M2-Branes Ending on M5-Branes

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Hodge theoretic reflections on the string landscape

Based on:

- S. A. Cherkis, CS, PRD 78 (2008) 066019, S. A. Cherkis, V. Dotsenko, CS, PRD 79 (2009) 086002, N. Akerblom, CS and M. Wolf, Nucl. Phys. B (2009), J. DeBellis, CS and R. J. Szabo, arXiv:1001.3275, ...

Effective Descriptions of Branes

While we have an effective description of D-branes, such a theory is missing for M-branes.

Lightning review of branes:

- D-branes appear in string theory as objects that open strings can end on. They correspond to BPS solutions in supergravity.
- IIA: D0, D2, D4, D6, D8, IIB: D(-1), D1, D3, D5, D7, D9
- Dp-brane: spatially p-dimensional object.
- Turn off gravity: we obtain a supersymmetric gauge theory.
- D-branes stacked together increases rank of gauge group.
- They can intersect and sometimes end on each other.
- Two different perspectives of the same configuration: duality.
- IIA string theory/IIA SUGRA: limit of M theory/11d SUGRA.
- In 11d, BPS solutions are M2- and M5-branes.
- An effective theory has been emerging over the last two years.

The Nahm Equation or D1-D3-Branes

In type IIB string theory, monopoles can be interpreted as D1-branes ending on D3-branes.

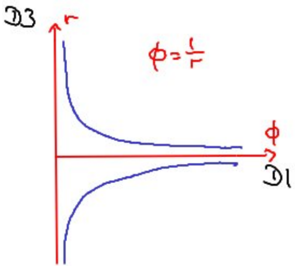
Consider a D3-brane in directions 0123.

A BPS solution to the SYM equations is a magnetic monopole $*F = D\phi$ with Higgs field $\phi \sim \frac{1}{r}$: A D1-brane appears.

As they are BPS, one trivially forms a stack of N D1-branes.

From the perspective of the D1-brane, the effective dynamics is described by the Nahm equations:

$$\frac{\mathrm{d}}{\mathrm{d}\phi}X^i + \varepsilon^{ijk}[X^j, X^k] = 0\ .$$



dim	0	1	2	3	4
$D1$	$\times$				$\times$
$D3$	$\times$	$\times$	$\times$	$\times$	

These equations have the following solution ("fuzzy funnel")

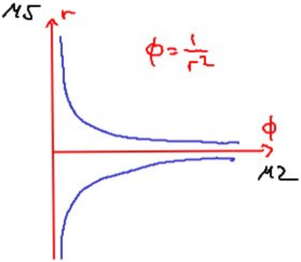
$$X^i = r(\phi)G^i\ , \quad r(\phi) = \frac{1}{\phi}\ , \quad G^i = \varepsilon^{ijk}[G^j, G^k]$$

The Basu-Harvey Equation or M2-M5-Branes

M2 branes ending on M5 branes should be described by Nahm-type equations.

M5-brane in directions 012345:

$$G^{\mu\nu}\nabla_\mu\nabla_\nu X^{a'} = 0$$
$$G^{\mu\nu}\nabla_\mu H_{\nu\rho\sigma} = 0$$



Ansatz for a soliton:

$$X^{6'} = \phi$$
$$H_{01\mu} = v_\mu \quad H_{\mu\nu\rho} = \varepsilon_{\mu\nu\rho\sigma} v^\sigma$$

Solution:

$$H_{01\mu} \sim \partial_\mu \phi \quad \phi \sim \frac{1}{r^2}$$

dim	0	1	2	3	4	5	6
$M2$	×					×	×
$M5$	×	×	×	×	×	×	

Perspective of M2: postulate four scalar fields X^μ, satisfying

$$\frac{\mathrm{d}}{\mathrm{d}\phi} X^\mu + \varepsilon^{\mu\nu\rho\sigma}[X^\nu, X^\rho, X^\sigma] = 0$$

Basu, Harvey, hep-th/0412310

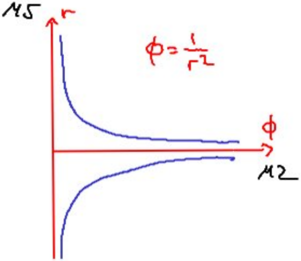
Basu-Harvey equation:

$$\frac{\mathrm{d}}{\mathrm{d}\phi}X^\mu + \varepsilon^{\mu\nu\rho\sigma}[X^\nu, X^\rho, X^\sigma] = 0$$

Solution (similar to D1-D3 case):

$$X^\mu = r(\phi)G^\mu \quad r(\phi) = \frac{1}{\sqrt{\phi}}$$

$$G^\mu = \varepsilon^{\mu\nu\rho\sigma}[G^\nu, G^\rho, G^\sigma]$$



Interpret this again as a fuzzy funnel, this time with a fuzzy S^3?

dim	0	1	2	3	4	5	6
$M2$	×					×	×
$M5$	×	×	×	×	×	×	

Motivating Questions and Outline

What is the gauge structure emerging from 3-brackets?

What is the full effective theory behind this BPS equation?

Does this theory share features with $\mathcal{N} = 4$ SYM theory?

Can one assign geometric meaning to such 3-brackets?

What about a duality with M5-branes?

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What is the algebra behind the triple bracket?

In analogy with Lie algebras, we can introduce 3-Lie algebras.

Basu-Harvey equation:

$$\frac{\mathrm{d}}{\mathrm{d}\phi}X^\mu + \varepsilon^{\mu\nu\rho\sigma}[X^\nu, X^\rho, X^\sigma] = 0\ , \quad X^\mu(\phi) \in \mathcal{A}$$

▷ $\mathcal{A}$ forms a vector space.

▷ $[\cdot,\cdot,\cdot]$ is a totally antisymmetric, linear map $\mathcal{A} \wedge \mathcal{A} \wedge \mathcal{A} \to \mathcal{A}$.

What is the algebra behind the triple bracket?
In analogy with Lie algebras, we can introduce 3-Lie algebras.

Basu-Harvey equation:

$$\frac{\mathrm{d}}{\mathrm{d}\phi}X^\mu + [A_\phi, X^\mu] + \varepsilon^{\mu\nu\rho\sigma}[X^\nu, X^\rho, X^\sigma] = 0\ , \quad X^\mu \in \mathcal{A}$$

▷ Gauge transformations from inner derivations:

The triple bracket forms a map $\delta : \mathcal{A} \wedge \mathcal{A} \to \mathrm{Der}(\mathcal{A}) =: \mathfrak{g}_\mathcal{A}$ via

$$\delta_{A\wedge B}(C) := [A, B, C]$$

Demand a "3-Jacobi identity," the fundamental identity:

$$\begin{aligned}\delta_{A\wedge B}(\delta_{C\wedge D}(E)) &:= [A, B, [C, D, E]] \\ &= [[A, B, C], D, E] + [C, [A, B, D], E] + [C, D, [A, B, E]]\end{aligned}$$

The inner derivations form indeed a Lie algebra:

$$[\delta_{A\wedge B}, \delta_{C\wedge D}](E) := \delta_{A\wedge B}(\delta_{C\wedge D}(E)) - \delta_{C\wedge D}(\delta_{A\wedge B}(E))$$

Bracket closes due to fundamental identity.

n-Lie algebras and Nambu-Poisson structures

Nambu-Poisson structures are special n-Lie algebra structures on $\mathcal{C}^\infty(\mathcal{M})$.

Definition

An n-Lie algebra is a vector space endowed with a totally antisymmetric, n-ary map satisfying the fundamental identity, an "n-Jacobi identity".

Definition

A Nambu-Poisson structure on a smooth manifold $\mathcal{M}$ is a totally antisymmetric, n-ary map $\mathcal{C}^\infty(\mathcal{M})^{\wedge n} \to \mathcal{C}^\infty(\mathcal{M})$ satisfying the fundamental identity

$$\{f_1, \dots, f_{n-1}, \{g_1, \dots, g_n\}\} = \{\{f_1, \dots, f_{n-1}, g_1\}, \dots, g_n\} + \dots + \{g_1, \dots, \{f_1, \dots, f_{n-1}, g_n\}\}$$

as well as the generalized Leibniz rule

$$\{f_1\, f_2, f_3, \dots, f_{n+1}\} = f_1\, \{f_2, \dots, f_{n+1}\} + \{f_1, \dots, f_{n+1}\}\, f_2\ .$$

A_4

Consider the vector space $\mathbb{R}^4$ with basis $\tau_1, ..., \tau_4$.
Then define the bracket $[\cdot,\cdot,\cdot]$ as the linear extension of

$$[\tau_a, \tau_b, \tau_c] = \sum_d \varepsilon_{abcd}\tau_d \quad .$$

Nambu-Poisson structure on S^3

Consider S^3 embedded into $\mathbb{R}^4$ with cartesian coordinates $x^1, ..., x^4$. Define the bracket $\{\cdot,\cdot,\cdot\}$ as the extension via linearity and generalized Leibniz rule of

$$\{x^\mu, x^\nu, x^\kappa\} = \sum_\lambda \varepsilon^{\mu\nu\kappa\lambda} x^\lambda \quad .$$

Short Remark: L_∞-algebras

This structures form a simple strong homotopy Lie algebra.

Note: we could combine $\mathcal{A}$ and $\mathfrak{g}_\mathcal{A}$ into one space $\mathcal{V}$ with two brackets $[\cdot,\cdot]$ and $[\cdot,\cdot,\cdot]$.

Jacobi-Identity and 3-Leibniz rule $\leftrightarrow$ Homotopy Jacobi identities.

$\mathcal{V}$ forms therefore an L_∞- or strong homotopy Lie algebra.

The Basu-Harvey equation is then precisely the homotopy Maurer-Cartan equation for the L_∞ algebra $\mathcal{V}\otimes\Omega^\bullet(\mathbb{R})$.

L_∞ algebras are behind most things coming up. Usefulness unclear.

C. I. Lazaroiu, D. McNamee, CS and A. Zejak, 0901.3905

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Cyclic Bilinear Pairings on Inner Derivations

There are two gauge invariant scalar products on the inner derivations of a 3-Lie algebra.

To write down an action, i.e. gauge invariant terms, we need an invariant pairing on $\mathcal{A}$:

$$(\cdot,\cdot):\mathcal{A}\otimes\mathcal{A}\to\mathbb{C}$$

Invariance under gauge transformations:

$$([A,B,C],D)+(C,[A,B,D])=0$$

On $\mathfrak{g}_{\mathcal{A}}$, there are now two pairings $(\!(\cdot,\cdot)\!)$:

1. The usual Killing form
2. A pairing induced from the pairing on $\mathcal{A}$:

$$(\!(\delta_{A\wedge B},\delta_{C\wedge D})\!)=(D,[A,B,C])$$

Key to constructing a maximally SUSY model later: use the latter.

Approaching the Effective Description of M2-Branes

Spacetime symmetries and BPS equations give helpful constraints on the description.

A stack of flat M2-branes in $\mathbb{R}^{1,10}$ should be effectively described by a conformal field theory with the following constraints:

Spacetime symmetries: $\mathrm{SO}(1,10) \to \mathrm{SO}(1,2) \times \mathrm{SO}(8)$
extended by $\mathcal{N} = 8$ SUSY.

Field content: $X = \Gamma_I X^I$, $I = 1, ..., 8$, and superpartners Ψ_α

Assumption

Take BPS/SUSY transformations from Basu-Harvey equation and therefore the matter fields take values in a metric 3-Lie algebra.

$$\delta X = \mathrm{i}\Gamma_I \bar{\varepsilon}\Gamma^I \Psi \qquad \delta\Psi = \partial_\mu X \Gamma^\mu \varepsilon - \tfrac{1}{6}[X, X, X]\varepsilon$$

Recipe: Try to close SUSY algebra. Constraints yield equations of motion for matter fields.

The Bagger-Lambert-Gustavsson Model

This model is an unconventional supersymmetric Chern-Simons matter theory.

BLG found that for SUSY, we need to introduce gauge symmetry.
$\Rightarrow$ Field content: $X \in \mathcal{A}$, $\Psi \in \mathcal{A}$ and gauge potential $A_\mu \in \mathfrak{g}_\mathcal{A}$.

The Bagger-Lambert-Gustavsson model

$$\begin{aligned} \mathcal{L}_{\rm BLG} \;=\; &+ \tfrac{k}{4\pi}\varepsilon^{\mu\nu\kappa}\left(\langle\!\langle A_\mu, \partial_\nu A_\kappa\rangle\!\rangle + \tfrac{1}{3}\langle\!\langle A_\mu, [A_\nu, A_\kappa]\rangle\!\rangle\right) \\ &- \tfrac{1}{2}(\nabla_\mu X, \nabla^\mu X)_{Cl} + \tfrac{\mathrm{i}}{2}(\bar{\Psi}, \Gamma^\mu \nabla_\mu \Psi) \\ &+ \tfrac{\mathrm{i}}{4}(\bar{\Psi}, [X, X, \Psi]) - \tfrac{1}{12}([X, X, X], [X, X, X])_{Cl} \end{aligned}$$

This model is invariant under the supersymmetry transformations:

$$\delta X = \mathrm{i}\Gamma_I \bar{\varepsilon}\Gamma^I \Psi \;, \qquad \delta\Psi = \nabla_\mu X \Gamma^\mu \varepsilon - \tfrac{1}{6}[X, X, X]\varepsilon \;,$$
$$\delta A_\mu = \mathrm{i}\bar{\varepsilon}\Gamma_\mu(\delta_{X\wedge\Psi})$$

Consistency checks

The BLG model passes a number of consistency checks.

$$\mathcal{L}_{\text{BLG}} = + \frac{k}{4\pi}\varepsilon^{\mu\nu\kappa}\left(\left(\!\left(A_\mu, \partial_\nu A_\kappa\right)\!\right) + \frac{1}{3}\left(\!\left(A_\mu, [A_\nu, A_\kappa]\right)\!\right)\right)$$
$$- \frac{1}{2}(\nabla_\mu X, \nabla^\mu X)_{Cl} + \frac{\mathrm{i}}{2}(\bar{\Psi}, \Gamma^\mu \nabla_\mu \Psi)$$
$$+ \frac{\mathrm{i}}{4}(\bar{\Psi}, [X, X, \Psi]) - \frac{1}{12}([X, X, X], [X, X, X])_{Cl}$$

Further results:

- The model is classically conformal and seems rather unique.
- If $\mathcal{N} = 8$ SUSY not anomalous $\Rightarrow$ vanishing β-function
- The model is parity invariant.
- Under some assumptions: reduction mechanism M2→D2.

(Mukhi, Papageorgakis,0803.3218)

- $k = 2$: moduli space matches 2 M2-branes at tip of $\mathbb{R}^8/\mathbb{Z}_2$.
- Recast into the ABJM version, it yields integrable spin chain.

Extending The Structure of a 3-Lie Algebra

The notion of a 3-Lie algebra is too restrictive and one has to find a generalized notion.

Problem: Given a three-algebra $\mathcal{A}$, if its bilinear form $(\cdot,\cdot)$ is positive definite, then $\mathcal{A}$ is A_4 or a direct sum thereof.

A_4 supposedly describes a stack of 2 M2-branes, not enough.

Mukhi, Papageorgakis, 0803.3218

Possible extensions:

(1) Assume, 3-Lie algebras appear accidentally $\Rightarrow$ ABJM model

(2) Give up positive definiteness of $(\cdot,\cdot)$ $\Rightarrow$ ghosts

(3) Relax conditions on 3-Lie algebras (+monopole operators)

Guideline: Demand gauge invariance and at least $\mathcal{N}=2$ SUSY:

$$\begin{aligned}([A,B,C],D) &= -([B,A,C],D)\\ &= -([A,B,D],C) = ([C,D,A],B)\end{aligned}$$

S. A. Cherkis, CS, 0807.0808

Generalized metric 3-Lie algebras or real 3-algebras

Same as a 3-Lie algebra, but relax $([A,B,C],D)$ from totally antisymmetric to the above symmetry properties.

Hermitian 3-Lie Algebras

Another generalization of 3-Lie algebras are the Hermitian ones yielding $\mathcal{N}=6$ SUSY.

Alternatively to our way of extending 3-Lie algebras:

Reduce supersymmetry to $\mathcal{N}=6$, i.e. assume the following:

$$\begin{aligned}
\delta\phi^i &= \sqrt{2}\bar{\varepsilon}^{ij}\bar{\psi}_j\ , \\
\delta\bar{\psi}_i &= -\mathrm{i}\sqrt{2}\sigma^\mu\varepsilon_{ij}\nabla_\mu\phi^j + [\phi^j,\phi^k;\bar{\phi}_j]\varepsilon_{ik} + [\phi^j,\phi^k;\bar{\phi}_i]\varepsilon_{jk}\ , \\
\delta A_\mu &= -\mathrm{i}\varepsilon_{ij}\sigma_\mu\phi^i\wedge\psi^j + \mathrm{i}\bar{\varepsilon}^{ij}\sigma_\mu\bar{\phi}_i\wedge\bar{\psi}_j\ .
\end{aligned}$$

where ε^{ij} is in the $\mathbf{6}$ of $SU(4)$. Closure of this algebra implies:

$$[A,B;C] = -[B,A;C] \qquad ([A,B;C],D) = (B,[C,D;A])\ .$$

$$[[C,D;E],A;B] - [[C,A;B],D;E] - [C,[D,A;B];E] + [C,D;[E,B;A]] = 0\ .$$

This leads to the ABJM model, a Chern-Simons-matter theory.

Aharony, Bergman, Jafferis, Maldacena, 0806.1218
Bagger, Lambert, 0807.0163

Unifying picture: de Medeiros, Figueroa-O'Farrill, Mendez-Escobar, Ritter, 0809.1086

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Recovering SYM Features: Marginal Deformations

The BLG model shares features with $\mathcal{N}=4$ SYM. What about marginal deformations?

Observation: BLG/ABJM seems similar to $\mathcal{N}=4$ SYM
($\rightarrow$ integrable spin chains).

$\mathcal{N}=4$ SYM admits (exactly) marginal deformations:

$$\mathcal{W} = \varepsilon_{ijk}\,\mathrm{tr}\,([\Phi^i,\Phi^j]_\beta\Phi^k)$$
$$[\Phi^i,\Phi^j]_\beta := \mathrm{e}^{\mathrm{i}\beta}\Phi^i\Phi^j - \mathrm{e}^{-\mathrm{i}\beta}\Phi^j\Phi^i$$

R. G. Leigh and M.J. Strassler, Nucl. Phys. B 447 (1995).

Conformality for β-deformed SYM to all orders in perturbation theory: S. Ananth, S. Kovacs, H. Shimada, JHEP 01 (2007) 046.

Such deformations correspond to deformations of $AdS_5\times S^5$.

Similar deformations for $AdS_4\times S^7$ in the literature.
What about BLG/ABJM and their deformations on quantum level?

Manifestly $\mathcal{N} = 2$ SUSY Formulation

There is a manifestly $\mathcal{N} = 2$ SUSY formulation, allowing for various deformations.

Approach: Take $\mathcal{N} = 1$, 4d superspace $\mathbb{R}^{1,3|4}$ and reduce to 3d.

Field content of the theory:

- The matter fields X^I, Ψ are encoded in four chiral multiplets:

$$\Phi^i(y) = \phi^i(y) + \sqrt{2}\theta\psi^i(y) + \theta^2 F^i(y) \ ,$$

- The gauge potential A_μ is contained in a vector superfield:

$$\begin{aligned} V(x) \ = \ & -\theta^\alpha\bar{\theta}^{\dot{\alpha}}(\sigma^\mu_{\alpha\dot{\alpha}}A_\mu(x) + \mathrm{i}\varepsilon_{\alpha\dot{\alpha}}\sigma(x)) \\ & + \mathrm{i}\theta^2(\bar{\theta}\bar{\lambda}(x)) - \mathrm{i}\bar{\theta}^2(\theta\lambda(x)) + \tfrac{1}{2}\theta^2\bar{\theta}^2 D(x) \ , \end{aligned}$$

$\mathcal{N} = 2$ superspace formulation of BLG (Cherkis, CS, 0807.0808)

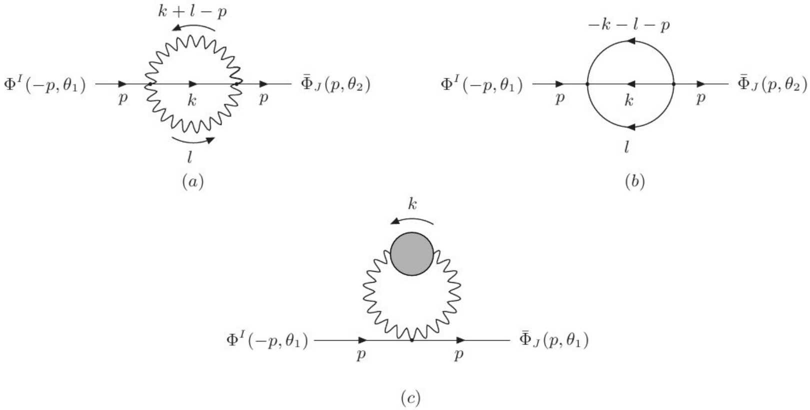
$$\mathcal{L} = \int \mathrm{d}^4\theta \ \kappa \left(\mathrm{i}(\!(V, (\bar{D}_\alpha D^\alpha V)\,)\!) + \tfrac{2}{3}(\!(V, \{(\bar{D}^\alpha V), (D_\alpha V)\})\!)\right)$$

$$+ (\bar{\Phi}_i, \mathrm{e}^{2iV} \cdot \Phi^i) + \alpha\left(\int \mathrm{d}^2\theta \ \varepsilon_{ijkl}([\Phi^i, \Phi^j, \Phi^k], \Phi^l) + c.c.\right)$$

Contributing diagrams

At 2 loop level, only three classes of diagrams contribute.

Contributing diagrams (only 2-pt contributions are divergent):



Potential flow of the couplings due to anomalous dimensions.

Results: The β-function for multitrace deformations

The BLG model is conformally invariant at two loops.

Example for a deformation:

$$\mathcal{W} = \Big[R^{(1)}_{ijkl}(\Phi^l, [\Phi^i, \Phi^j, \Phi^k]) + R^{(2)}_{ijkl}(\Phi^i, \Phi^j)(\Phi^k, \Phi^l)\Big]$$

Total anomalous dimension:

$$\begin{aligned}\gamma_i{}^j \ = \ \frac{1}{8\pi^2\kappa^2}\Big\{&\big[k_2 + k_1^2 + \tfrac{1}{12}(2k_2 + N_f k_3)\big]\delta_i{}^j \\ &+ \ 8\kappa^2\Big[R^{(1)}_{iklm}\big(-c_3 R^{jklm}_{(1)} + 2c_2 R^{jmlk}_{(1)} + 2c_1 R^{jmlk}_{(2)}\big) \\ &+ R^{(2)}_{iklm}\big(d\, R^{jklm}_{(2)} + 2R^{jmlk}_{(2)} + 2c_1 R^{jmlk}_{(1)}\big)\Big]\Big\}\end{aligned}$$

Quick test: BLG. $R^{(2)}_{ijkl} = 0$, $\mathcal{A} = A_4$, therefore $R^{(1)}_{ijkl} = \lambda\varepsilon_{ijkl}$ and

$$d = 4 \quad k_1 = 0 \quad k_2 = -3 \quad k_3 = 6 \quad c_1 = 0 \quad c_2 = c_3 = -6$$

The β-function reads as (the phase does not flow)

$$\beta^{(1)}_{ijkl} = -\tfrac{3}{4\pi^2\kappa^2}\big[1 - (4!\kappa)^2|\lambda|^2\big]R^{(1)}_{ijkl} \quad \text{so} \quad |\lambda| = \frac{1}{4!\kappa}$$

Discussion of results

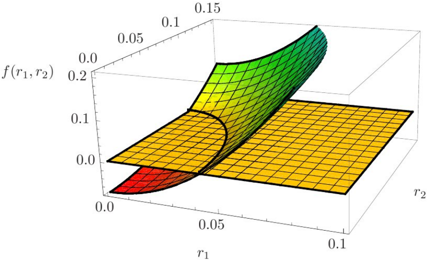
The running of the coupling is exactly as expected.

For simplicity, we take $\mathcal{A} = A_4$ and the superpotential

$$R^{(1)}_{ijkl} = \frac{\lambda_1}{\kappa}\,\varepsilon_{ijkl} \quad \text{and} \quad R^{(2)}_{ijkl} = \frac{\lambda_2}{\kappa}\,\delta_{ij}\delta_{kl}\ , \qquad \lambda_i = r_i \mathrm{e}^{\varphi_i}$$

The β-function at two loops reads as (phases do not flow)

$$\beta^{(\ell)}_{ijkl} = \frac{f(r_1, r_2)}{\kappa^2} R^{(\ell)}_{ijkl} \qquad f(r_1, r_2) := -\frac{3}{4\pi^2}\left[1 - 96\left(6r_1^2 + r_2^2\right)\right]$$



BLG: $r_1 = \frac{1}{24}, r_2 = 0$

points on ellipse:
IR fixed points

Recover β-deformations

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Geometries we will focus on

Aim: To make sense of fuzzy S^3 and the geometry of the Nambu-Heisenberg algebra.

We will focus on the two most obvious geometries:

- D1-D3 yields fuzzy S^2.

  ▷ What is the definition of fuzzy S^3 appearing for M2-M5?

  This has many interesting implications: BLG, ...

- Best-known NC geometry: Moyal plane $[x^1, x^2] \sim \mathbb{1}$.

  ▷ What is the NC geometry of $[x^1, x^2, x^3] \sim \mathbb{1}$?

  This Nambu-Heisenberg algebra was found as the WV equation of M5-branes in certain backgrounds

  C. Chu and D. J. Smith, arXiv:0901.1847

Axioms of Quantization

Quantization is nontrivial and far from being fully understood.

Classical level: states are points on a Poisson manifold $\mathcal{M}$.
observables are functions on $\mathcal{M}$.

Quantum level: states are rays in a complex Hilbert space $\mathscr{H}$.
observables are hermitian operators on $\mathscr{H}$.

Full Quantization

A full quantization is a map $\hat{-} : \mathcal{C}^\infty(\mathcal{M}) \to \mathrm{End}\,(\mathscr{H})$ satisfying

1. $f \mapsto \hat{f}$ is linear over $\mathbb{C}$, $f = f^* \Rightarrow \hat{f} = \hat{f}^\dagger$.
2. the constant function $f = 1$ is mapped to the identity on $\mathscr{H}$.
3. Correspondence principle: $\{f_1, f_2\} = g \Rightarrow [\hat{f}_1, \hat{f}_2] = \hat{g}$.
4. The quantized coordinate functions act irreducibly on $\mathscr{H}$.

Problem:

Groenewold-van Howe: no full quant. for $T^*\mathbb{R}^n$ or S^2 (T^2 OK)

Loopholes to the obstructions to full quantizations

There are three interesting weakenings of the axioms for quantization.

Three approaches to weaken the axioms of a full quantization:

- Drop irreducibility
- Quantize a subset of $\mathcal{C}^\infty(\mathcal{M})$
- Correspondence principle applies only to $\mathcal{O}(\hbar)$

The first two yield prequantization and geometric quantization.
The last approach leads eventually to deformation quantization.

We will use Berezin quantization (or fuzzy geometry),
a hybrid of geometric and deformation quantization.

Berezin-Toeplitz Quantization of $\mathbb{C}P^1 \simeq S^2$

The fuzzy sphere is the Berezin quantization of $\mathbb{C}P^1$.

Hilbert space

$\mathscr{H}$ is the space of global holomorphic sections of a certain line bundle: $\mathscr{H} = H^0(\mathcal{M}, L)$. For $\mathcal{M} = \mathbb{C}P^1$: $L := \mathcal{O}(k)$.

$$\mathscr{H}_k \cong \mathrm{span}(z_{\alpha_1} \dots z_{\alpha_k}) \cong \mathrm{span}(\hat{a}^\dagger_{\alpha_1} \dots \hat{a}^\dagger_{\alpha_k} |0\rangle)$$

Coherent states (Algorithm by Rawnsley)

For any $z \in \mathcal{M}$: coherent st. $|z\rangle \in \mathscr{H}$. Here: $|z\rangle = \frac{1}{k!}(\bar{z}_\alpha \hat{a}^\dagger_\alpha)^k |0\rangle$.

Quantization

The coherent states can be used as a bridge in two ways:

$$f(z) = \sigma(\hat{f}) = \frac{\langle z|\hat{f}|z\rangle}{\langle z|z\rangle} \quad \text{or} \quad \hat{f} = \int \frac{\omega^n}{n!} \frac{|z\rangle\langle z|}{\langle z|z\rangle} f \; .$$

Axioms of Generalized Quantization

We propose a generalization of the quantization axioms to Nambu-Poisson manifolds.

Problem is notoriously difficult, and many people tried to extend geometric quantization. Berezin quantization should be easier. Keep: a complex Hilbert space $\mathscr{H}$ and $\mathsf{End}\,(\mathscr{H})$ as observables.

Generalized quantization axioms

A NP quantization is a map $\hat{\ }: \Sigma \to \mathsf{End}\,(\mathscr{H}),\ \Sigma \subset \mathcal{C}^\infty(M)$ satisfying

1. $f \mapsto \hat{f}$ is linear over $\mathbb{C}$, $f = f^* \Rightarrow \hat{f} = \hat{f}^\dagger$.
2. the constant function $f = 1$ is mapped to the identity on $\mathscr{H}$.
3. Correspondence principle:

$$\lim_{\hbar\to 0} \left\| \frac{\mathrm{i}}{\hbar}\,\sigma\big([\hat{f}_1,\dots,\hat{f}_n]\big) - \{f_1,\dots,f_n\} \right\|_{L^2} = 0$$

If $\mathcal{M}$ is a Poisson manifold, this holds for Berezin quantization.

A natural n-Lie bracket

Truncating the Nambu-Poisson algebra allows for an unusual n-Lie bracket.

On the algebra of polynomials, one can often truncate the Nambu-Poisson algebra to obtain a corresponding n-Lie algebra.

Then one can introduce

$$[\hat{A}_1, \ldots, \hat{A}_n] := \sigma^{-1}(-\mathrm{i}\hbar\{\sigma(\hat{A}_1), \ldots, \sigma(\hat{A}_n)\}_K) \ ,$$

and the correspondence principle holds always automatically.

Interesting is the comparison of this to the totally antisymmetric operator product.

Quantization of S^4

Our quantization of S^4 yields the noncommutative spheres of Guralnik/Ramgoolam.

Observation:

Using the Clifford algebra $Cl(\mathbb{R}^4)$, we find embedding $S^4 \hookrightarrow \mathbb{C}P^3$:

$$x^\mu = \frac{R}{|z|^2} \gamma^\mu_{\alpha\beta} \bar{z}^\alpha z^\beta \ , \quad \sum_\mu x^\mu x^\mu = R^2 \ .$$

Embedding not holomorphic, otherwise: factor out ideal:

$$\mathcal{M} = \{z \in \mathbb{C}P^n | \, p(z) = 0\} \to \mathscr{H}_\mathcal{M} = \{|\mu\rangle \in \mathscr{H}_{\mathbb{C}P^n} | \, \hat{p}|\mu\rangle = 0\}$$

CS, hep-th/0612173

Make the following idea rigorous:

$$\hat{x}^\mu := \frac{R}{|z|^2} \gamma^\mu_{\alpha\beta} \frac{\hat{a}^\dagger_\alpha \hat{a}^\dagger_{\gamma_1} \dots \hat{a}^\dagger_{\gamma_{k-1}} |0\rangle\langle 0| \hat{a}_\beta \hat{a}_{\gamma_1} \dots \hat{a}_{\gamma_{k-1}}}{k!}$$

This satisfies $\sum_\mu \hat{x}^\mu \hat{x}^\mu \sim R^2 \mathbb{1}$ and on linear level is identical to the totally antisymmetric operator product. This quantization yields the Guralnik/Ramgoolam spheres. Hyperboloids, ...

Z. Guralnik and S. Ramgoolam, hep-th/0101001

Quantization of $\mathbb{R}^3$

The quantized Nambu-Heisenberg algebra corresponds to the space $\mathbb{R}^3_\lambda$.

What is the geometry of $[\hat{x}, \hat{y}, \hat{z}] = -\mathrm{i}\,\hbar\,\mathbb{1}$?

No 3-bracket ensuring the correspondence principle.
$\Rightarrow$ 3-algebra structure only at linear level.

One possible interpretation as $\mathbb{R}^3_\lambda$:
Take a fuzzy sphere with Hilbert space $H^0(\mathbb{C}P^1, \mathcal{O}(k))$. Define:

$$[\hat{x}^1, \hat{x}^2, \hat{x}^3] = \sum_{i,j,k} \varepsilon^{ijk}\,\hat{x}^i\,\hat{x}^j\,\hat{x}^k = -\mathrm{i}\,\frac{6R^3}{k}\,\mathbb{1}_{\mathscr{H}_k}$$

Radius of this fuzzy sphere: $R_{F,k} = \sqrt{1+\frac{2}{k}}\;\sqrt[3]{\frac{\hbar\,k}{6}}$.
Now "discretely foliate" $\mathbb{R}^3$ by fuzzy spheres. $\Rightarrow \mathbb{R}^3_\lambda$.

A. B. Hammou, M. Lagraa, M. M. Sheikh-Jabbari, hep-th/0110291

Motivating Questions and Outline

What is the gauge structure emerging from 3-brackets?

What is the full effective theory behind this BPS equation?

Does this theory share features with $\mathcal{N} = 4$ SYM theory?

Can one assign geometric meaning to such 3-brackets?

- What about a duality with M5-branes?

The following is work in progress and still sketchy.

Recall: Monopoles are solutions to the Bogomolny equation

$$F_{ij} = \varepsilon_{ijk} D_k \phi$$

ADHMN construction ($\sim$ Fourier-Mukai transform)

1. Take solutions X^i to the Nahm eq. $\frac{\mathrm{d}}{\mathrm{d}s} X^i + \varepsilon^{ijk}[X^j, X^k] = 0$
2. Construct Dirac operator $\nabla\!\!\!/ := \frac{\mathrm{d}}{\mathrm{d}s} + \sigma_i X^i$.
   Because of the Nahm equation, $\mathbb{1} \sim \nabla\!\!\!/^\dagger \nabla\!\!\!/ > 0$.
3. Twist the Dirac operator to read as $\nabla\!\!\!/ := \frac{\mathrm{d}}{\mathrm{d}s} + \sigma_i(x^i + X^i)$.
4. Monopole: $A_i = \int \mathrm{d}s\, \psi^\dagger \partial_i \psi$, $\phi = \int \mathrm{d}s\, \psi^\dagger s \psi$, if $\nabla\!\!\!/^\dagger \psi = 0$.

Note: We ignore here boundary conditions, norm and dim. of ψ.

Lift of the ADHMN Construction to M2-branes

A Dirac operator yielding the Basu-Harvey equation is readily found.

IIB	0	1	2	3	4	5	6
$D1$	×						×
$D3$	×	×	×	×			

$$\nabla\!\!\!/ = \frac{\mathrm{d}}{\mathrm{d}s} + \sigma_i X^i$$

IIA	0	1	2	3	4	5	6
$D2$	×					×	×
$D4$	×	×	×	×		×	

$$\nabla\!\!\!/ = \gamma_5 \frac{\mathrm{d}}{\mathrm{d}s} + \gamma_4 \gamma_i X^i$$

M	0	1	2	3	4	5	6
$M2$	×					×	×
$M5$	×	×	×	×	×	×	

$$\nabla\!\!\!/ = \gamma_5 \frac{\mathrm{d}}{\mathrm{d}s} + \gamma_{\mu\nu} [X^\mu, X^\nu, \cdot\,]$$

The condition $\mathbb{1} \sim \nabla\!\!\!/^\dagger \nabla\!\!\!/ > 0$ yields:

$$\frac{\mathrm{d}}{\mathrm{d}s} X^\mu + \varepsilon^{\mu\nu\rho\sigma} [X^\nu, X^\rho, X^\sigma] = 0$$

$$\varepsilon_{\mu\nu\rho\sigma} [X^\mu, X^\nu, [X^\rho, X^\sigma, \cdot\,] = 0$$

Twisting the Dirac Operator

The problem here is that the B-field belongs to an abelian gerbe.

We want to "construct" solutions $B_{\mu\nu}$, ϕ to $\partial_{[\mu} B_{\nu\rho]} = \varepsilon_{\mu\nu\rho\sigma}\partial_\sigma\phi$

How do we twist the Dirac operator for this purpose?

Note: B-field really belongs to an abelian gerbe with connection.

For a local description, go to loop space: $x^\mu(\tau)$, $\dot{x}^\mu(\tau)$.

Twisted Dirac operator:

$$\nabla\!\!\!/ = \gamma_5 \frac{\mathrm{d}}{\mathrm{d}s} + \gamma_{\mu\nu} x^\mu \dot{x}^\nu + \gamma_{\mu\nu}[X^\mu, X^\nu, \cdot\,]$$

Work with the following fields:

$$A_\mu := B_{\mu\nu}\dot{x}^\nu = \int \mathrm{d}s\, \psi^\dagger \partial_\mu \psi\ , \quad \phi = \int \mathrm{d}s\, \psi^\dagger s \psi$$

Auxiliary Fields on Loop Space

The auxiliary fields allow for solutions and reduce nicely to the Bogomolny equations.

$$A_\mu := B_{\mu\nu}\dot{x}^\nu = \int \mathrm{d}s\, \psi^\dagger \partial_\mu \psi \ , \quad \phi = \int \mathrm{d}s\, \psi^\dagger s \psi$$

Note that consistency requires $\dot{x}^\mu A_\mu = 0$, so we impose $\dot{x}^\mu \partial_\mu f = 0$ for all loop fields f. We can then rewrite:

$$\partial_{[\mu} B_{\nu\rho]} = \varepsilon_{\mu\nu\rho\sigma}\partial_\sigma \phi \rightarrow \partial_{[\mu} A_{\nu]} = \varepsilon_{\mu\nu\rho\sigma}\dot{x}^\rho \partial_\sigma \phi \ .$$

Results:

- Constructed fields A_μ, ϕ satisfy this equation automatically.
- $\dot{x}^\mu \partial_\mu r^2 = 0 \rightarrow x^\mu \dot{x}^\mu = 0$: loops are located on spheres.
- $X^\mu = 0$-solution: $\phi = \frac{1}{\sqrt{x^\mu x^\mu \dot{x}^\nu \dot{x}^\nu - (x^\mu \dot{x}^\mu)^2}}$, gauge choice: $\dot{x}^\mu \dot{x}^\mu = x^\mu x^\mu$ yields solution to self-dual string.
- Back to IIA: $\dot{x}^4 = r$: Bogomolny, matches $X^\natural = r\tau^4$.

alternative approach: Gustavsson 0802.3456

Summary:

- We have a candidate for an effective description of M2-branes.
- It shares many features with $\mathcal{N}=4$ SYM theory.
- Naive extension of quantization to Nambu-Poisson manifolds.
- NC interpretation of fuzzy 3-funnel and NH algebra.
- M5-brane geometry in M2-M5 + background: $\mathbb{R}^{1,2}_\lambda \times \mathbb{R}^3_\lambda$.
- We have a sketchy analogue of the ADHMN construction.

Future directions:

- Quantization of S^3 via gerbes.
- Understand Nahm transform for M2-M5.
- Study classical integrability in this setting.

M2-Branes Ending on M5-Branes

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Hodge theoretic reflections on the string landscape