Monopoles, Integrability and M-Theory

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Based on:

- Bogomolny, Atiyah, Drinfeld, Hitchin, Manin, Nahm
- Witten, Howe, Lambert, Basu, Harvey, Bagger
- CS, arXiv:1007.3301, CMP ...

Motivation - Physics

We want to learn more about M-theory.

- String Theory: Point particles $\rightarrow$ (excitations of) strings
- Consistency requires that spacetime is 10-dimensional.
- Also $p+1$-dimensional objects appear: Dp-branes
- type IIA: D$(2p)$-branes, type IIB: D$(2p+1)$-branes
- turning off gravity, they are described by a gauge theory: "Effective descript.": max. supersymmetric Yang-Mills theory
- Several string theories (IIA, IIB, ...), unified in M-theory.
- In M-theory: only M2-branes and M5-branes.
- Effective description of M2-branes proposed in 2007.
- This created lots of interest: BLG-model: >440 citations, ABJM-model: >555 citations
- Open question: technicalities, M5-branes

Motivation - Mathematics

Find an algorithm for the construction of self-dual string solutions

- Higher Gauge Theories: 2-connections
- Multisymplectic geometry, geometric quantization, ...
- New integrable structures, twistor spaces, etc.
- D-branes mathematically very useful. M-branes too?

Outline

We will discuss the construction of monopoles and lift each ingredient to M-theory.

- Monopoles and the Bogomolny equation
- String theoretic interpretation of monopoles
- Monopoles in M-theory: Self-Dual Strings
- 3-Lie algebras
- Principal $\mathsf{U}(1)$-bundles, abelian gerbes and loop space
- ADHMN construction and its lift
- Examples of self-dual string solutions
- Non-abelian tensor multiplet on loop space

Magnetic monopoles

Magnetic monopoles are BPS solutions to the super Yang-Mills equations.

Consider a principal $\mathsf{U}(N)$-bundle over a spacetime $\mathbb{R}^d$

Endow with connection A, i.e. locally: Lie-algebra valued one-form

Maxwell/Yang-Mills (YM) equations:

$$\mathrm{d}_A F = 0 \quad \text{and} \quad \mathrm{d}_A \star F = J \quad \text{with} \quad F := \mathrm{d}_A A := \mathrm{d}A + A \wedge A$$

BPS-equations

- first-order deq., solutions are a subset of the solutions to YM.
- The action functional evaluated at a solution is finite.
- The solution is invariant under some of the supersymmetries.
- Solutions classify perturbative sectors of the quantum theory.

Examples:

Instanton: $d = 4$: $F = \star F$, $J = 0$

Monopole: $d = 3$: $F = \star \mathrm{d}\Phi$, $J = \star[\Phi, \mathrm{d}_A \Phi]$

For convenience, change to index notation as follows:

$$
\begin{array}{lcl}
\mathrm{d}_A = \mathrm{d} + A & \rightarrow & \nabla_\mu = \partial_\mu + A_\mu \ , \\
F = \mathrm{d}A + A \wedge A & \rightarrow & F_{\mu\nu} = [\nabla_\mu, \nabla_\nu] = \partial_\mu A_\nu - \partial_\nu A_\mu + [A_\mu, A_\nu] \ , \\
0 = \mathrm{d}_A \star F & \rightarrow & 0 = \sum_\mu \nabla_\mu F_{\mu\nu} = \nabla_\mu F_{\mu\nu} \ , \\
0 = \mathrm{d}_A F & \rightarrow & 0 = \varepsilon_{\mu\nu\kappa\ldots} \nabla_\mu F_{\nu\kappa} \ , \\
F = \star F & \rightarrow & F_{\mu\nu} = \tfrac{1}{2}\varepsilon_{\mu\nu\rho\sigma} F_{\rho\sigma} \ , \\
F = \star \mathrm{d}_A \Phi & \rightarrow & F_{ij} = \tfrac{1}{2}\varepsilon_{ijk} \nabla_k \Phi \ .
\end{array}
$$

D-branes and super Yang-Mills theory

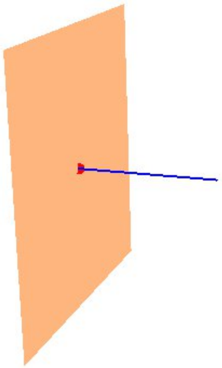
D-branes are effectively described in terms of super Yang-Mills theory.

- A flat D9-brane fills $\mathbb{R}^{10}$, described by $\mathsf{U}(1)$ Yang-Mills theory.
- N flat D9-branes are described by $\mathsf{U}(N)$ Yang-Mills theory.
- Lower dimensional Dp-branes are obtained as follows:
  - Split index $\mu = 0, ..., 9 \to (m = 0, ..., p, i = p+1, ..., 9)$
  - Put $\partial_i = 0$. That is: $A_\mu \to (A_m, X^i)$
  - Eigenvalues of X^i describe transverse movement.
  - Alternatively: $\mathbb{R}^{10} \to \mathbb{R}^{p+1} \times T^{9-p}$, take zero modes
- If D-branes are contained in or end on other D-branes in a BPS way, fix some of the X^i and take BPS equations.
- Often, we are interested in vacuum solutions which are time independent.

D1-D3-Branes and the Nahm Equation

D1-branes ending on D3-branes can be described by the Nahm equation.

dim	0	1	2	3	...	6
$D1$	$\times$					$\times$
$D3$	$\times$	$\times$	$\times$	$\times$		



N D1-branes ending on D3-branes:

Time independent: Monopole.

$X^i \in \mathsf{U}(N)$: transverse fluctuations

Nahm equation: $(s = x^6)$

$$\frac{\mathrm{d}}{\mathrm{d}s} X^i + \varepsilon^{ijk}[X^j, X^k] = 0$$

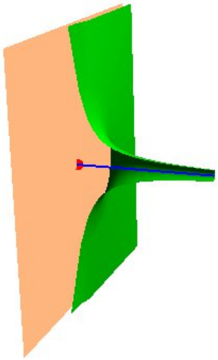
Solution: $X^i = r(s) G^i$ with

$$r(s) = \frac{1}{s}\ , \quad G^i = \varepsilon^{ijk}[G^j, G^k]$$

D1-D3-Branes and the Nahm Equation

The D1-branes end on the D3-branes by forming a fuzzy funnel.

dim	0	1	2	3	...	6
$D1$	$\times$					$\times$
$D3$	$\times$	$\times$	$\times$	$\times$		



Solution: $X^i = r(s) G^i$

$$r(s) = \frac{1}{s}\ , \quad G^i = \varepsilon^{ijk}[G^j, G^k]$$

The D1-branes form a fuzzy funnel:

G^i form irrep of $\mathsf{SU}(2)$:
coordinates on fuzzy sphere S^2_F

(Berezin-quantized sphere S^2_F:
Hybrid of deformation and
geometric quantization)

D1-worldvolume polarizes: $2d \to 4d$

Lifting D1-D3-Branes to M2-M5-Branes

The lift to M-theory is performed by a T-duality and an M-theory lift

IIB	0	1	2	3	4	5	6
$D1$	×						×
$D3$	×	×	×	×			

T-dualize along x^5:

IIA	0	1	2	3	4	5	6
$D2$	×					×	×
$D4$	×	×	×	×		×	

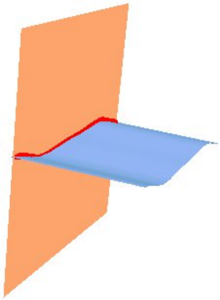
Interpret x^4 as M-theory direction:

M	0	1	2	3	4	5	6
$M2$	×					×	×
$M5$	×	×	×	×	×	×	

The Basu-Harvey lift of the Nahm Equation

M2-branes ending on M5-branes yield a Nahm equation with a cubic term.

M	0	1	2	3	4	5	6
$M2$	$\times$					$\times$	$\times$
$M5$	$\times$	$\times$	$\times$	$\times$	$\times$	$\times$	



Basu, Harvey, hep-th/0412310

A Self-Dual String appears.

Substitute SO(3)-inv. Nahm eqn.

$$\frac{\mathrm{d}}{\mathrm{d}s}X^i + \varepsilon^{ijk}[X^j, X^k] = 0$$

by the SO(4)-invariant equation

$$\frac{\mathrm{d}}{\mathrm{d}s}X^\mu + \varepsilon^{\mu\nu\rho\sigma}[X^\nu, X^\rho, X^\sigma] = 0$$

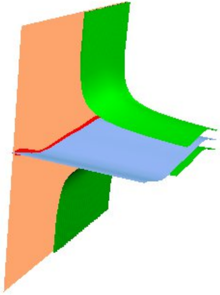
Solution: $X^\mu = r(s)G^\mu$ with

$$r(s) = \frac{1}{\sqrt{s}}\ , \quad G^\mu = \varepsilon^{\mu\nu\rho\sigma}[G^\nu, G^\rho, G^\sigma]$$

The Basu-Harvey lift of the Nahm Equation

M2-branes ending on M5-branes yield a Nahm equation with a cubic term.

M	0	1	2	3	4	5	6
$M2$	$\times$					$\times$	$\times$
$M5$	$\times$	$\times$	$\times$	$\times$	$\times$	$\times$	



Solution: $X^\mu = r(s) G^\mu$

$$r(s) = \frac{1}{\sqrt{s}}\ , \quad G^\mu = \varepsilon^{\mu\nu\rho\sigma}[G^\nu, G^\rho, G^\sigma]$$

The M2-branes form a fuzzy funnel:

G^μ form a rep of SO(4):
coordinates on fuzzy sphere S^3_F

M2-worldvolume polarizes: $3d \to 6d$

What is this triple bracket?

What is the algebra behind the triple bracket?

In analogy with Lie algebras, we can introduce 3-Lie algebras.

3-Lie algebra

A 3-Lie algebra is a vector space $\mathcal{A}$ endowed with a totally antisymmetric, trilinear map $[\cdot,\cdot,\cdot] : \mathcal{A}^{\wedge 3} \to \mathcal{A}$, which satisfies for any $A, B, C, D, E \in \mathcal{A}$ the fundamental identity

$$[A, B, [C, D, E]] = [[A, B, C], D, E] + [C, [A, B, D], E] + [C, D, [A, B, E]].$$

There is an associated Lie algebra of inner derivations:

Triple bracket forms a map $D : \mathcal{A} \wedge \mathcal{A} \to \mathrm{Der}(\mathcal{A}) =: \mathfrak{g}_{\mathcal{A}}$ via

$$D(A, B) \triangleright C := [A, B, C]$$

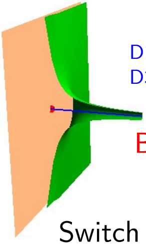
These inner derivations form indeed a Lie algebra:

$$[D(A,B), D(C,D)] \triangleright E := D(A,B) \triangleright (D(C,D) \triangleright E) - D(C,D) \triangleright (D(A,B) \triangleright E)$$

Bracket closes due to fundamental identity.

Monopoles and Self-Dual Strings

Lifting monopoles to M-theory yields self-dual strings.



	0	1	2	3	4	5	6
D1	×						×
D3	×	×	×	×			

BPS configuration!

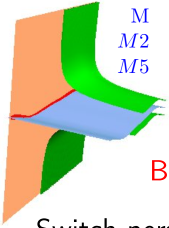
Switch perspective: D1→ D3:
Bogomolny monopole eqn.:

$$F_{ij} = \varepsilon_{ijk} \nabla_k \Phi \;\Rightarrow\; \nabla^2 \Phi = 0$$

Single D3: Dirac monopole

$$\Phi = \frac{1}{r} \;\Rightarrow\; r(s) = \frac{1}{s}$$

⇒ matching profile!



M	0	1	2	3	4	5	6
$M2$	×					×	×
$M5$	×	×	×	×	×	×	

BPS configuration!

Switch perspective: M2→ M5:
Self-dual string eqn.:

$$H_{\mu\nu\rho} = \varepsilon_{\mu\nu\rho\sigma} \partial_\sigma \Phi \;\Rightarrow\; \partial^2 \Phi = 0$$

Only single M5 known:

$$\Phi = \frac{1}{r^2} \;\Rightarrow\; r(s) = \frac{1}{\sqrt{s}}$$

⇒ matching profile!

Manifold M with cover $(U_i)_i$. Principal U(1)-bundle over M:

$$F \in \Omega^2(M, \mathfrak{u}(1)) \ ,$$
$$A_{(i)} \in \Omega^1(U_i, \mathfrak{u}(1)) \text{ with } F = \mathrm{d}A_{(i)}$$
$$g_{ij} \in \Omega^0(U_i \cap U_j, \mathsf{U}(1)) \text{ with } A_{(i)} - A_{(j)} = \mathrm{d}\log g_{ij}$$

Consider monopole in $\mathbb{R}^3$, but describe it on S^2 around monopole:

S^2 with patches U_+, U_-, $U_+ \cap U_- \sim S^1$: $g_{+-} = \mathrm{e}^{-\mathrm{i}n\phi}$, $n \in \mathbb{Z}$

$$c_1 = \frac{\mathrm{i}}{2\pi} \int_{S^2} F = \frac{\mathrm{i}}{2\pi} \int_{S^1} A^+ - A^- = \frac{1}{2\pi} \int_0^{2\pi} n\mathrm{d}\phi = n$$

Monopole charge: n

Self-Dual Strings and Abelian Gerbes

Self-dual strings are described by abelian gerbes.

Manifold M with cover $(U_i)_i$. Abelian (local) gerbe over M:

$$H \in \Omega^3(M, \mathfrak{u}(1)) \; ,$$
$$B_{(i)} \in \Omega^2(U_i, \mathfrak{u}(1)) \text{ with } H = \mathrm{d}B_{(i)}$$
$$A_{(ij)} \in \Omega^1(U_i \cap U_j, \mathfrak{u}(1)) \text{ with } B_{(i)} - B_{(j)} = \mathrm{d}A_{ij}$$
$$h_{ijk} \in \Omega^0(U_i \cap U_j \cap U_k, \mathfrak{u}(1)) \text{ with } A_{(ij)} - A_{(ik)} + A_{(jk)} = \mathrm{d}h_{ijk}$$

Note: Local gerbe: principal $\mathsf{U}(1)$-bundles on intersections $U_i \cap U_j$.

Consider S^3, patches U_+, U_-, $U_+ \cap U_- \sim S^2$: bundle over S^2
Reflected in: $H^2(S^2, \mathbb{Z}) \cong H^3(S^3, \mathbb{Z}) \cong \mathbb{Z}$

$$\frac{\mathrm{i}}{2\pi} \int_{S^3} H = \frac{\mathrm{i}}{2\pi} \int_{S^2} B_+ - B_- = \ldots = n$$

Charge of self-dual string: n

Abelian Gerbes and loop space

By going to loop space, one can reduce differential forms by one degree.

Consider the following double fibration:

$$\begin{array}{ccc} & \mathcal{L}M \times S^1 & \\ {}^{ev}\swarrow & & \searrow^{\int_{S^1}} \\ M & & \mathcal{L}M \end{array}$$

Identify $T\mathcal{L}M = \mathcal{L}TM$, then: $x \in \mathcal{L}M \Rightarrow \dot{x}(\tau) \in \mathcal{L}TM$

Transgression

$$\mathcal{T} : \Omega^{k+1}(M) \rightarrow \Omega^k(\mathcal{L}M)\ , \quad \mathcal{T} = \int_{S^1} ! \circ ev^*$$

$$(\mathcal{T}\omega)_x(v_1(\tau), \ldots, v_k(\tau)) := \int_{S^1} \mathrm{d}\tau\, \omega(v_1(\tau), \ldots, v_k(\tau), \dot{x}(\tau))$$

An abelian local gerbe over M is a principal $\mathsf{U}(1)$-bundle over $\mathcal{L}M$.

Note: Most of the time, we will work on $\mathcal{L}M \times S^1$.

The ADHMN construction

There is a map between solutions to the Nahm equations and monopoles.

Nahm transform: Instantons on $T^4 \mapsto$ instantons on $(T^4)^*$

Roughly here:

T^4: $\begin{cases} 3 \text{ rad. } 0 \\ 1 \text{ rad. } \infty \ : \ \text{D1 WV} \end{cases}$ and $(T^4)^*$: $\begin{cases} 3 \text{ rad. } \infty \ : \ \text{D3 WV} \\ 1 \text{ rad. } 0 \end{cases}$

Introduce (twisted) "Dirac operators": (σ^i gen. $\mathsf{SU}(2) = \mathsf{Spin}(3)$)

$$\nabla\!\!\!/_{s,x} = -\mathbb{1}\tfrac{\mathrm{d}}{\mathrm{d}s} + \sigma^i \otimes (\mathrm{i}X^i + x^i \mathbb{1}_k) \ , \quad \bar{\nabla\!\!\!/}_{s,x} := \mathbb{1}\tfrac{\mathrm{d}}{\mathrm{d}s} + \sigma^i \otimes (\mathrm{i}X^i + x^i \mathbb{1}_k)$$

Properties:

$\Delta_{s,x} := \bar{\nabla\!\!\!/}_{s,x} \nabla\!\!\!/_{s,x} > 0$, $[\Delta_{s,x}, \sigma^i] = 0 \Leftrightarrow X^i$ satisfy Nahm eqn.

Normalized zero modes: $\bar{\nabla\!\!\!/}_{s,x} \psi_{s,x,\alpha} = 0$, $\mathbb{1} = \int_{\mathcal{I}} \mathrm{d}s\, \bar{\psi}_{s,x} \psi_{s,x}$ yield:

$$A_\mu := \int_{\mathcal{I}} \mathrm{d}s\, \bar{\psi}_{s,x} \frac{\partial}{\partial x^\mu} \psi_{s,x} \quad \text{and} \quad \Phi := -\mathrm{i} \int_{\mathcal{I}} \mathrm{d}s\, \bar{\psi}_{s,x}\, s\, \psi_{s,x}$$

This is a solution to the Bogomolny monopole equations!

Examples: Dirac monopoles

One can easily construct Dirac monopole solutions using the ADHMN construction.

Charge 1: U(1) Nahm eqn: $\partial_s X^i = 0$, so put $X^i = 0$. Zero mode:

$$\psi_+ = \mathrm{e}^{-sR} \frac{\sqrt{R + x^3}}{x^1 - \mathrm{i}x^2} \begin{pmatrix} x^1 - \mathrm{i}x^2 \\ R - x^3 \end{pmatrix}$$

Monopole solution:

$$\Phi^+ = -\frac{\mathrm{i}}{2R}\ ,\ A_i^+ = \frac{\mathrm{i}}{2(x^1 + x^2)^2} \left(x^2 \left(1 - \frac{x^3}{R} \right), -x^1 \left(1 - \frac{x^3}{R} \right), 0 \right)$$

Charge 2: U(2) Nahm eqn. nontrivial. Choose:

$$X^i = -\frac{1}{s} T^i \quad \text{with} \quad T^i = \frac{\sigma^i}{2\mathrm{i}} = -\bar{T}^i$$

Resulting solution:

$$\Phi^+ = -\frac{\mathrm{i}}{R}\ ,\ A_i^+ = \ldots$$

Lift of the "Dirac operator"

There is a natural lift of the Dirac operator to M-theory.

IIB (tw.): $(\sigma^i : \text{Spin}(3))$

$$\nabla^{\text{IIB}}_{s,x} = -\mathbb{1}\frac{\mathrm{d}}{\mathrm{d}s} + \sigma^i(\mathrm{i}X^i + x^i\mathbb{1}_k)$$

IIB	0	1	2	3	4	5	6
$D1$	×						×
$D3$	×	×	×	×			

IIA (tw.): $(\gamma^4\gamma^i : \text{Spin}(3) \subset \text{Spin}(4))$

$$\nabla^{\text{IIA}}_{s,x} = -\gamma_5\mathbb{1}_k\frac{\mathrm{d}}{\mathrm{d}s} + \gamma^4\gamma^i(X^i - \mathrm{i}x^i)$$

IIA	0	1	2	3	4	5	6
$D2$	×					×	×
$D4$	×	×	×	×		×	

M (untw.): $(\gamma^{\mu\nu} : \text{Spin}(4))$

$$\nabla^{\text{M}}_{s} = -\gamma_5\frac{\mathrm{d}}{\mathrm{d}s} + \tfrac{1}{2}\gamma^{\mu\nu}D(X^\mu, X^\nu)$$

M	0	1	2	3	4	5	6
$M2$	×					×	×
$M5$	×	×	×	×	×	×	

M-theory (twisted):

$$\nabla^{\text{M}}_{s,x(\tau)} = -\gamma_5\frac{\mathrm{d}}{\mathrm{d}s} + \gamma^{\mu\nu}\left(\tfrac{1}{2}D^{(\rho)}(X^\mu, X^\nu) - \mathrm{i}x^\mu(\tau)\dot{x}^\nu(\tau)\right)$$

Lifted ADHMN Construction

The lifted ADHMN construction yields solutions to the loop space self-dual string eqns.

Recall: $\Delta^{\rm IIB} := \bar{\nabla\!\!\!/}^{\rm IIB}\nabla\!\!\!/^{\rm IIB}$, $[\Delta^{\rm IIB},\sigma^i]=0 \Leftrightarrow X^i$ satisfy Nahm eqn.

Here: $\Delta^{\rm M} := \bar{\nabla\!\!\!/}^{\rm M}\nabla\!\!\!/^{\rm M}$, $[\Delta,\gamma^{\mu\nu}]=0 \Leftarrow X^\mu$ satisfy BH eqn.

Our Dirac operator involved loop space, so we need to transgress:

$$H=\left(\varepsilon_{\mu\nu\rho\sigma}\frac{\partial}{\partial x^\sigma}\Phi\right)\mathrm{d}x^\mu\wedge\mathrm{d}x^\nu\wedge\mathrm{d}x^\rho$$

is turned into

$$F_{\mu\nu}(x(\tau)):=\frac{\partial}{\partial x^{[\mu}}A_{\nu]}(x(\tau))=\varepsilon_{\mu\nu\rho\sigma}\dot{x}^\rho(\tau)\frac{\partial}{\partial x^\sigma}\Phi(x(\tau))$$

From normalized, $\mathcal{A}$-valued zero modes $\psi_{s,x(\tau)}$ of $\bar{\nabla\!\!\!/}^{\rm M}$ construct

$$A_\mu=\int\mathrm{d}s\,\bar{\psi}_{s,x(\tau)}\frac{\partial}{\partial x^\mu}\psi_{s,x(\tau)}\ ,\quad \Phi=-\mathrm{i}\int\mathrm{d}s\,\bar{\psi}_{s,x(\tau)}\,s\,\psi_{s,x(\tau)}$$

Verification of the Construction

Verifying the construction is rather straightforward.

$$F_{\mu\nu} = \int \mathrm{d}s\, (\partial_{[\mu}\bar{\psi}_s)\partial_{\nu]}\psi_s$$

$$= \int \mathrm{d}s \int \mathrm{d}t\, (\partial_{[\mu}\bar{\psi}_s) \left(\psi_s\bar{\psi}_t - \nabla_s^{\mathrm{M}} G^{\mathrm{M}}(s,t)\bar{\nabla}_t^{\mathrm{M}}\right) \partial_{\nu]}\psi_t$$

$$= \int \mathrm{d}s \int \mathrm{d}t\, \bar{\psi}_s \left(\gamma^{\mu\kappa}\dot{x}^\kappa G^{\mathrm{M}}(s,t)\gamma^{\nu\lambda}\dot{x}^\lambda - \gamma^{\nu\kappa}\dot{x}^\kappa G^{\mathrm{M}}(s,t)\gamma^{\mu\lambda}\dot{x}^\lambda\right)\psi_t$$

Identity : $[\gamma^{\mu\kappa},\gamma^{\nu\lambda}]\dot{x}^\kappa\dot{x}^\lambda = -2\varepsilon_{\mu\nu\rho\sigma}\gamma^{\sigma\kappa}\gamma_5\dot{x}^\rho\dot{x}^\kappa$

$$F_{\mu\nu} = -\varepsilon_{\mu\nu\rho\sigma}\int \mathrm{d}s \int \mathrm{d}t\, \bar{\psi}_s \left(2\gamma^{\sigma\kappa}\gamma_5 G^{\mathrm{M}}(s,t)\dot{x}^\rho\dot{x}^\kappa\right)\psi_t$$

$$= -\mathrm{i}\varepsilon_{\mu\nu\rho\sigma}\dot{x}^\rho \int \mathrm{d}s \int \mathrm{d}t \Big((\partial_\sigma\bar{\psi}_s)\left(\psi_s\bar{\psi}_t - \nabla_s^{\mathrm{M}} G^{\mathrm{M}}(s,t)\bar{\nabla}_t^{\mathrm{M}}\right) t\,\psi_t +$$

$$\bar{\psi}_s\, s\, \left(\psi_s\bar{\psi}_t - \nabla_s^{\mathrm{M}} G^{\mathrm{M}}(s,t)\bar{\nabla}_t^{\mathrm{M}}\right)\partial_\sigma\psi_t\Big)$$

$$= -\mathrm{i}\varepsilon_{\mu\nu\rho\sigma}\dot{x}^\rho \int \mathrm{d}s\, (\partial_\sigma\bar{\psi}_s)\, s\,\psi_s + \bar{\psi}_s\, s\,\partial_\sigma\psi_s$$

$$= \varepsilon_{\mu\nu\rho\sigma}\dot{x}^\rho\partial_\sigma\Phi$$

Reduction to the ADHMN Construction

The lift reduces in the expected way to the ADHMN construction.

On $\mathcal{L}S^3 \subset \mathcal{L}\mathbb{R}^4$: $x^\mu x^\mu = \dot{x}^\mu \dot{x}^\mu = R^2$, $x^\mu \dot{x}^\mu = 0$.

Reduction (cf. Mukhi/Papageorgakis, 0803.3218):

$$\langle X^4 \rangle = \frac{r}{\ell_p^{3/2}} e_4 = g_{\mathrm{YM}} e_4 \ , \quad \dot{x}^4(\tau_0) = R \Rightarrow \dot{x}^i(\tau_0) = x^4(\tau_0) = 0$$

$$F_{\mu\nu} = \varepsilon_{\mu\nu\rho\sigma} \dot{x}^\rho \frac{\partial}{\partial x^\sigma} \Phi_{\mathrm{SDS}} \quad \rightarrow \quad F_{ij} = \varepsilon_{ijk} \frac{\partial}{\partial x^k} R \Phi_{\mathrm{SDS}} + \dots$$

$$\frac{\mathrm{d}}{\mathrm{d}s} X^\mu = \tfrac{1}{3!} \varepsilon^{\mu\nu\rho\sigma} [X^\nu, X^\rho, X^\sigma] \quad \rightarrow \quad \frac{\mathrm{d}}{\mathrm{d}s} X^i = \tfrac{1}{2} \varepsilon^{ijk} R [X^j, X^k] + \dots$$

$$\nabla\!\!\!/^{\mathrm{M}} = -\gamma_5 \frac{\mathrm{d}}{\mathrm{d}s} + \gamma^{\mu\nu} \left(\tfrac{1}{2} D^{(\rho)}(X^\mu, X^\nu) - \mathrm{i} x^\mu(\tau) \dot{x}^\nu(\tau) \right)$$

$$\rightarrow \ -\gamma_5 \frac{\mathrm{d}}{\mathrm{d}s} + \gamma^{\mu\nu} \left(\tfrac{1}{2} D^{(\rho)}(X^\mu, X^\nu) - \mathrm{i} x^\mu(\tau_0) \dot{x}^\nu(\tau_0) \right)$$

$$= -\gamma_5 \frac{\mathrm{d}}{\mathrm{d}s} + R \gamma^{4i} \left(X^{i\alpha} D^{(\rho)}(e_\alpha, e_4) - \mathrm{i} x^i(\tau_0) \right) + \dots = \nabla\!\!\!/^{\mathrm{IIA}} + \dots$$

Charge 1: Choose again trivial Nahm data. Zero modes:

$$\psi \sim \mathrm{e}^{-R^2 s}\begin{pmatrix} \mathrm{i}\left(R^2 + x^2\dot{x}^1 - x^1\dot{x}^2 - x^4\dot{x}^3 + x^3\dot{x}^4\right) \\ x^3(\dot{x}^1 + \mathrm{i}\dot{x}^2) + x^4(\dot{x}^2 - \mathrm{i}\dot{x}^1) - (x^1 + \mathrm{i}x^2)(\dot{x}^3 - \mathrm{i}\dot{x}^4) \\ 0 \\ 0 \end{pmatrix}$$

Solution:

$$\Phi = \frac{\mathrm{i}}{2R^2}\ ,\ \ F = \frac{2\mathrm{i}\sin\theta^1\sin^2\theta^2(\dot{\theta}^2\,\mathrm{d}\phi\wedge\mathrm{d}\theta^1 - \dot{\theta}^1\,\mathrm{d}\phi\wedge\mathrm{d}\theta^2 + \dot{\phi}\,\mathrm{d}\theta^1\wedge\mathrm{d}\theta^2)}{\sqrt{\dot{\phi}^2 + 2(\dot{\theta}^1)^2 + 4(\dot{\theta}^2)^2 - (\dot{\phi}^2 + 2(\dot{\theta}^1)^2)\cos(2\theta^2) - 2\dot{\phi}^2\cos(2\theta^1)\sin^2\theta^2}}$$

This solves the loop-space self-dual string equation.

Regression:

$$\begin{aligned} H &= F|_{\dot{\theta}^1=1,\dot{\theta}^2=0,\dot{\phi}=0}\wedge\sin\theta^2\mathrm{d}\theta^1 - F|_{\dot{\theta}^1=0,\dot{\theta}^2=1,\dot{\phi}=0}\wedge\mathrm{d}\theta^2 \\ &\quad + F|_{\dot{\theta}^1=0,\dot{\theta}^2=0,\dot{\phi}=1}\wedge\sin\theta^1\sin\theta^2\mathrm{d}\phi \\ &= 6\mathrm{i}\sin\theta^1\sin^2\theta^2\,\mathrm{d}\theta^1\wedge\mathrm{d}\theta^2\wedge\mathrm{d}\phi\ , \end{aligned}$$

This is indeed the expected solution.

Charge 2:

Nahm data:

$$X^\mu = \frac{e_\mu}{\sqrt{2s}}~, \quad e_\mu \text{ generate } A_4$$

Solution:

$$\Phi(x) = \frac{\mathrm{i}}{R^2}$$

As expected: twice the charge of the case $k = 1$.

Remarks

Our lift of the ADHMN construction is very natural and rather straightforward.

- The lift of the Dirac operator was natural considering the corresponding brane configurations.
- It is natural to go to loop space to describe self-dual strings.
- The construction nicely involves the Basu-Harvey equation.
- It reduces nicely to the ADHMN construction.
- The construction does produce transgressed self-dual strings.
- A regression can be performed to get original self-dual string.

Recall the transgression map:

$$(\mathcal{T}\omega)_x(v_1(\tau),\dots,v_k(\tau)) := \int_{S^1} \mathrm{d}\tau\, \omega(v_1(\tau),\dots,v_k(\tau),\dot{x}(\tau))$$

Equations found by Lambert, Papageorgakis, 1007.2982:

$$\nabla^2 X^I - \tfrac{\mathrm{i}}{2}[\bar{\Psi},\Gamma_\nu\Gamma^I\Psi,C^\nu] - [X^J,C^\nu,[X^J,C_\nu,X^I]] = 0$$
$$\Gamma^\mu\nabla_\mu\Psi - [X^I,C^\nu,\Gamma_\nu\Gamma^I\Psi] = 0$$
$$\nabla_{[\mu}H_{\nu\lambda\rho]} + \tfrac{1}{4}\varepsilon_{\mu\nu\lambda\rho\sigma\tau}[X^I,\nabla^\tau X^I,C^\sigma] + \tfrac{\mathrm{i}}{8}\varepsilon_{\mu\nu\lambda\rho\sigma\tau}[\bar{\Psi},\Gamma^\tau\Psi,C^\sigma] = 0$$
$$F_{\mu\nu} - D(C^\lambda,H_{\mu\nu\lambda}) = 0$$
$$\nabla_\mu C^\nu = D(C^\mu,C^\nu) = 0$$
$$D(C^\rho,\nabla_\rho X^I) = D(C^\rho,\nabla_\rho\Psi) = D(C^\rho,\nabla_\rho H_{\mu\nu\lambda}) = 0$$

Factorization of $C^\rho = C\dot{x}^\rho$. Here, 3-Lie algebra transgression:

$$(\mathcal{T}\omega)_x(v_1(\tau),\dots,v_k(\tau)) := \int_{S^1} \mathrm{d}\tau\, D(\omega(v_1(\tau),\dots,v_k(\tau),\dot{x}(\tau)),C)$$

The non-abelian tensor multiplet on loop space

The corresponding equations can all be rewritten on loop space.

Transgression of other fields (missing in Huang, Huang, 1008.3834)

$$\breve{\Psi} = \dot{x}^\rho \Gamma_\rho D(C, \Psi) \quad \text{and} \quad \breve{X}^I = D(C, X^I)$$

Equations of motion (SYM-like):

$$\nabla_\mu F^{\mu\nu} + \tfrac{1}{2}[\breve{X}^I, \nabla^\nu \breve{X}^I] - \tfrac{\mathrm{i}}{4}\left(-[\bar{\breve{\Psi}}, \Gamma^\nu \breve{\Psi}] + [\bar{\breve{\Psi}}, \dot{x}^\nu \dot{x}^\rho \Gamma_\rho \breve{\Psi}]\right) = 0$$

$$\nabla^2 \breve{X}^I - \tfrac{\mathrm{i}}{2}[\bar{\breve{\Psi}}, \dot{x}^\nu \Gamma_\nu \Gamma^I \breve{\Psi}] - [\breve{X}^J, [\breve{X}^J, \breve{X}^I]] = 0$$

$$\Gamma^\mu \dot{x}^\nu \Gamma_\nu \nabla_\mu \breve{\Psi} - \Gamma^I [\breve{X}^I, \breve{\Psi}] = 0$$

Supersymmetry transformations (SYM-like):

$$\delta \breve{X}^I = \mathrm{i}\bar{\varepsilon}\Gamma^I \dot{x}^\rho \Gamma_\rho \breve{\Psi}$$

$$\delta \breve{\Psi} = \dot{x}_\nu \Gamma^{\nu\mu} \Gamma^I \nabla_\mu \breve{X}^I \varepsilon + \tfrac{1}{2\times 3!}\Gamma_{\mu\nu}\Gamma_{\mathrm{ch}} F^{\mu\nu}\varepsilon - \tfrac{1}{2}\Gamma^{IJ}[\breve{X}^I, \breve{X}^J]\varepsilon$$

$$\delta A_\mu = \mathrm{i}\bar{\varepsilon}\Gamma_\mu \breve{\Psi}$$

$$\delta C^\mu = 0$$

Remarks

The loop space tensor multiplet fits well into the picture.

- Note that this is work in progress (with C. Papageorgakis)
- Get SYM theory on loop space from the tensor multiplet
- C-field blocks modes of the theory, need to get rid of it
- Our loop space self-dual string equation extends compatibly:

$$F_{\mu\nu} = \varepsilon_{\mu\nu\rho\sigma}\dot{x}^\rho D(C, \nabla^\sigma X^6)$$

- ADHMN construction for two M5-branes using this equation
- Right direction, more work necessary to get rid of C etc.

Summary:

- ✓ Reformulation of self-dual string equation on loop space
- ✓ Generalized ADHMN construction for self-dual string
- ✓ Explicit construction of $k = 1$ and $k = 2$ examples
- ✓ Reformulate non-abelian tensor multiplet eqns. on loop space
- ✓ Partially generalized ADHMN construction

Future directions:

- ▷ Extend constructions to non-commutative/non-abelian cases
- ▷ Study classical integrability in more detail
- ▷ Quantization of S^3 via gerbes and groupoids

Monopoles, Integrability and M-Theory

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