Generalized Berezin Quantization, Bergman Metrics and Fuzzy Laplacians

Christian Sämann

School of Mathematics, Trinity College Dublin

Annual Meeting of the EU Network in NC Geometry 2008

- CS, JHEP **02** (2008) 111 [hep-th/0612173]
- C. I. Lazaroiu, D. McNamee and CS, 0804.4555 [hep-th]

Outline

- Fuzzy toric varieties from fuzzy $\mathbb{C}P^n$
  - Fuzzification $\cong$ Quantization of the toric base
- Donaldson's algorithm for calculating Calabi-Yau metrics
- Generalized Berezin quantization
  - Berezin and Berezin-Toeplitz quantization
  - Relation to generalized Berezin quantization
  - Integral formulas in Berezin quantization
  - Why calculating CY metrics is not easier here
- Quantized harmonic analysis and Laplace operators
- Fuzzy scalar field theories on Hodge manifolds
- Conclusion

Fuzzy $\mathbb{C}P^n$

Quantization of $\mathbb{C}P^n$ particularly nice, as it is a homogeneous space.

Underlying idea (naïve approach):

- Group theoretic: Truncate the spectrum of the Laplace operator and deform the product to obtain a closed algebra.
- Complex geometry: Quantize $\mathbb{C}^{n+1}$ and use the induced result on $\mathbb{C}P^n$.

Quantization of $\mathbb{C}^{n+1}$: $(w_\alpha, \bar{w}_\beta) \to (\hat{a}^\dagger_\alpha, \hat{a}_\beta)$

Functions on $\mathbb{C}P^n$: normalize and use Hopf fibration:

$$0 \to \mathrm{U}(1) \to S^{2n+1} \to \mathbb{C}P^n \to 0 \; .$$

This yields the quantization map:

$$\frac{1}{|w|^{2k}} w_{\alpha_1} \ldots w_{\alpha_k} \bar{w}_{\beta_1} \ldots \bar{w}_{\beta_k} \to \hat{a}^\dagger_{\alpha_1} \ldots \hat{a}^\dagger_{\alpha_k} |0\rangle\langle 0| \hat{a}_{\beta_1} \ldots \hat{a}_{\beta_k}$$

Homogeneous coordinates $\leftrightarrow$ creation/annihilation operators.

Fuzzy projective algebraic varieties

Coordinate rings of projective algebraic varieties sit inside the ones of a $\mathbb{C}P^n$.

A projective algebraic variety is a subspace of $\mathbb{C}P^n$ described by a finite set of polynomial equations.

Example: $\mathbb{C}P^n$. Equivalently: $\mathrm{Proj} B_{n+1}$, $B_{n+1} = \mathbb{C}[\mathbb{C}^{n+1}]$

Example: $W\mathbb{C}P^2(1,1,2) : (z_0, z_1, z_2) \sim (\lambda z_0, \lambda z_1, \lambda^2 z_2)$

Embedding into $\mathbb{C}P^3$: $(w_0, w_1, w_2, w_3) = (z_0^2, z_0 z_1, z_1^2, z_2)$

Coordinates no longer independent, but satisfy $w_1^2 - w_0 w_2 = 0$.

This generates an ideal I in B_3, and as a projective algebraic variety, we have $W\mathbb{C}P^2(1,1,2) = \mathrm{Proj}(B_3/I)$.

Fuzzy projective algebraic varieties

How to factor out an ideal.

Recall the quantization map:

$$\frac{1}{|w|^{2k}} w_{\alpha_1} \ldots w_{\alpha_k} \bar{w}_{\beta_1} \ldots \bar{w}_{\beta_k} \rightarrow \hat{a}^\dagger_{\alpha_1} \ldots \hat{a}^\dagger_{\alpha_k} |0\rangle\langle 0| \hat{a}_{\beta_1} \ldots \hat{a}_{\beta_k}$$

Decompose:

the coordinate ring $B_3 = \sum_{k=0}^\infty B_{3,k}$, $\quad z_{\alpha_1} \ldots z_{\alpha_k} \in B_{3,k}$

the Fock space $\mathcal{F} = \sum_{k=0}^\infty \mathcal{F}_k$, $\quad \hat{a}^\dagger_{\alpha_1} \ldots \hat{a}^\dagger_{\alpha_k} |0\rangle \in \mathcal{F}_k$

Quantization maps $B_{3,k} \cdot \bar{B}_{3,k}$ to $\mathcal{F}_k \cdot \mathcal{F}^*_k$

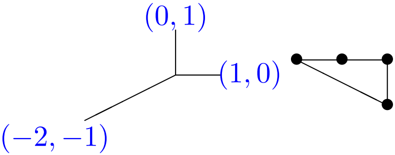
To quantize $\mathrm{Proj}(B_3/I)$, map $B_{3,k}/I \cdot \bar{B}_{3,k}/I$ to $\mathcal{F}_k/\hat{I} \cdot \mathcal{F}^*_k/\hat{I}$

If $f(w_0, \ldots, w_n)$ generates I, then $\mathcal{F}_k/\hat{I}$ refers to those elements $|\mu\rangle \in \mathcal{F}_k$ which satisfy $\hat{f}(\hat{a}_0, \ldots, \hat{a}_n)|\mu\rangle = 0$.

Fuzzy toric varieties

Quantization of a toric variety corresponds to quantization of its toric base.

A toric variety is characterized by a number of $\mathrm{U}(1)$ actions, which are specified by a toric fan. This fan in turn gives rise to a certain polytope, called the toric base.
Example: $W\mathbb{C}P^2(1,1,2) : (z_0, z_1, z_2) \sim (\lambda z_0, \lambda z_1, \lambda^2 z_2)$



$z_0^2 \quad z_0 z_1 \quad z_1^2 \qquad w_0 \quad w_1 \quad w_2$

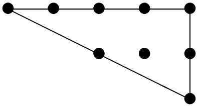
$\qquad\qquad\quad\; z_2 \qquad\qquad\qquad\quad w_3$

The dots correspond to monomials in z_α and in w_α, which in turn correspond to basis elements of $\mathcal{F}_1/\hat{I}$: the states $\hat{a}_0^\dagger|0\rangle, \ldots, \hat{a}_3^\dagger|0\rangle$ all satisfy the ideal equation $(\hat{a}_0\hat{a}_2 - \hat{a}_1^2)|\mu\rangle = 0$.

Fuzzy toric varieties

Quantization of a toric variety corresponds to quantization of its toric base.

Now quantize $W\mathbb{C}P^2(1,1,2)$ at level $k=2$. The toric base gets a finer subdivision:



z_0^4	$z_0^3 z_1$	$z_0^2 z_1^2$	$z_0 z_1^3$	z_1^4	w_0^2	$w_0 w_1$	$w_0 w_2 = w_1^2$	$w_1 w_2$	w_2^2
		$z_2 z_0^2$	$z_2 z_0 z_1$	$z_2 z_1^2$			$w_3 w_0$	$w_3 w_1$	$w_3 w_2$
				z_2^2					w_3^2

The dots correspond again to basis elements of $\mathcal{F}_2/\hat{I}$. The original dimension of $\mathcal{F}_2$ was 10, the two states $\hat{a}_0^\dagger \hat{a}_2^\dagger|0\rangle$ and $\hat{a}_1^\dagger \hat{a}_1^\dagger|0\rangle$, which do not satisfy the equation $(\hat{a}_0\hat{a}_2 - \hat{a}_1^2)|\mu\rangle = 0$ are mapped to the same point in the toric polytope, corresponding to the state $\hat{a}_0^\dagger \hat{a}_2^\dagger|0\rangle - \frac{1}{2}\hat{a}_1^\dagger \hat{a}_1^\dagger|0\rangle$.

$\Rightarrow$Quantizing a toric variety amounts to quantizing its toric base.

Donaldson's algorithm for computing Calabi-Yau metrics

The algorithm employs methods strongly reminiscent of our quantization procedure.

Calabi-Yau manifold:

- Kähler manifold with vanishing first Chern class
- admits a Levi-Civita connection with $\mathrm{SU}(n)$ holonomy
- has trivial canonical bundle
- admits a nowhere-vanishing holomorphic $(n, 0)$-form
- admits a Ricci-flat metric

They appear in superstring compactifications as $M^4 \times CY$.

To extract information about the low-energy physics of such compactifications, we need the metric. Unfortunately, no Ricci-flat metric is known for any Calabi-Yau manifold.

Donaldson has proposed an elegant algorithm for computing such metrics.

The metric on a CY manifold is trivially obtained from the Kähler two-form ω, which comes from the Kähler potential K: $\omega = \partial\bar{\partial}K$.

Consider a CY manifold which is a projective algebraic variety, e.g. the Fermat Quintic in $\mathbb{C}P^4$ defined by: $f = w_0^5 + ... + w_4^5 = 0$. Approximate the Kähler potential by

$$K = \ln h^{\alpha_1 \dots \alpha_k \beta_1 \dots, \beta_k} w_{\alpha_1} \dots w_{\alpha_k} \bar{w}_{\beta_1} \dots \bar{w}_{\beta_k} =: \ln h^{IJ} w_I \bar{w}_J \ .$$

Factoring out the ideal $I(f)$: $w_{\alpha_1} \dots w_{\alpha_k} \to$ basis set (s_i).
A metric determined by $K = \ln h^{ij} s_i \bar{s}_j$ is called balanced, if

$$(h^{ij})^{-1} = \langle s_i, s_j \rangle = \frac{N_k}{\mathrm{vol}_C Y} \int_{CY} \mathrm{d}\mu_{CY}(x) \frac{s_i \bar{s}_j}{h^{kl} s_k \bar{s}_l} \ .$$

Theorem: For each $k \geq 0$, the balanced metric exists and is unique. As $k \to \infty$, the sequence of balanced metrics converges to the Ricci-flat metric.

Iterative procedure:

$$h_{n+1,ij} = \frac{N_k}{\mathrm{vol}_C Y} \int_{CY} \mathrm{d}\mu(x) \frac{s_i \bar{s}_j}{h_n^{kl} s_k \bar{s}_l} \ .$$

Integration done numerically (hep-th/0606261, 0612075, ...)

The approximation of the Kähler potential $K = \ln h^{ij} s_i \bar{s}_j$ corresponds to quantizing the CY manifold with our procedure. As in the quantized picture, integration goes over into a trace, can we perform the calculation actually much easier in our scheme?

To see, how our quantization procedure might be applied here, let us look more closely at the procedure. $\longrightarrow$ Geometric Quantization

X compact complex manifold
L polarization of X, i.e. a positive (ample) holomorphic line bundle
Kähler metric on X $\leftrightarrow$ hermitian metric h on L up to rescaling:

$$\omega = \frac{\mathrm{i}}{2\pi} F = -\partial\bar{\partial} \ln h(\sigma, \sigma)\ , \quad \sigma \in H^0(L) \text{ over patch } \sigma(x) \neq 0$$

Replace L by very ample: L^k (basis of sections yields $X \hookrightarrow \mathbb{C}P^n$)

Consequences: $h_k = h^{\otimes k}$ and $\nabla_k = \nabla^{\otimes k}$, $E = H^0(L^k) = \mathrm{span}(s_i)$

(X, ω, L, h): prequantized Hodge manifold.

Let $\mathbb{L}$ be the total space of L and $\mathbb{L}_0 = \mathbb{L} \backslash o$. Also: $\pi : \mathbb{L} \to X$.

$$s(\pi(q)) =: \hat{q}(s) q \ , \quad q \in \mathbb{L}_0 \ , \quad s \in H^0(L)$$

$\hat{q}(s)$: "How much does one have to scale s to pass through q"
By Riesz's theorem, there is a unique holomorphic section e_q with:

$$(e_q, s) = \hat{q}(s)$$

Define $G_{ij} = (s_i, s_j)$, then

$$e_q = G^{ji} \overline{\hat{q}(s_i)} s_j$$

Introduce the Rawnsley's coherent state projectors:

$$P_x := \frac{|e_q)(e_q|}{(e_q|e_q)} \ , \quad q \in L_x \backslash \{0\}$$

P_x depends only on $x \in X$, L and G.

Rawnsley coherent state projector - Integral formula

A simple integral identity can be derived for the Rawnsley coherent states.

$$P_x := \frac{|e_q)(e_q|}{(e_q|e_q)}\ , \quad q \in L_x \backslash \{0\}$$

Introduce the ε-function

$$\varepsilon(x) := h(q,q)||e_q||^2 = G^{ij}h(x)(s_i(x), s_j(x))$$

Consider the scalar product given by $\mu(x)$ and h:

$$(s,t) = \int_X \mathrm{d}\mu(x) h(x)(s(x), t(x)) = \int_X \mathrm{d}\mu(x)\varepsilon(x)(s|P_x|t)\ .$$

Thus:

$$\int_X \mathrm{d}\mu(x)\varepsilon(x) P_x = \mathrm{id}_E$$

Scalar product on E balanced:

$$\varepsilon(x) = \frac{\mu(X)}{N+1}$$

Linear operator $C \in \mathsf{End}\,(E)$. Define its lower Berezin symbol:

$$\sigma(C)(x) := \operatorname{tr}(CP_x) = \frac{(e_q|C|e_q)}{(e_q|e_q)}$$

Call $\sigma(\mathsf{End}\,(E)) =: \Sigma$.
Define the Berezin quantization of $f \in \Sigma$ as: $\sigma^{-1}(f)$.

The Toeplitz quantization $T : C^\infty(X) \to \mathsf{End}\,(E)$ is defined:

$$T(f) = \int_X \frac{\omega^n}{n!} \varepsilon(x) f(x) P_x$$

Properties: $T(\bar{f}) = T(f)^\dagger$ and $T(1_X) = \mathbb{1}_E$

Berezin and Berezin-Toeplitz quantization

There are two quantization procedures making use of P_x.

Connection between Berezin and Berezin-Toeplitz quantization:

$$\text{Berezin transform}: \quad \beta := \sigma \circ T$$

Altogether:

$$\begin{array}{ccc} C^\infty(X) & \xrightarrow{\;T\;} & \mathsf{End}\,(E) \\ \downarrow \beta & & \downarrow = \\ \Sigma & \xleftarrow{\;\sigma\;} & \mathsf{End}\,(E) \end{array}$$

Quantization of $\mathbb{C}P^n$: $P_x = \frac{1}{k!}(z_i a_i^\dagger)^k |0\rangle\langle 0| (\bar{z}_i a_i)^k$

Alternatively (polarization tensors): P_x such that $Y_{lm} \rightarrow \hat{Y}_{lm}$:

$$T|_\Sigma = \sigma^{-1}, \; \beta = \Pi_\Sigma$$

Generalized Berezin quantization

The Berezin quantization can be generalized to arbitrary scalar products.

If $(\ ,\ )$ is induced from h on L and $\Omega = \frac{\omega^n}{n!}$, then σ is injective.

What happens if we change the scalar product?

$$(s,t)' = (As,t) = (s,At)\ ,$$

where A Hermitian, positive-definite. Consequences:

$$e'_q = A^{-1} e_q \quad , \quad P'_x = \frac{1}{\sigma(A^{-1}(x))} A^{-1} P_x \quad , \quad \sigma'(C) = \frac{\sigma(CA^{-1})}{\sigma(A^{-1})}$$

$\Rightarrow$ Two (generalized) Berezin quantizations agree, if the operator A is proportional to the identity:
$\sigma'(C) = \sigma(C) \Rightarrow \forall_C : \sigma(C)\sigma(A) = \sigma(A)\sigma(C)$
$\Rightarrow \forall_C : \sigma(AC) = \sigma(CA) \Rightarrow \forall_C : [A,C] = 0 \Rightarrow A = \lambda \mathbb{1}_E$.
(Similarly: generalized Toeplitz quantization.)

Consider (X, L), $E_k = H^0(L^k)$
Homogeneous coordinate ring: $R(X, L) = \oplus_{k=0}^{\infty} E_k$
We have $R \cong_{[k]} B/I$, where $B = \oplus_{k=0}^{\infty} E_1^{\odot k}$

Two ways of introducing a scalar product on E_k:

$$\langle r, t\rangle_k = \int_X \frac{\omega^n}{n!} h^{\otimes k}(r, t) \quad \Rightarrow \quad \text{ordinary Berezin/Toeplitz theory}$$

$$(r_1 \odot \ldots r_k, t_1 \odot \ldots t_k)_B = \frac{1}{k!} \delta_{k,l} \sum_{\sigma \in S_k} (r_1, t_{\sigma(1)})_1 \ldots (r_k, t_{\sigma(k)})_1$$

Choose $(s_\alpha, s_\beta) = \delta_{\alpha\beta}$: metric on E_k is implied in using
$(w_\alpha, \bar{w}_\beta) \rightarrow (\hat{a}_\alpha^\dagger, \hat{a}_\beta)$, $\hat{f}|\mu\rangle = 0$, $(\mu, \nu) := \langle \mu | \nu \rangle$

To obtain integral formulas, recall the overcompleteness relation for coherent states $\int_X \mathrm{d}\mu(x)\varepsilon(x)P_x = \mathrm{id}_E$, which implies that

$$\int_X \mathrm{d}\mu(x)\varepsilon(x)f(x) = \int_X \mathrm{d}\mu(x)\varepsilon(x)\,\mathrm{tr}\,(P_x\hat{f}) = \mathrm{tr}\,(\hat{f})\ .$$

For balanced metrics, $\varepsilon(x) = 1$, otherwise: introduce the operator

$$\hat{\rho} = T\left(\frac{1}{\mathrm{vol}_\omega(X)\varepsilon(x)}\right) = \frac{1}{\mathrm{vol}_\omega(X)}\int_X \frac{\omega^n}{n!}P_x\ ,$$

such that

$$\frac{1}{\mathrm{vol}_\omega(X)}\int_X \frac{\omega^n}{n!}f(x) = \frac{1}{\mathrm{vol}_\omega(X)}\int_X \frac{\omega^n}{n!}\,\mathrm{tr}\,(P_x\hat{f}) = \mathrm{tr}\,(\hat{\rho}\hat{f})\ .$$

Thus: can integrate over arbitrary measures in quantum picture.

To obtain useful integral formulas, which are needed for iterating

$$h_{n+1,ij} = \frac{N_k}{\mathrm{vol}_C Y} \int_{CY} \mathrm{d}\mu(x) \frac{s_i \bar{s}_j}{h_n^{kl} s_k \bar{s}_l} \ ,$$

we would have to compute the matrix A which takes us from

$$\int_X \frac{\omega^n}{n!} h^{\otimes k}(r,t) \quad \Rightarrow \quad \frac{1}{k!}\delta_{k,l} \sum_{\sigma \in S_k} (r_1, t_{\sigma(1)})_1 \ldots (r_k, t_{\sigma(k)})_1$$

This involves computing even more integrals.
Trivial only for balanced case.

$\Rightarrow$ Never try to be smarter than Donaldson, unless you have a very good reason.

First idea: the quantum versions $O^B : \mathrm{End}\,(E) \to \mathrm{End}\,(E)$ of a differential operator $O : \Sigma \to \Sigma$ should act on quantized functions as they do on ordinary functions:

$$O^B \hat{f} := \sigma^{-1}(\Pi_{L^2}(O(\sigma(\hat{f}))))$$

We call this the Berezin push of an operator. (Analogously, define the Berezin pull $O^B \to O$.)

This definition can be used to perform approximate harmonic analysis on projective varieties.

Approximate harmonic analysis on Fermat curves

The Berezin-push can be used to analyze the spectrum of Δ approximatly.

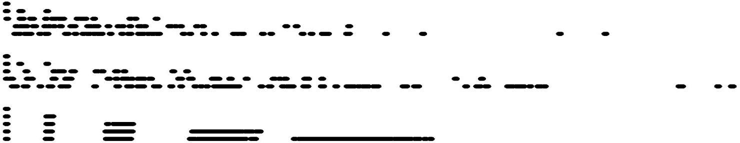
Example:

Fermat curve: projective algebraic variety $X_p \subset \mathbb{C}P^2$ given by

$$f(w_0, w_1, w_2) = w_0^p + w_1^p + w_2^p = 0 \ .$$

Endow X_p with the Bergman metric obtained by pulling back the Fubini-Study metric from $\mathbb{C}P^2$, which determines the Laplacian. Calculate the matrix $\Delta_{(ij)(kl)}$ with $\Delta^B(s_k \bar{s}_l) = \Delta_{(ij)(kl)} s_i \bar{s}_k$ and determine its eigenvalues.

Examples: $X_2, X_3, \mathbb{C}P^2$:



However: If O hermitian with respect to $(\ ,\ )_\omega$, its Berezin push is not hermitian with respect to $\langle\ ,\ \rangle_{HS}$!

Second idea: preserve hermiticity:

$$\langle T(f), C\rangle_{HS} = \mathrm{tr}\left(T(f)^\dagger C\right) = \int_X d\mu(x)\epsilon(x)\bar{f}(x)\sigma(C) = (f, \sigma(C))_{\varepsilon,\omega}$$

Thus: $\sigma^\oplus = T \circ M_{\frac{1}{\varepsilon}}$, and define the Berezin-Toeplitz lift of O:

$$\hat{O} = \sigma^\oplus \circ O \circ \sigma = T \circ M_{\frac{1}{\varepsilon}} \circ O \circ \sigma\ .$$

This procedure preserves hermiticity, but $\hat{1} \neq \mathbb{1}_{\mathsf{End}\,(E)}$.

Fuzzy scalar field theory

Scalar field theories can be regularized using quantized spaces.

Scalar field theory on (X, ω) is defined by a functional $S[\phi]$:

$$S[\phi] := \frac{1}{\mathrm{vol}_\omega(X)} \int_X \frac{\omega^n}{n!} \left(\phi \Delta \phi + V(\phi)\right) \ , \quad \phi \in \mathcal{C}^\infty(X, \mathbb{R}) \ ,$$

where $V(\phi) = \sum_{s=0}^d a_s \phi^s$. The corresponding quantum version:

$$S_q[\phi] := \frac{1}{\mathrm{vol}_\omega(X)} \mathrm{tr} \left(\Phi \hat{\Delta} \Phi + V(\Phi) \right) \ , \quad \Phi \in \mathsf{End}\,(E) \ ,$$

The functional integral $\int \mathcal{D}[\phi]$ in the partition function

$$Z_q = \int \mathcal{D}[\Phi] \mathrm{e}^{-S[\Phi]}$$

becomes a well-defined finite-dimensional integral.

$\Rightarrow$ We have all the necessary ingredients for defining precisely quantum field theories on arbitrary (quantum) Hodge manifolds.

We achieved the following:

- Explicit relation between intrinsic and extrinsic quantization
- Formulas for integrals and differential operators on quantizable manifolds
- Huge new classes of regularization of scalar field theories

Future directions:

- Extending all our results to supermanifolds
- Study quantized toric varieties (flip transitions, etc.)
- Quantization of singular projective varieties
- Continue looking for fuzzy shortcut for Donaldson's algorithm

Generalized Berezin Quantization, Bergman Metrics and Fuzzy Laplacians

Christian Sämann

School of Mathematics, Trinity College Dublin

Annual Meeting of the EU Network in NC Geometry 2008