formalism

Essentially:

Classical BRST-BV formalism = Chevalley–Eilenberg resolution + Koszul–Tate resolution

We have

$$S_{\rm BV}~, \quad Q_{\rm BV} := \{S_{\rm BV}, -\}~, \quad Q_{\rm BV}^2 = 0$$

Question: What is the L_∞-algebra dual to the BV complex?

Recall dualization (invert sign) and shift by one:

Lie algebra $\mathfrak{g} \longleftrightarrow (\mathfrak{g}[1], Q = \dots) \longleftrightarrow$ dg algebra $(\mathscr{C}^\infty(\mathfrak{g}[1]), Q)$

Translate coordinate functions to elements of vector spaces.

Example: Gauge theory

- Field Φ of degree 0 $\longleftrightarrow$ $\Phi \in \mathsf{L}_1$
- Ghost c of degree 1 $\longleftrightarrow$ $c \in \mathsf{L}_0$
- antifield Φ^+ of degree -1 $\longleftrightarrow$ $\Phi^+ \in \mathsf{L}_2$
- antifield of ghost c^+ of degree -2 $\longleftrightarrow$ $c^+ \in \mathsf{L}_3$
- etc. for higher gauge theories

Altogether:

$\cdots$	L_0	L_1	L_2	L_3	$\cdots$
$\cdots$	gauge transf.	physical fields	equations of motion	Noether identities	$\cdots$

For Yang–Mills theory:

- Manifold M, Lie algebra $\mathfrak{g}$, Coord. functions: A, A^+, c, c^+
- Symplectic form: $\omega = \int_M \{\langle \delta A, \delta A^+ \rangle_{\mathfrak{g}} - \langle \delta c, \delta c^+ \rangle_{\mathfrak{g}}\}$
- Action: $S = \int_M \left\{ \frac{1}{2}\langle F, \star F\rangle_{\mathfrak{g}} - \langle A^+, \nabla c\rangle_{\mathfrak{g}} + \frac{1}{2}\langle c^+, [c,c]\rangle_{\mathfrak{g}} \right\}$
- Homological vector field: $Q := \{S, -\}$ with $Q^2 = 0$
- This is the dual of an L_∞-algebra.

L_∞-algebra picture:

Complex:

$$\underbrace{\Omega^0(M,\mathfrak{g})}_{\mathsf{L}_0} \xrightarrow{\mu_1 := \mathrm{d}} \underbrace{\Omega^1(M,\mathfrak{g})}_{\mathsf{L}_1} \xrightarrow{\mu_1 := \mathrm{d}\star\mathrm{d}} \underbrace{\Omega^{d-1}(M,\mathfrak{g})}_{\mathsf{L}_2} \xrightarrow{\mu_1 := \mathrm{d}} \underbrace{\Omega^d(M,\mathfrak{g})}_{\mathsf{L}_3}$$

Higher Products:

$$\mu_1(c_1) := \mathrm{d}c_1\ , \quad \mu_1(A_1) := \mathrm{d}\star\mathrm{d}A_1\ , \quad \mu_1(A_1^+) := \mathrm{d}A_1^+\ ,$$
$$\mu_2(c_1,c_2) := [c_1,c_2]\ , \quad \mu_2(c_1,A_1) := [c_1,A_1]\ , \quad \mu_2(c_1,A_2^+) := [c_1,A_2^+]\ ,$$
$$\mu_2(c_1,c_2^+) := [c_1,c_2^+]\ , \quad \mu_2(A_1,A_2^+) := [A_1,A_2^+]\ ,$$
$$\mu_2(A_1,A_2) := \mathrm{d}\star[A_1,A_2] + [A_1,\star\mathrm{d}A_2] + [A_2,\star\mathrm{d}A_1]\ ,$$
$$\mu_3(A_1,A_2,A_3) := [A_1,\star[A_2,A_3]] + [A_2,\star[A_3,A_1]] + [A_3,\star[A_1,A_2]]$$

Homotopy Maurer–Cartan action with $a = A$ is Yang–Mills action!
hMC action with $a = c_0 + A + A^+ + c^+$ is BV action!

Classical BV:

$S_{\mathrm{BV}} \in \mathcal{C}^\infty(\mathfrak{F})$ solving classical master equation:

$$\{S_{\mathrm{BV}}, S_{\mathrm{BV}}\} = 0$$

with

$$Q := \{S_{\mathrm{BV}}, -\} \quad \text{with} \quad Q^2 = 0$$

Full BV:

$S^\hbar_{\mathrm{BV}} \in \mathcal{C}^\infty(\mathfrak{F})[[\hbar]]$ solving quantum master equation:

$$\hbar \Delta S^\hbar_{\mathrm{BV}} + \{S^\hbar_{\mathrm{BV}}, S^\hbar_{\mathrm{BV}}\} = 0$$

yields

$$Q^\hbar := \hbar\Delta + \{S^\hbar_{\mathrm{BV}}, -\} \quad \text{with} \quad (Q^\hbar)^2 = 0$$

- $Q^\hbar$ is differential, but not derivation!
- Defines quantum L_∞-algebra:

$$\sum_{i+j=n,\sigma,g_1,g_2} \pm \mu^{g_1}_{i+1}(\mu^{g_2}_j(\ell_{\sigma(1)},\ldots,\ell_{\sigma(j)}),\ell_{\sigma(j+1)},\ldots,\ell_{\sigma(n)}) - \hbar \sum_a \mu^{g_1+g_2-1}_{i+2}(\tau^a,\tau_a,\ell_1,\ldots,\ell_i) = 0$$

Examples of Applications

Quasi-isomorphisms

Question: When are two L_∞-algebras essentially the same?

- Morphisms in dga-picture clear:

$$\mathcal{C}^\infty(E) \xrightarrow{\Phi} \mathcal{C}^\infty(E')~, \quad Q' \circ \Phi = \Phi \circ Q$$

- Morphisms of L_∞-algebras $\phi : \mathsf{L} \to \mathsf{L}'$ induced:

$$\phi_i : \mathsf{L}^{\wedge i} \to \mathsf{L}~, \quad |\phi_i| = 1 - i~, \quad \phi_{1*} : H^\bullet_{\mu_1}(\mathsf{L}) \to H^\bullet_{\mu'_1}(\mathsf{L}')$$

- L_∞-algebras L and L' quasi-isomorphic:
  There is a $\phi : \mathsf{L} \to \mathsf{L}'$ with $\phi_1 : H^\bullet_{\mu_1}(\mathsf{L}) \cong H^\bullet_{\mu_1}(\mathsf{L}')$

Equivalent field theories FT∼FT'	$\leftrightarrow$	Quasi-isomorphic L_∞-algebras $\mathsf{L} \approxeq \mathsf{L}'$

Classical BV formalism:

- Manifold M, Lie algebra $\mathfrak{g}$,
- Coords: $A \in \Omega^1(M,\mathfrak{g})$, $B \in \Omega^2_+(M,\mathfrak{g})$, A^+, B^+_+, c, c^+
- Symplectic form ω obvious/canonical, Action:

$$S = \int_M \Big\{ \langle F, B_+ \rangle_\mathfrak{g} + \tfrac{\varepsilon}{2} \langle B_+, B_+ \rangle_\mathfrak{g} - \\ - \langle A^+, \nabla c \rangle_\mathfrak{g} - \langle B^+_+, [B_+, c] \rangle_\mathfrak{g} + \tfrac{1}{2} \langle c^+, [c,c] \rangle_\mathfrak{g} \Big\}$$

- Yields L_∞-algebra $\mathsf{L}_{\mathrm{YM_1BV}}$

Morphism of L_∞-algebras:

- Easy to check: $H^\bullet_{\mu_1}(\mathsf{L}_{\mathrm{YM_2BV}}) \cong H^\bullet_{\mu_1}(\mathsf{L}_{\mathrm{YM_1BV}})$
- Moreover: We have $\Phi : \mathscr{C}^\infty(\mathfrak{F}_{\mathrm{YM_1BV}}) \to \mathscr{C}^\infty(\mathfrak{F}_{\mathrm{YM_2BV}})$ with

$$\Phi(c) := c\,, \quad \Phi(B_+) := -\tfrac{1}{\varepsilon} F_+\,, \quad \Phi(A) := A\,, \\ \Phi(B^+_+) := 0\,, \quad \Phi(A^+) = A^+\,, \quad \Phi(c^+) := c^+\,.$$

- This satisfies $Q_{\mathrm{YM_2BV}} \circ \Phi = \Phi \circ Q_{\mathrm{YM_1BV}}$

Definition: L_∞-algebra with $\mu_1 = 0$: minimal (not $\approxeq$ to "smaller")

Minimal Model Theorem

Every L_∞-algebra is quasi-isomorphic to a minimal one.

In field theory:

- L_∞-algebra L encoding field theory
- quadratic term $\langle a, \mu_1(a)\rangle$: μ_1 encodes inverse propagator
- Minimal model $H^\bullet_{\mu_1}(\mathsf{L}) \approxeq \mathsf{L}$ encodes equivalent FT'
- FT' has trivial propagator

Conclusion:

The higher products of the minimal model yield
the tree level amplitudes of a field theory:

$$\langle \Psi_1, \mu_i^\circ(\Psi_2, \dots, \Psi_{i+1})\rangle \quad \longleftrightarrow \quad \langle \Psi_1 \Psi_2 \dots \Psi_{i+1}\rangle$$

Explicitly:

$$\langle \Psi_1, \mu_i^\circ(\Psi_2, \ldots, \Psi_{i+1}) \rangle \quad \longleftrightarrow \quad \langle \Psi_1 \Psi_2 \ldots \Psi_{i+1} \rangle$$

$$\begin{aligned}
\mu_1^\circ(\Psi_1) &:= 0 \,, \\
\mu_2^\circ(\Psi_1, \Psi_2) &:= (\mathsf{p} \circ \mu_2)(\phi_1(\Psi_1), \phi_1(\Psi_2)) \,, \\
&\vdots \\
\mu_i^\circ(\Psi_1, \ldots, \Psi_i) &:= \sum_{j=2}^{i} \frac{1}{j!} \sum_{k_1+\cdots+k_j=i} \sum_{\sigma} \pm (\mathsf{p} \circ \mu_j)\big(\phi_{k_1}\big(\Psi_{\sigma(1)}, \ldots, \Psi_{\sigma(k_1)}\big), \ldots, \phi_{k_j}\big(\ldots\big)\big)
\end{aligned}$$

Recursion for currents ϕ_i:

$$\begin{aligned}
\phi_1(\Psi_1) &:= \mathsf{e}(\Psi_1) \,, \\
\phi_2(\Psi_1, \Psi_2) &:= -(\mathsf{h} \circ \mu_2)(\phi_1(\Psi_1), \phi_1(\Psi_2)) \,, \\
&\vdots \\
\phi_i(\Psi_1, \ldots, \Psi_i) &:= -\sum_{j=2}^{i} \frac{1}{j!} \sum_{k_1+\cdots+k_j=i} \sum_{\sigma} \pm (\mathsf{h} \circ \mu_j)\big(\phi_{k_1}\big(\Psi_{\sigma(1)}, \ldots, \Psi_{\sigma(k_1)}\big), \ldots, \phi_{k_j}\big(\ldots\big)\big)
\end{aligned}$$

Note:

- Explicit formulas for computing minimal model:
  - Formulas are recursive.
  - In case of Yang–Mills theory: Berends–Giele relations
  - Our perspective: generalizes this to all Lagrangian field theories
- Generalize to quantum L_∞-algebras
  - Formulas are recursive.
  - Exist for all Lagrangian field theories
  - Interesting mathematical challenges

Our perspective:

perturbative QFT $\leftrightarrow$ algebraic problem
$+$ analytical complications

Conclusions

Summary:

- The BV-formalism assigns to every Lagrangian field theory:
  - an equivalent classical L_∞-algebra
  - an equivalent quantum L_∞-algebra
- Minimal models $\leftrightarrow$ Scattering amplitudes
- Very useful:
  - Classical/quantum equivalence of field theories
  - Recursion relations for scattering amplitudes
  - Algebraic understanding of Feynman diagrams

Soon to come:

▷ Quantum recursion relations (WIP)

▷ MHV amplitudes from quasi-isomorphisms (WIP)

▷ Applications to Integrable Systems (WIP)

▷ Better algebraic understanding of Feynman diagrams

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Christian Saemann

School of Mathematical and Computer Sciences
Heriot-Watt University, Edinburgh

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