Generalized Berezin-Toeplitz Quantization and Aspects of the Bagger-Lambert-Gustavsson theory

Christian Sämann

School of Mathematics, Trinity College Dublin

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Based on papers written over the last three years in collaboration with
Sergey Cherkis, Vladimir Dotsenko, Seckin Kurkcuoglu, Calin Lazaroiu, Olaf Lechtenfeld, Dan McNamee, Sean Murray, Denjoe O'Connor

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1. Berezin-Toeplitz quantization of Hodge manifolds

Fuzzy $\mathbb{C}P^n$

Quantization of $\mathbb{C}P^n$ particularly nice, as it is a homogeneous space.

Underlying idea (naïve approach):

- Group theoretic: Truncate the spectrum of the Laplace operator and deform the product to obtain a closed algebra.
- Complex geometry: Quantize $\mathbb{C}^{n+1}$ and use the induced result on $\mathbb{C}P^n$.

Quantization of $\mathbb{C}^{n+1}$: $(w_\alpha, \bar{w}_\beta) \rightarrow (\hat{a}^\dagger_\alpha, \hat{a}_\beta)$

Functions on $\mathbb{C}P^n$: normalize and use Hopf fibration:

$$0 \rightarrow \mathrm{U}(1) \rightarrow S^{2n+1} \rightarrow \mathbb{C}P^n \rightarrow 0 \ .$$

This yields the quantization map:

$$\frac{1}{|w|^{2k}} w_{\alpha_1} \ldots w_{\alpha_k} \bar{w}_{\beta_1} \ldots \bar{w}_{\beta_k} \rightarrow \hat{a}^\dagger_{\alpha_1} \ldots \hat{a}^\dagger_{\alpha_k} |0\rangle\langle 0| \hat{a}_{\beta_1} \ldots \hat{a}_{\beta_k}$$

Homogeneous coordinates $\leftrightarrow$ creation/annihilation operators.

A projective algebraic variety is a subspace of $\mathbb{C}P^n$ described by a finite set of polynomial equations.

Example: $\mathbb{C}P^n$. Equivalently: $\mathrm{Proj} B_{n+1}$, $B_{n+1} = \mathbb{C}[\mathbb{C}^{n+1}]$

Example: $W\mathbb{C}P^2(1,1,2) : (z_0, z_1, z_2) \sim (\lambda z_0, \lambda z_1, \lambda^2 z_2)$

Embedding into $\mathbb{C}P^3$: $(w_0, w_1, w_2, w_3) = (z_0^2, z_0 z_1, z_1^2, z_2)$

Coordinates no longer independent, but satisfy $w_1^2 - w_0 w_2 = 0$.

This generates an ideal I in B_3, and as a projective algebraic variety, we have $W\mathbb{C}P^2(1,1,2) = \mathrm{Proj}(B_3/I)$.

Fuzzy projective algebraic varieties
Quantization of a toric variety corresponds to quantization of its toric base.

Recall the quantization map:

$$\frac{1}{|w|^{2k}} w_{\alpha_1} \ldots w_{\alpha_k} \bar{w}_{\beta_1} \ldots \bar{w}_{\beta_k} \rightarrow \hat{a}^\dagger_{\alpha_1} \ldots \hat{a}^\dagger_{\alpha_k} |0\rangle\langle 0| \hat{a}_{\beta_1} \ldots \hat{a}_{\beta_k}$$

Decompose:
the coordinate ring $B_3 = \sum_{k=0}^\infty B_{3,k}$, $\quad z_{\alpha_1} \ldots z_{\alpha_k} \in B_{3,k}$
the Fock space $\mathcal{F} = \sum_{k=0}^\infty \mathcal{F}_k$, $\quad \hat{a}^\dagger_{\alpha_1} \ldots \hat{a}^\dagger_{\alpha_k} |0\rangle \in \mathcal{F}_k$
Quantization maps $B_{3,k} \cdot \bar{B}_{3,k}$ to $\mathcal{F}_k \cdot \mathcal{F}_k^*$

Idea:

To quantize $\mathrm{Proj}(B_3/I)$, map $B_{3,k}/I \cdot \bar{B}_{3,k}/I$ to $\mathcal{F}_k/\hat{I} \cdot \mathcal{F}_k^*/\hat{I}$.
If $f(w_0, \ldots, w_n)$ generates I, then $\mathcal{F}_k/\hat{I}$ refers to those elements $|\mu\rangle \in \mathcal{F}_k$ which satisfy $\hat{f}(\hat{a}_0, \ldots, \hat{a}_n)|\mu\rangle = 0$.

Quantization of toric variety: quantization of its toric base.
In this manner: even singular varieties. CS: hep-th/0612173

Donaldson's algorithm for computing Calabi-Yau metrics

The algorithm employs methods strongly reminiscent of our quantization procedure.

Calabi-Yau manifold:

Kähler manifolds appearing in superstring compactifications as $M^4 \times CY$, admit Ricci-flat metric in every Kähler class.

To extract information about the low-energy physics: need metric.
$\Rightarrow$ Donaldson's algorithm:
Expand Kähler potential K: $\omega = \partial\bar{\partial}K$ to order k as

$$K = \ln h^{\alpha_1 \dots \alpha_k \beta_1 \dots, \beta_k} w_{\alpha_1} \dots w_{\alpha_k} \bar{w}_{\beta_1} \dots \bar{w}_{\beta_k}$$

Factoring out the ideal $I(f)$: $w_{\alpha_1} \dots w_{\alpha_k} \longrightarrow$ basis set (s_i).
A metric determined by $K = \ln h^{ij} s_i \bar{s}_j$ is called balanced, if

$$(h^{ij})^{-1} = \langle s_i, s_j \rangle = \frac{N_k}{\mathrm{vol}_C Y} \int_{CY} \mathrm{d}\mu_{CY}(x) \frac{s_i \bar{s}_j}{h^{kl} s_k \bar{s}_l} .$$

Theorem: For each $k \geq 0$, the balanced metric exists and is unique. For $k \to \infty$: convergence to the Ricci-flat metric.
Algorithm: Solve by iteration at each level k, convergence fast.

To see, how our quantization procedure might be applied here, let us look more closely at the procedure. $\longrightarrow$ Geometric Quantization

X compact complex manifold
L polarization of X, i.e. a positive (ample) holomorphic line bundle
Kähler metric on $X \leftrightarrow$ hermitian metric h on L up to rescaling:

$$\omega = \frac{\mathrm{i}}{2\pi} F = -\partial\bar{\partial} \ln h(\sigma,\sigma) \ , \quad \sigma \in H^0(L) \text{ over patch } \sigma(x) \neq 0$$

Replace L by very ample: L^k (basis of sections yields $X \hookrightarrow \mathbb{C}P^n$)

Consequences: $h_k = h^{\otimes k}$ and $\nabla_k = \nabla^{\otimes k}$, $E = H^0(L^k) = \mathrm{span}(s_i)$

(X, ω, L, h): prequantized Hodge manifold.

Let $\mathbb{L}$ be the total space of L and $\mathbb{L}_0 = \mathbb{L}\backslash o$. Also: $\pi : \mathbb{L} \to X$.

$$s(\pi(q)) =: \hat{q}(s) q \ , \quad q \in \mathbb{L}_0 \ , \quad s \in H^0(L)$$

$\hat{q}(s)$: "How much does one have to scale s to pass through q"
By Riesz's theorem, there is a unique holomorphic section e_q with:

$$(e_q, s) = \hat{q}(s)$$

Define $G_{ij} = (s_i, s_j)$, then

$$e_q = G^{ji}\overline{\hat{q}(s_i)} s_j$$

Introduce the Rawnsley's coherent state projectors:

$$P_x := \frac{|e_q)(e_q|}{(e_q|e_q)} \ , \quad q \in L_x\backslash\{0\}$$

P_x depends only on $x \in X$, L and G.

A simple integral identity can be derived for the Rawnsley coherent states.

$$P_x := \frac{|e_q)(e_q|}{(e_q|e_q)}\ , \quad q \in L_x \backslash \{0\}$$

Introduce the ε-function

$$\varepsilon(x) := h(q,q)||e_q||^2 = G^{ij} h(x)(s_i(x), s_j(x))$$

Consider the scalar product given by $\mu(x)$ and h:

$$(s,t) = \int_X \mathrm{d}\mu(x) h(x)(s(x), t(x)) = \int_X \mathrm{d}\mu(x)\varepsilon(x)(s|P_x|t)\ .$$

Thus:

$$\int_X \mathrm{d}\mu(x)\varepsilon(x) P_x = \mathrm{id}_E$$

Scalar product on E balanced:

$$\varepsilon(x) = \frac{\mu(X)}{N+1}$$

Berezin and Berezin-Toeplitz quantization

There are two quantization procedures making use of P_x.

Linear operator $C \in \mathsf{End}\,(E)$. Define its lower Berezin symbol:

$$\sigma(C)(x) := \operatorname{tr}(CP_x) = \frac{(e_q|C|e_q)}{(e_q|e_q)}$$

Call $\sigma(\mathsf{End}\,(E)) =: \Sigma$.
Define the Berezin quantization of $f \in \Sigma$ as: $\sigma^{-1}(f)$.

The Toeplitz quantization $T : C^\infty(X) \to \mathsf{End}\,(E)$ is defined:

$$T(f) = \int_X \frac{\omega^n}{n!} \varepsilon(x) f(x) P_x$$

Properties: $T(\bar{f}) = T(f)^\dagger$ and $T(1_X) = \mathbb{1}_E$

Generalized Berezin quantization

The Berezin quantization can be generalized to arbitrary scalar products.

If $(\ ,\ )$ is induced from h on L and $\Omega = \frac{\omega^n}{n!}$, then σ is injective.

What happens if we change the scalar product?

$$(s,t)' = (As,t) = (s,At)\ ,$$

where A Hermitian, positive-definite. Consequences:

$$e'_q = A^{-1}e_q \quad , \quad P'_x = \frac{1}{\sigma(A^{-1}(x))}A^{-1}P_x \quad , \quad \sigma'(C) = \frac{\sigma(CA^{-1})}{\sigma(A^{-1})}$$

$\Rightarrow$ Two (generalized) Berezin quantizations agree, if the operator A is proportional to the identity:
$\sigma'(C) = \sigma(C) \Rightarrow \forall_C : \sigma(C)\sigma(A) = \sigma(A)\sigma(C)$
$\Rightarrow \forall_C : \sigma(AC) = \sigma(CA) \Rightarrow \forall_C : [A,C] = 0 \Rightarrow A = \lambda \mathbb{1}_E$.
(Similarly: generalized Toeplitz quantization.)

Berezin-Bergman quantization

This quantization procedure is a special case of generalized Berezin quantization.

Consider (X, L), $E_k = H^0(L^k)$
Homogeneous coordinate ring: $R(X, L) = \oplus_{k=0}^{\infty} E_k$
We have $R \cong_{[k]} B/I$, where $B = \oplus_{k=0}^{\infty} E_1^{\odot k}$

Two ways of introducing a scalar product on E_k:

$$\langle r, t\rangle_k = \int_X \frac{\omega^n}{n!} h^{\otimes k}(r, t) \quad \Rightarrow \quad \text{ordinary Berezin/Toeplitz theory}$$

$$(r_1 \odot \ldots r_k, t_1 \odot \ldots t_k)_B = \frac{1}{k!}\delta_{k,l} \sum_{\sigma \in S_k} (r_1, t_{\sigma(1)})_1 \ldots (r_k, t_{\sigma(k)})_1$$

Choose $(s_\alpha, s_\beta) = \delta_{\alpha\beta}$: metric on E_k is implied in using
$(w_\alpha, \bar{w}_\beta) \rightarrow (\hat{a}_\alpha^\dagger, \hat{a}_\beta)$, $\hat{f}|\mu\rangle = 0$, $(\mu, \nu) := \langle \mu | \nu \rangle$

To obtain integral formulas, recall the overcompleteness relation for coherent states $\int_X \mathrm{d}\mu(x)\varepsilon(x)P_x = \mathrm{id}_E$, which implies that

$$\int_X \mathrm{d}\mu(x)\varepsilon(x)f(x) = \int_X \mathrm{d}\mu(x)\varepsilon(x)\,\mathrm{tr}\,(P_x\hat{f}) = \mathrm{tr}\,(\hat{f})\ .$$

For balanced metrics, $\varepsilon(x) = 1$, otherwise: introduce the operator

$$\hat{\rho} = T\left(\frac{1}{\mathrm{vol}_\omega(X)\varepsilon(x)}\right) = \frac{1}{\mathrm{vol}_\omega(X)}\int_X \frac{\omega^n}{n!}P_x\ ,$$

such that

$$\frac{1}{\mathrm{vol}_\omega(X)}\int_X \frac{\omega^n}{n!}f(x) = \frac{1}{\mathrm{vol}_\omega(X)}\int_X \frac{\omega^n}{n!}\,\mathrm{tr}\,(P_x\hat{f}) = \mathrm{tr}\,(\hat{\rho}\hat{f})\ .$$

Thus: can integrate over arbitrary measures in quantum picture.

To obtain useful integral formulas, which are needed for iterating

$$h_{n+1,ij} = \frac{N_k}{\mathrm{vol}_C Y} \int_{CY} \mathrm{d}\mu(x) \frac{s_i \bar{s}_j}{h_n^{kl} s_k \bar{s}_l} \; ,$$

we would have to compute the matrix A which takes us from

$$\int_X \frac{\omega^n}{n!} h^{\otimes k}(r,t) \quad \Rightarrow \quad \frac{1}{k!} \delta_{k,l} \sum_{\sigma \in S_k} (r_1, t_{\sigma(1)})_1 \ldots (r_k, t_{\sigma(k)})_1$$

This involves computing even more integrals.
Trivial only for balanced case.

⇒ Never try to be smarter than Donaldson, unless you have a very good reason for doing so.

First idea: the quantum versions $O^B : \mathsf{End}\,(E) \to \mathsf{End}\,(E)$ of a differential operator $O : \Sigma \to \Sigma$ should act on quantized functions as they do on ordinary functions:

$$O^B \hat{f} := \sigma^{-1}(\Pi_{L^2}(O(\sigma(\hat{f}))))$$

We call this the Berezin push of an operator. (Analogously, define the Berezin pull $O^B \to O$.)

This definition can be used to perform approximate harmonic analysis on projective varieties.

Approximate harmonic analysis on Fermat curves

The Berezin-push can be used to analyze the spectrum of Δ approximatly.

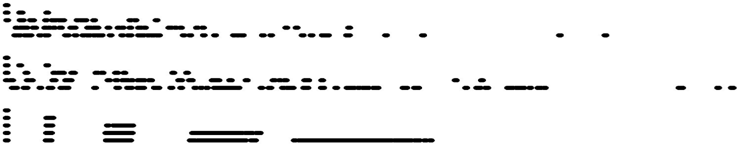
Example:

Fermat curve: projective algebraic variety $X_p \subset \mathbb{C}P^2$ given by

$$f(w_0, w_1, w_2) = w_0^p + w_1^p + w_2^p = 0\ .$$

Endow X_p with the Bergman metric obtained by pulling back the Fubini-Study metric from $\mathbb{C}P^2$, which determines the Laplacian. Calculate the matrix $\Delta_{(ij)(kl)}$ with $\Delta^B(s_k\bar{s}_l) = \Delta_{(ij)(kl)} s_i \bar{s}_k$ and determine its eigenvalues.

Examples: $X_2, X_3, \mathbb{C}P^2$:



However: If O hermitian with respect to $(\ ,\ )_\omega$, its Berezin push is not hermitian with respect to $\langle\ ,\ \rangle_{HS}$!

Second idea: preserve hermiticity:

$$\langle T(f), C\rangle_{HS} = \mathrm{tr}\,(T(f)^\dagger C) = \int_X d\mu(x)\epsilon(x)\bar{f}(x)\sigma(C) = (f, \sigma(C))_{\varepsilon,\omega}$$

Thus: $\sigma^\oplus = T \circ M_{\frac{1}{\varepsilon}}$, and define the Berezin-Toeplitz lift of O:

$$\hat{O} = \sigma^\oplus \circ O \circ \sigma = T \circ M_{\frac{1}{\varepsilon}} \circ O \circ \sigma\ .$$

This procedure preserves hermiticity, but $\hat{1} \neq \mathbb{1}_{\mathsf{End}\,(E)}$.

Fuzzy scalar field theory

Scalar field theories can be regularized using quantized spaces.

Scalar field theory on (X, ω) is defined by a functional $S[\phi]$:

$$S[\phi] := \frac{1}{\mathrm{vol}_\omega(X)} \int_X \frac{\omega^n}{n!} \left(\phi \Delta \phi + V(\phi)\right) \ , \quad \phi \in \mathcal{C}^\infty(X, \mathbb{R}) \quad ,$$

where $V(\phi) = \sum_{s=0}^{d} a_s \phi^s$. The corresponding quantum version:

$$S_q[\phi] := \frac{1}{\mathrm{vol}_\omega(X)} \mathrm{tr} \left(\Phi \hat{\Delta} \Phi + V(\Phi)\right) \ , \quad \Phi \in \mathsf{End}\,(E) \quad ,$$

The functional integral $\int \mathcal{D}[\phi]$ in the partition function

$$Z_q = \int \mathcal{D}[\Phi] \mathrm{e}^{-S[\Phi]}$$

becomes a well-defined finite-dimensional integral.

$\Rightarrow$ We have all the necessary ingredients for defining precisely quantum field theories on arbitrary (quantum) Hodge manifolds.

What remains the same:

- Kodaira embedding theorem translates (LeBrun, Poon, Wells)
- Rawnsley coherent states can be translated
- Our quantization procedures, up to technicalities

New features:

- Non-split supermanifolds require more detailed analysis
- Normalizations, as supermanifolds can have zero volume
- Regularization of SUSY theories not completely understood

Interestingly, regularizing SUSY field theories works when using CY supermanifolds: the volume form of $\mathbb{C}^{1|2}$ and the local volume form on $\mathbb{C}P^{1|2}$ agree.

Lazaroiu, McNamee, CS, 0811.4743

Applying matrix model techniques to fuzzy field theories

Fuzzy scalar field theory is significantly harder than matrix models usually considered.

Fuzzy scalar field theory: $Z = \int \mathrm{d}\mu_D(\Phi)\ \mathrm{e}^{-\operatorname{tr}\left(a[L_i,\Phi][L_i,\Phi]+b\,\Phi^2+c\,\Phi^4\right)}$

One-Hermitian Matrix Model

$$Z \;=\; \int \mathrm{d}\mu_D(\Phi)\ \mathrm{e}^{-\operatorname{tr}\left(b\,\Phi^2+c\,\Phi^4\right)}$$

Solution: splitting $\Phi = \Omega\Lambda\Omega^\dagger$, $\Lambda \;=\; \mathrm{diag}(\lambda_1,\ldots,\lambda_N)$ as well as $\int \mathrm{d}\mu_D(\Phi) \;=\; \int \prod_{i=1}^N \mathrm{d}\lambda_i \Delta^2(\Lambda) \int \mathrm{d}\mu_H(\Omega)$ yields

$$Z \;=\; \int \prod_{i=1}^N \mathrm{d}\lambda_i\ \mathrm{e}^{-2\sum_{i>j}\ln|\lambda_i-\lambda_j|-b\sum_i \lambda_i^2+c\sum_i \lambda_i^4}$$

From here: saddle point, orthogonal polynomials, etc.

Difficulty in our case: multiple external matrices.

(Single one solvable as found by Itzykson and Di Francesco)

The angular variables can be integrated out in the perturbative series.

Introduce $K_{ab} := \mathrm{tr}\,([L_i,\tau_a][L_i,\tau_b])$, $\Phi^a = \mathrm{tr}\,(\tau^a\Omega\Lambda\Omega^\dagger)$. Then:

$$\mathrm{e}^{a\Phi^a K_{ab}\Phi^b} = 1 + a\Phi^a K_{ab}\Phi^b + \frac{a^2}{2}\Phi^a K_{ab}\Phi^b\,\Phi^c K_{cd}\Phi^d + \dots$$

To integrate over $\mathrm{d}\mu_H(\Omega)$ we need to compute terms like

$$\int \mathrm{d}\mu_H(\Omega) K_{ab}\,\mathrm{tr}\,(\tau^a\Omega\Lambda\Omega^\dagger)\,\mathrm{tr}\,(\tau^b\Omega\Lambda\Omega^\dagger)$$

Recall: $$\int \mathrm{d}\mu_H(\Omega)\ \ [\rho(\Omega)]_{ij}\ [\rho^\dagger(\Omega)]_{kl} = \frac{1}{\dim(\rho)}\delta_{il}\delta_{jk}$$

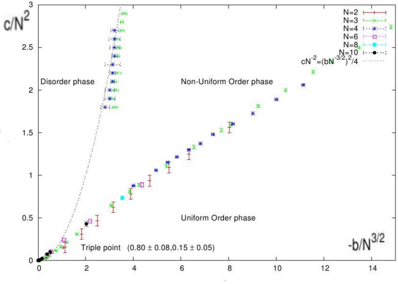
$$\mathrm{tr}\,\big((\tau^a\Omega\Lambda\Omega^\dagger)\otimes(\tau^b\Omega\Lambda\Omega^\dagger)\big) = \mathrm{tr}\,\big((\tau^a\otimes\tau^b)(\Omega\otimes\Omega)(\Lambda\otimes\Lambda)(\Omega^\dagger\otimes\Omega^\dagger)\big)$$

Thus: $$\int \mathrm{d}\mu_H(\Omega) K_{ab}\Phi^a\Phi^b = K_{ab}\sum_\rho \tfrac{1}{\dim(\rho)}\,\mathrm{tr}\,_\rho(\tau^a\otimes\tau^b)\,\mathrm{tr}\,_\rho(\Lambda\otimes\Lambda)$$

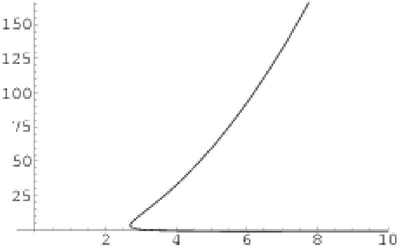
Results from saddle point approximation in large-N limit

Phase diagrams show rough agreement.

The boundary of the region of validity of the one-cut solution is consistent with the data.



Numerics, fuzzy ϕ^4-theory



Analytical results, multi-trace MM

Triple pt: $(-2.3 \pm 0.2, 0.52 \pm 0.02)$ Turning pt: $(-2.7, 0.25)$

$\Rightarrow$ Turning point corresponds to triple point.

Past work:

- Explicit relation between intrinsic and extrinsic quantization
- Formulas for integrals and differential operators on quantized manifolds
- Huge new classes of regularization of scalar field theories
- Demonstrated the applicability of matrix model techniques
- Quantization procedures extended to supermanifolds
- Regularization of examples of SUSY field theories

Future directions:

- Improve understanding of regularization in SUSY case
- Study quantized toric varieties (flip transitions, etc.)
- Quantization of singular projective varieties
- Continue looking for fuzzy shortcut for Donaldson's algorithm

2. Multiple M2-branes: The Bagger-Lambert-Gustavsson model

A stack of flat M2-branes in $\mathbb{R}^{1,10}$ should be effectively described by a conformal field theory with the following constraints:

Spacetime symmetries: $\mathrm{SO}(1,10) \rightarrow \mathrm{SO}(1,2) \times \mathrm{SO}(8)$
extended by $\mathcal{N} = 8$ SUSY.

Field content: X^I, $I = 1, ..., 8$, and superpartners Ψ_α

Assumption

SUSY transformations from Basu-Harvey equation and therefore the matter fields take values in a metric 3-Lie algebra

metric 3-Lie algebras

$\mathcal{A}$ a real vector space with a bracket $[\cdot,\cdot,\cdot]:\Lambda^3\mathcal{A}\to\mathcal{A}$ satisfying

$$[A,B,[C,D,E]] = [[A,B,C],D,E]+[C,[A,B,D],E]+[C,D,[A,B,E]] \quad \text{(FI)}$$

and a bilinear symmetric map $(\cdot,\cdot)_{\mathcal{A}}:\mathcal{A}\otimes\mathcal{A}\to\mathcal{A}$ satisfying

$$([A,B,C],D)_{\mathcal{A}}+(C,[A,B,D])_{\mathcal{A}} = 0 \quad \text{(Cmp)}$$

Action of $\mathfrak{g}_{\mathcal{A}}:=\mathcal{A}\wedge\mathcal{A}$ on $\mathcal{A}$ given by linearly extending

$$(A\wedge B)\rhd C:=[A,B,C]\ ,\quad A,B,C\in\mathcal{A}$$

Because of (FI), the commutator of two such actions is again of this type. Therefore: Lie algebra structure on $\mathfrak{g}_{\mathcal{A}}$.

Two invariant pairings on $\mathfrak{g}_{\mathcal{A}}$: $(A\wedge B,C\wedge D)_{\mathfrak{g}}:=([A,B,C],D)_{\mathcal{A}}$ and induced Killing form.

The Bagger-Lambert-Gustavsson model

This model is an unconventional supersymmetric Chern-Simons matter theory.

BLG found that for SUSY, we need to introduce gauge symmetry.
$\Rightarrow$ additional gauge potential A_μ taking values in $\mathfrak{g}_\mathcal{A}$.

Simplify: Clifford alg. $Cl(\mathbb{R}^{1,10})$, $X := \Gamma_I X^I$, $\{\Gamma_I, \Gamma_J\} = 2\eta_{IJ}$
$(A,B)_{\mathcal{A}\otimes\mathcal{C}} := \frac{1}{32}\operatorname{tr}_\mathcal{C}\big((A,B)_\mathcal{A}\big)$, $[\cdot,\cdot,\cdot]$ linearly extended

The Bagger-Lambert-Gustavsson model

$$\begin{aligned}\mathcal{L}_{\rm BLG} \;=\; &+\tfrac{1}{2}\varepsilon^{\mu\nu\kappa}\left((A_\mu,\partial_\nu A_\kappa)_\mathfrak{g}+\tfrac{1}{3}(A_\mu,[A_\nu,A_\kappa])_\mathfrak{g}\right)\\ &-\tfrac{1}{2}(\nabla_\mu X,\nabla^\mu X)_{\mathcal{A}\otimes\mathcal{C}}+\tfrac{\mathrm{i}}{2}(\bar\Psi,\Gamma^\mu\nabla_\mu\Psi)_\mathcal{A}\\ &+\tfrac{\mathrm{i}}{4}(\bar\Psi,[X,X,\Psi])_\mathcal{A}-\tfrac{1}{12}([X,X,X],[X,X,X])_{\mathcal{A}\otimes\mathcal{C}}\end{aligned}$$

$$\delta X = \mathrm{i}\Gamma_I\bar\varepsilon\Gamma^I\Psi\ ,\quad \delta\Psi = \nabla_\mu X\Gamma^\mu\varepsilon-\tfrac{1}{6}[X,X,X]\varepsilon\ ,$$
$$\delta A_\mu = \mathrm{i}\bar\varepsilon\Gamma_\mu(X\wedge\Psi)$$

Manifestly $\mathcal{N}=2$ SUSY formulation

There is a manifestly $\mathcal{N}=2$ SUSY formulation, allowing for various deformations.

Take $\mathcal{N}=1$, $d=4$ superspace $\mathbb{R}^{1,3|4}$ and dim. reduce along x^2.

Superfields on $\mathbb{R}^{1,2|4}$:

$$\begin{aligned}\Phi^i(y) &= \phi^i(y)+\sqrt{2}\theta\psi^i(y)+\theta^2F^i(y)\ ,\\ V(x) &= -\theta^\alpha\bar\theta^{\dot\alpha}(\sigma^\mu_{\alpha\dot\alpha}A_\mu(x)+\mathrm{i}\varepsilon_{\alpha\dot\alpha}\sigma(x))\\ &\quad +\mathrm{i}\theta^2(\bar\theta\bar\lambda(x))-\mathrm{i}\bar\theta^2(\theta\lambda(x))+\tfrac{1}{2}\theta^2\bar\theta^2D(x)\ ,\end{aligned}$$

$\mathcal{N}=2$ superspace formulation of BLG (Cherkis, CS, 0807.0808)

$$\begin{aligned}L=\int \mathrm{d}^4\theta\ \kappa\left(\mathrm{i}(V,(\bar D_\alpha D^\alpha V))_{\mathfrak{g}}+\tfrac{2}{3}(V,\{(\bar D^\alpha V),(D_\alpha V)\})_{\mathfrak{g}}\right)\\ +(\bar\Phi_i,\mathrm{e}^{2iV}\rhd\Phi^i)_{\mathcal{A}}+\alpha\left(\int \mathrm{d}^2\theta\ \varepsilon_{ijkl}([\Phi^i,\Phi^j,\Phi^k],\Phi^l)_{\mathcal{A}}+c.c.\right)\end{aligned}$$

This Lagrangian is not manifestly gauge invariant.

Admissible 3-algebraic structures

Gauge invariance leads to much freedom.

Demanding gauge invariance in above theory yields the condition:

$$
\begin{aligned}
([A,B,C],D)_{\mathcal{A}} &= -([B,A,C],D)_{\mathcal{A}} \\
&= -([A,B,D],C)_{\mathcal{A}} = ([C,D,A],B)_{\mathcal{A}}
\end{aligned}
$$

With (FI) and (Cmp), this defines Generalized 3-Lie algebras.

Example:
Take a Clifford algebra $Cl(\mathbb{R}^{2d})$ generated by γ_a. Define:

$$[\gamma_a,\gamma_b,\gamma_c] := [[\gamma_a,\gamma_b]\gamma_{ch},\gamma_c]\ , \quad (\gamma_a,\gamma_b)_{\mathcal{A}} = \operatorname{tr}(\gamma_a^\dagger\gamma_b)$$

Other Generalizations: hermitian and unitary 3-algebras.

All these fit into a nice, unifying picture.
(Medeiros, Figueroa-O'Farrill, Mendez-Escobar, Ritter, 0809.1086)

Classifications of Matrix Representations of 3-Algebras

Using the elementary operations in matrix algebras, representations can be constructed.

Matrix representation of a (metric) 3-algebra:

Take a matrix $*$-algebra equipped with a trace form. Construct a 3-bracket on this algebra from matrix products and the involution and use the Hilbert-Schmidt scalar product $(A, B) = \mathrm{tr}\,(A^\dagger B)$.

Classification of all such representations in the real and hermitian case using MuPad done in Cherkis, Dotsenko, CS, 0812.3127

Example: The Real case. $[A, B, C] :=$

$$
\begin{aligned}
I &: \alpha([[A^*, B], C] + [[A, B^*], C] + [[A, B], C^*] - [[A^*, B^*], C^*]) \\
II &: \alpha([[A, B^*], C] + [[A^*, B], C]) \\
III &: \alpha(AB^* - BA^*)C + \beta C(A^*B - B^*A) \\
IV &: \alpha([[A, B], C] + [[A^*, B^*], C] + [[A^*, B], C^*] + [[A, B^*], C^*]) \\
&\quad + \beta([[A, B], C^*] + [[A^*, B], C] + [[A, B^*], C] + [[A^*, B^*], C^*])\ .
\end{aligned}
$$

Manifestly $\mathcal{N}=4$ supersymmetric formulation

In projective superspace, one can make $\mathcal{N}=4$ SUSY in the BLG model manifest.

Projective superspace in 4d

$\mathcal{N}=2$ SUSY covariant derivatives on $\mathbb{R}^{1,3|8}$:

$$\{D_{i\alpha}, D_{j\beta}\} = 0 \quad \{\bar{D}^i_{\dot\alpha}, D^j_{\dot\beta}\} = 0 \quad \{D_{i\alpha}, \bar{D}^j_{\dot\alpha}\} = -2\mathrm{i}\delta_i^j \sigma^\mu_{\alpha\dot\alpha}\partial_\mu$$

add $\zeta \in U_0 \subset \mathbb{C}P^1$ parameterizing $\mathcal{N}=2$ within $\mathcal{N}=1$:

$$\nabla_\zeta = D_1 + \zeta D_2 \ , \quad \bar\nabla_\zeta = -\zeta \bar{D}^1 + \bar{D}^2$$

Projective superspace: $\mathbb{R}^{1,3|8} \times \mathbb{C}P^1$ "divided by" ∇_ζ, $\bar\nabla_\zeta$.

Field content of the BLG model encoded after dim. red. as:

2 hyper- or $\mathcal{O}(4)$ multiplets: $\eta = \bar\Phi\frac{1}{\zeta^2} + \bar\Sigma\frac{1}{\zeta} + X - \zeta\Sigma + \zeta^2\Phi$

1 vector or tropical multiplet: $\mathcal{V}(\zeta,\bar\zeta) = \sum_{n=-\infty}^{\infty} v_n \zeta^n$

Supersymmetric action:

$$\int \mu\,\kappa\Big(\mathrm{i}(\mathcal{V}, (\bar{\mathcal{D}}_\alpha \mathcal{D}^\alpha \mathcal{V}))_{\mathfrak{g}} + \tfrac{2}{3}(\mathcal{V}, \{(\bar{\mathcal{D}}^\alpha \mathcal{V}), (\mathcal{D}_\alpha \mathcal{V})\})_{\mathfrak{g}}\Big) + \big(\bar\eta_k, \mathrm{e}^{2\mathrm{i}\mathcal{V}} \rhd \eta_k\big)_{\mathcal{A}}$$

L_∞-algebras and homotopy Maurer-Cartan equations

The eom of the BLG model can be reformulated as homotopy Maurer-Cartan equations.

L_∞- or strongly homotopy Lie algebras

- Introduced by Stasheff (1963) “only way to extend Lie algebras”
- appear in string FT, top. conf. FT, Morse theory

Definition:

R-module L, with family of R-multilinear maps $\mu_n : L^{\times n} \to L$ s.t.:

$$\mu_n(x_{\sigma(1)} \dots x_{\sigma(n)}) = \epsilon(\sigma)\mu_n(x_1 \dots x_n)$$

$$\sum_{i=1}^{n} \sum_{\sigma \in Sh(i,n-i)} (-1)^{i(n+1)} \epsilon(\sigma) \mu_{n-i+1}(\mu_i(x_{\sigma(1)}, \dots, x_{\sigma(i)}), x_{\sigma(i+1)}, \dots, x_{\sigma(n)}) = 0$$

There is also a graded version, then μ_n is of degree $2-n$.

Note: μ_1 is a differential, $\mu_1, \mu_2 \neq 0 \to$ diff. (grad.) Lie algebra

Interestingly, n-Lie algebras are (ungraded) L_∞-algebras (Hanlon, Wachs 1995, Dzhumadil’daev, math/0202043)

L_∞-algebras and homotopy Maurer-Cartan equations

The eom of the BLG model can be reformulated as homotopy Maurer-Cartan equations.

homotopy Maurer-Cartan equation

Given a (graded) L_∞-algebra $\mathcal{L} = \oplus_i \mathcal{L}_i$,

$$\sum_{\ell \geq 0} \frac{(-1)^{\ell(\ell+1)/2}}{\ell!} \mu_\ell(\varphi^{\otimes \ell}) = 0 \ , \quad \varphi \in \mathcal{L}$$

is invariant under the gauge transformations

$$\delta\varphi = -\sum_{\ell \geq 1} \frac{(-1)^{\ell(\ell-1)/2}}{(\ell-1)!} \mu_\ell(\alpha \otimes \varphi^{\ell-1}) \ , \quad \alpha \in \mathcal{L}_0$$

Andrei Losev: "All classical equations of motion are of hMC form."

Example: The Nahm equation $\nabla_s X^i + \varepsilon^{ijk}[X^j, X^k] = 0$

$\mathcal{L} := \Omega^\bullet(\mathbb{R}, Cl(\mathbb{R}^3)) \otimes su(N)$, also $X = \gamma_i X^i$, $\widetilde{\mathrm{d}s} = 1$, $\widetilde{\gamma_i} = 1$

$\mu_2(\lambda_1, \lambda_2) := [\lambda_1, \lambda_2]$ $\quad \mu_2(\lambda, A) := [\lambda, A]$ $\quad \mu_2(\lambda, X) := [\lambda, X]$

$\mu_2(A, X) := [A, X]$ $\quad \mu_2(X, X) := [X, X]\mathrm{d}s$

This reproduces eom and gauge symmetry correctly.

BLG equations of motion (bosonic part):

$$\begin{aligned}\nabla_\mu \nabla^\mu X + \tfrac{1}{2}\Gamma[X,X,\Gamma[X,X,X]] &= 0\\ [\nabla_\mu,\nabla_\nu] + \varepsilon_{\mu\nu\kappa}(\operatorname{tr}_{\mathcal{C}}(X\wedge(\nabla^\kappa X))) &= 0\end{aligned}$$

Start with 3-Lie algebra L and introduce the module

$$\mathcal{L} := \Omega^\bullet(\mathbb{R}^3)\otimes_{\mathbb{C}} Cl_8 \otimes (L\oplus \mathfrak{g}_L)$$

define gradings:

$$\deg(\Omega^0(\mathbb{R}^3)\otimes_{\mathbb{C}} Cl_{8,0}\otimes\mathfrak{g}_L) = 0$$

$$\deg(\Omega^1(\mathbb{R}^3)\otimes_{\mathbb{C}} Cl_{8,0}\otimes\mathfrak{g}_L) = \deg(\Omega^0(\mathbb{R}^3)\otimes_{\mathbb{C}} Cl_{8,1}\otimes L) = 1$$

$$\deg(\Omega^2(\mathbb{R}^3)\otimes_{\mathbb{C}} Cl_{8,0}\otimes\mathfrak{g}_L) = \deg(\Omega^3(\mathbb{R}^3)\otimes_{\mathbb{C}} Cl_{8,1}\otimes L) = 2$$

The fields will live in the following subspaces:

$$A\in\Omega^1(\mathbb{R}^3)\otimes Cl_{8,0}\otimes\mathfrak{g}_L \qquad X\in\Omega^0(\mathbb{R}^3)\otimes Cl_{8,1}\otimes L$$

$$\lambda\in\Omega^0(\mathbb{R}^3)\otimes Cl_{8,0}\otimes\mathfrak{g}_L$$

L_∞-algebras and homotopy Maurer-Cartan equations

The eom of the BLG model can be reformulated as homotopy Maurer-Cartan equations.

BLG equations of motion (bosonic part):

$$\begin{aligned}\nabla_\mu\nabla^\mu X + \tfrac{1}{2}\Gamma[X,X,\Gamma[X,X,X]] &= 0\\ [\nabla_\mu,\nabla_\nu] + \varepsilon_{\mu\nu\kappa}(\operatorname{tr}_{\mathcal{C}}(X\wedge(\nabla^\kappa X))) &= 0\end{aligned}$$

Define the following brackets:

$$\mu_1(A) := \mathrm{d}A \qquad \mu_2(A,A) := [[A\wedge A]]\ ,$$

$$\mu_2(X,X) := *\tau(X\wedge \mathrm{d}X) \qquad \mu_3(A,X,X) := *\tau(X\wedge[A,X])$$

$$\mu_1(X) := \Delta X\omega \qquad \mu_2(A,X) := \partial_\mu[A^\mu,X]\omega + [A_\mu,\partial^\mu X]\omega$$

$$\mu_3(A,A,X) := [A_\mu,[A^\mu,X]]\omega \qquad \mu_5(X^{\otimes 5}) := \Gamma[X,X,\Gamma[X,X,X]]$$

further brackets consistently from the homotopy Jacobi identities.

The hMC equations $\sum_{\ell\geq 0}\frac{(-1)^{\ell(\ell+1)/2}}{\ell!}\mu_\ell(\varphi^{\otimes\ell}) = 0$ reproduce the BLG model together with its gauge invariance. (SUSY extension)

Lazaroiu, McNamee, Saemann, Zejak, [0901.????]

Past work:

- Identification of extended 3-algebraic strctures
- Classification of categorical matrix representations
- Manifestly $\mathcal{N}=2$ and $\mathcal{N}=4$ supersymmetric formulations of the BLG-like models
- Identification of L_∞-algebra structure
- BLG eoms rewritten as homotopy Maurer-Cartan equations

Future directions:

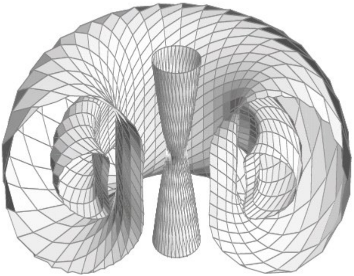
- What is the role of L_∞-algebras? Extendable? Classifications?
- Which 3-algebras yield Hamiltonians of integrable spin chains?
- Extend SUSY models by Yang-Mills term, analyze
- Lift the Nahm/Fourier-Mukai transform to M-theory
- Ultimately: find analogous models for M5 branes
- Lift a D-brane correspondence to M-theory...

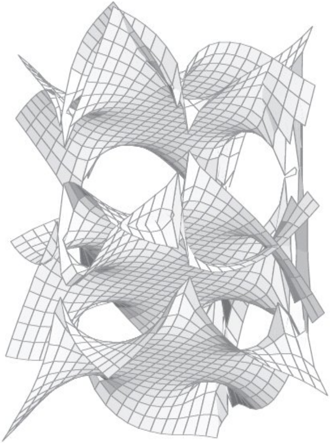
3. Example of possible future research: Lift a D-brane correspondence to M-theory

Underlying Idea of Twistor String Theory

To make contact with string theory, we need to extend this picture supersymmetrically.

Marrying Twistor- and Calabi-Yau geometry





... with supermanifolds: Witten, hep-th/0312171

Supertwistor Space

The supertwistor space $\mathcal{P}^{3|\mathcal{N}}$ is a holomorphic vector bundle of rank $3|4\mathcal{N}$ over $\mathbb{C}P^1$.

The Supertwistor Space $\mathcal{P}^{3|\mathcal{N}}$

Start from $\mathbb{C}P^{3|\mathcal{N}}$, take out $\mathbb{C}P^{1|\mathcal{N}}$ at infinity:

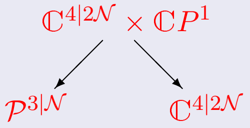
$$\mathcal{P}^{3|\mathcal{N}} := \mathbb{C}^2 \otimes \mathcal{O}(1) \oplus \mathbb{C}^{\mathcal{N}} \otimes \Pi\mathcal{O}(1) \to \mathbb{C}P^1$$

Incidence Relations

$$\omega^\alpha = x^{\alpha\dot{\alpha}}\lambda_{\dot{\alpha}}$$

$$\eta_i = \eta_i^{\dot{\alpha}}\lambda_{\dot{\alpha}}$$

Double Fibration



First Chern Class of $\mathcal{P}^{3|4}$

$T\mathbb{C}P^1$ 2, $\mathcal{O}(1)$ 1, $\Pi\mathcal{O}(1)$ -1, in total: $c_1 = 0$.
Therefore, there exists a holomorphic measure $\Omega^{3,0|4,0}$.

Penrose-Ward Transform on $\mathcal{P}_\tau^{3|4}$

Imposing reality conditions simplifies the situation significantly.

Introducing a real structure τ, the double fibration collapses:

$$\begin{array}{ccccc} & \mathbb{C}^{4|2\mathcal{N}} \times \mathbb{C}P^1 & & & \\ \swarrow & & \searrow & \longrightarrow & \mathcal{P}_\tau^{3|\mathcal{N}} \to \mathbb{R}_\tau^{4|2\mathcal{N}} \\ \mathcal{P}^{3|\mathcal{N}} & & \mathbb{C}^{4|2\mathcal{N}} & & \end{array}$$

($\tau_{\pm 1}$ related to Kleinian and Euclidean metrics on $\mathbb{R}_\tau^{4|2\mathcal{N}}$.)

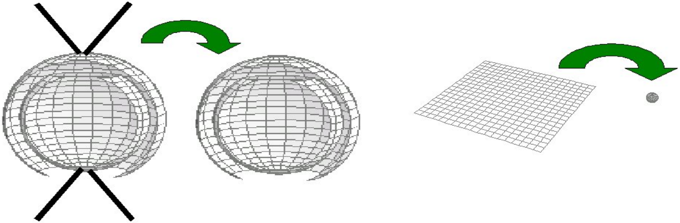
Now: Field expansion of hCS gauge potential $\mathcal{A}^{0,1}$ available:

$$\begin{aligned} \mathcal{A}_\alpha =& \lambda^{\dot\alpha} A_{\alpha\dot\alpha}(x) + \eta_i \chi^i_\alpha(x) + \gamma \tfrac{1}{2!} \eta_i\eta_j \hat\lambda^{\dot\alpha} \phi^{ij}_{\alpha\dot\alpha}(x) + \\ & \gamma^2 \tfrac{1}{3!} \eta_i\eta_j\eta_k \hat\lambda^{\dot\alpha} \hat\lambda^{\dot\beta} \tilde\chi^{ijk}_{\alpha\dot\alpha\dot\beta}(x) + \gamma^3 \tfrac{1}{4!} \eta_i\eta_j\eta_k\eta_l \hat\lambda^{\dot\alpha} \hat\lambda^{\dot\beta} \hat\lambda^{\dot\gamma} G^{ijkl}_{\alpha\dot\alpha\dot\beta\dot\gamma}(x) \\ \mathcal{A}_{\bar\lambda} =& \gamma^2 \eta_i\eta_j \phi^{ij}(x) - \gamma^3 \eta_i\eta_j\eta_k \hat\lambda^{\dot\alpha} \tilde\chi^{ijk}_{\dot\alpha}(x) + 2\gamma^4 \eta_i\eta_j\eta_k\eta_l \hat\lambda^{\dot\alpha} \hat\lambda^{\dot\beta} G^{ijkl}_{\dot\alpha\dot\beta}(x) \end{aligned}$$

Popov, CS, ATMP 9 (2005) 931

This field expansion makes the equivalence hCS $\leftrightarrow$ SDYM manifest.

Dimensional reduction to a matrix model: $\mathbb{R}^{4|8} \to \mathbb{R}^{0|8}$:



- Matrix Model from $\mathcal{N}=4$ hCS theory (MQM):

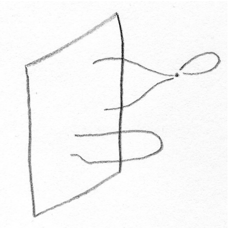
$$S \;:=\; \int_{\mathbb{C}P^1_{\mathrm{ch}}} \Omega_{\mathrm{red}} \wedge \mathrm{tr}\,\varepsilon^{\alpha\beta}\mathcal{X}_\alpha\left(\bar{\partial}\mathcal{X}_\beta + \left[\mathcal{A}^{0,1}_{\mathbb{C}P^1}, \mathcal{X}_\beta\right]\right)$$

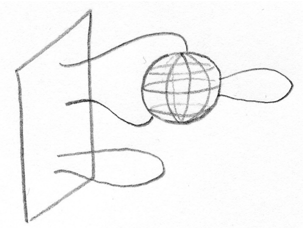
$$\Omega_{\mathrm{red}} \;:=\; \Omega^{3,0|4,0}|_{\mathbb{C}P^1_{\mathrm{ch}}} \qquad \Omega_{\mathrm{red}\pm} \;=\; \pm\mathrm{d}\lambda_\pm \wedge \mathrm{d}\eta_1^\pm \ldots \mathrm{d}\eta_4^\pm$$

- Matrix Model from $\mathcal{N}=4$ SDYM theory:

$$S := \mathrm{tr}\left(G^{\dot\alpha\dot\beta}\left(-\tfrac{1}{2}\varepsilon^{\alpha\beta}[A_{\alpha\dot\alpha}, A_{\beta\dot\beta}]\right) + \tfrac{\varepsilon}{2}\phi^{ij}[A_{\alpha\dot\alpha}, [A^{\alpha\dot\alpha}, \phi_{ij}]] + \ldots\right)$$

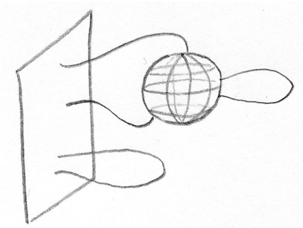
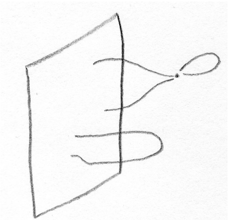
- Perspective of D(-1)-branes
- Supersymmetrically extend ADHM eqns.:
  $A_{\alpha\dot\alpha} \to A_{\alpha\dot\alpha} + \eta^i_{\dot\alpha}\chi_{i\alpha}$ and $w_{\dot\alpha} \to w_{\dot\alpha} + \eta^i_{\dot\alpha}\psi_i$
- Drop the D(-1)-D3-strings, i.e. $w_{\dot\alpha} \stackrel{!}{=} 0$
- $\Rightarrow$ SDYM MM equations
- How to obtain the full picture?
- Incorporate D(-1)-D3-strings in MM in hCS: D1-D5-strings.

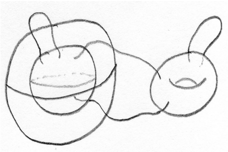
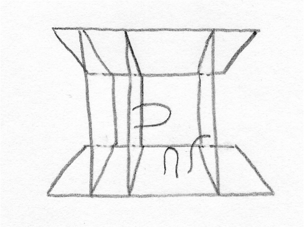




D-brane configuration equivalences

We had topological-physical D-brane equivalences for ADHM and Nahm construction.

 $\Longleftrightarrow$ 

 $\Longleftrightarrow$ 

But: There are many more.

Problem: Extend correspondences and lift to M-theory.

Generalized Berezin-Toeplitz Quantization and Aspects of the Bagger-Lambert-Gustavsson theory

Christian Sämann

School of Mathematics, Trinity College Dublin

University of Edinburgh, January, 19th 2009