

# On Stacks of M2-Branes

Christian Sämann



*School of Mathematics, Trinity College Dublin*

Irish Quantum Field Theory Meeting, May, 15th 2009

Based on:

- S. A. Cherkis, CS, [PRD 78 \(2008\) 066019](#)
- S. A. Cherkis, V. Dotsenko, CS, [PRD 79 \(2009\) 086002](#)
- C. I. Lazaroiu, D. McNamee, CS and A. Zejak, [arX:0901.3905](#)

- **Review part**
  - The **Nahm** equation or **D1-D3** branes
  - The **Basu-Harvey** equation or **M2-M5** branes
  - **Stacks** of flat M2-branes: The **BLG** model
- **Superspace formulations** of BLG-like models
  - Manifestly  $\mathcal{N} = 2$  supersymmetric formulation
  - Manifestly  $\mathcal{N} = 4$  supersymmetric formulation
- **Generalized 3-Lie algebras** and BLG-like models
  - The **structure** of generalized 3-Lie algebras
  - The **unifying picture** by Figueroa-O'Farrill et al.
  - **Representations** on  $*$ -algebras
- The framework of **strong homotopy Lie algebras**
  - $L_\infty$  algebras and **homotopy Maurer-Cartan (hMC) equations**
  - The Nahm and the **Basu-Harvey** equations as hMC equations
  - The **BLG** equation as hMC equations

# The Nahm Equation or D1-D3-Branes

In type IIB string theory, monopoles can be seen as D1-branes ending on D3-branes.

Consider a **D3-brane** in directions **0234**.

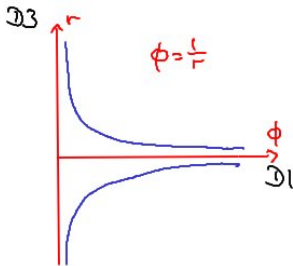
A BPS solution to the SYM equations is the magnetic monopole with Higgs field

$\phi \sim \frac{1}{r}$ : A **D1-brane** appears.

As they are BPS, one trivially forms a stack of  $N$  **D1-branes**.

From the perspective of the **D1-brane**, the effective dynamics is described by the **Nahm equations**:

$$\frac{d}{d\phi} X^i + \varepsilon^{ijk} [X^j, X^k] = 0 .$$



	dim	0	1	2	3	4
$D1$		×	×			
$D3$		×		×	×	×

These equations have the following solution (“**fuzzy funnel**”)

$$X^i = r(\phi) G^i , \quad r(\phi) = \frac{1}{\phi} , \quad G^i = \varepsilon^{ijk} [G^j, G^k]$$

# The Basu-Harvey Equation or M2-M5-Branes

M2 branes ending on M5 branes should be described by Nahm-type equations.

**M5-brane** in directions 013456:

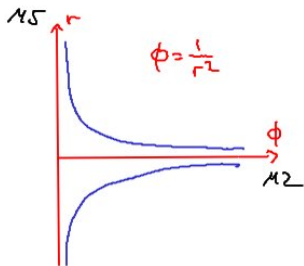
$$G^{mn} \nabla_m \nabla_n X^{a'} = 0$$

$$G^{mn} \nabla_m H_{npq} = 0$$

Ansatz for a soliton:

$$X^{5'} = \phi$$

$$H_{01m} = v_m \quad H_{mnpq} = \varepsilon_{mnpq} v^q$$



Solution:

$$H_{01m} \sim \partial_m \phi \quad \phi \sim \frac{1}{r^2}$$

dim	0	1	2	3	4	5	6
M2	×	×	×				
M5	×	×		×	×	×	×

**Perspective of M2:** postulate four scalar fields  $X^i$ , satisfying

$$\frac{d}{d\phi} X^i + \varepsilon^{ijkl} [X^j, X^k, X^l] = 0$$

Basu, Harvey, hep-th/0412310

# The Basu-Harvey Equation or M2-M5-Branes

M2 branes ending on M5 branes should be described by Nahm-type equations.

Basu-Harvey equation:

$$\frac{d}{d\phi} X^i + \varepsilon^{ijkl} [X^j, X^k, X^l] = 0$$

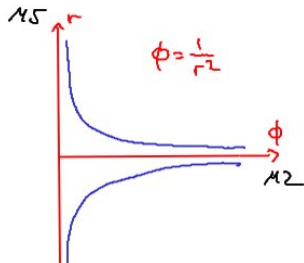
**Solution** (similar to D1-D3 case):

$$X^i = r(\phi) G^i \quad r(\phi) = \frac{1}{\sqrt{\phi}}$$

$$G^i = \varepsilon^{ijkl} [G^j, G^k, G^l]$$

Interprete this again as a **fuzzy funnel**, this time with a fuzzy  $S^3$  at every point  $\phi$  (not quite...).

$$R \sim \sqrt{N} \quad \text{dofs} \sim R^3 \sim N^{3/2} \quad \checkmark$$



dim	0	1	2	3	4	5	6
$M2$	$\times$	$\times$	$\times$				
$M5$	$\times$	$\times$		$\times$	$\times$	$\times$	$\times$

# Metric 3-Lie Algebras

3-Lie algebras come with a triple bracket and an induced Lie algebra structure.

metric 3-Lie algebras (Filippov, 1985)

$\mathcal{A}$  a real vector space with a bracket  $[\cdot, \cdot, \cdot] : \Lambda^3 \mathcal{A} \rightarrow \mathcal{A}$  satisfying

$$[A, B, [C, D, E]] = \\ [[A, B, C], D, E] + [C, [A, B, D], E] + [C, D, [A, B, E]] \quad (\text{FI})$$

and a bilinear symmetric map  $(\cdot, \cdot)_{\mathcal{A}} : \mathcal{A} \otimes \mathcal{A} \rightarrow \mathbb{R}$  satisfying

$$([A, B, C], D)_{\mathcal{A}} + (C, [A, B, D])_{\mathcal{A}} = 0 \quad (\text{Cmp})$$

There is a map from  $\mathcal{A} \wedge \mathcal{A}$  to  $\text{Der}(\mathcal{A})$  given by linearly extending

$$D_{A \wedge B}(C) := [A, B, C], \quad A, B, C \in \mathcal{A}$$

The inner derivations  $\mathfrak{g}_{\mathcal{A}} := \text{im}(D_{\mathcal{A} \wedge \mathcal{A}})$  form a Lie algebra.

Two invariant pairings on  $\mathfrak{g}_{\mathcal{A}}$ :  $(A \wedge B, C \wedge D)_{\mathfrak{g}} := ([A, B, C], D)_{\mathcal{A}}$   
and induced Killing form.

# The Metric 3-Lie Algebra $A_4$

The 3-Lie algebra  $A_4$  is the most important 3-Lie algebra in the context of BLG.

Consider the vector space  $\mathbb{R}^4$  with basis  $\tau_1, \dots, \tau_4$ .

Then define the bracket  $[\cdot, \cdot, \cdot]$  as the linear extension of

$$[\tau_a, \tau_b, \tau_c] = \sum_d \varepsilon_{abcd} \tau_d \ .$$

The bilinear symmetric map  $(\cdot, \cdot)_{A_4}$  is given as the lin. extension of

$$(\tau_a, \tau_b)_{A_4} = \delta_{ab} \ .$$

The **associated Lie algebra** is  $\mathfrak{so}(4) \cong \mathfrak{su}(2) \times \mathfrak{su}(2)$ ;

its bilinear pairing  $(\cdot, \cdot)_{\mathfrak{g}}$  has **split signature**.

# Approaching the Effective Description of M2-Branes

Spacetime symmetries and BPS equations give helpful constraints on the description.

A stack of flat **M2-branes** in  $\mathbb{R}^{1,10}$  should be effectively described by a conformal field theory with the following constraints:

Spacetime symmetries:  $SO(1, 10) \rightarrow SO(1, 2) \times SO(8)$   
extended by  $\mathcal{N} = 8$  **SUSY**.

Field content:  $X = \Gamma_I X^I$ ,  $I = 1, \dots, 8$ , and superpartners  $\Psi_\alpha$

## Assumption

Take **BPS/SUSY transformations** from **Basu-Harvey** equation and therefore the matter fields take values in a **metric 3-Lie algebra**.

$$\delta X = i\Gamma_I \bar{\epsilon} \Gamma^I \Psi \quad \delta \Psi = \partial_\mu X \Gamma^\mu \epsilon - \frac{1}{6} [X, X, X] \epsilon$$

**Recipe:** Try to close SUSY algebra. Constraints yield equations of motion for matter fields.



# The Bagger-Lambert-Gustavsson Model

This model is an unconventional supersymmetric Chern-Simons matter theory.

BLG found that for **SUSY**, we need to introduce gauge symmetry.  
 $\Rightarrow$  Field content:  $X \in \mathcal{A}$ ,  $\Psi \in \mathcal{A}$  and gauge potential  $A_\mu \in \mathfrak{g}_\mathcal{A}$ .

## The Bagger-Lambert-Gustavsson model

$$\begin{aligned}\mathcal{L}_{\text{BLG}} = & + \frac{1}{2} \varepsilon^{\mu\nu\kappa} ((A_\mu, \partial_\nu A_\kappa)_\mathfrak{g} + \frac{1}{3} (A_\mu, [A_\nu, A_\kappa])_\mathfrak{g}) \\ & - \frac{1}{2} (\nabla_\mu X, \nabla^\mu X)_{\mathcal{A} \otimes Cl} + \frac{i}{2} (\bar{\Psi}, \Gamma^\mu \nabla_\mu \Psi)_\mathcal{A} \\ & + \frac{i}{4} (\bar{\Psi}, [X, X, \Psi])_\mathcal{A} - \frac{1}{12} ([X, X, X], [X, X, X])_{\mathcal{A} \otimes Cl}\end{aligned}$$

This model is invariant under the supersymmetry transformations:

$$\begin{aligned}\delta X &= i\Gamma_I \bar{\varepsilon} \Gamma^I \Psi, & \delta \Psi &= \nabla_\mu X \Gamma^\mu \varepsilon - \frac{1}{6} [X, X, X] \varepsilon, \\ \delta A_\mu &= i\bar{\varepsilon} \Gamma_\mu (X \wedge \Psi)\end{aligned}$$

# Consistency checks

The BLG model passes a number of consistency checks.

$$\begin{aligned}\mathcal{L}_{\text{BLG}} = & + \frac{1}{2} \varepsilon^{\mu\nu\kappa} \left( (A_\mu, \partial_\nu A_\kappa)_{\mathfrak{g}} + \frac{1}{3} (A_\mu, [A_\nu, A_\kappa])_{\mathfrak{g}} \right) \\ & - \frac{1}{2} (\nabla_\mu X, \nabla^\mu X)_{\mathcal{A} \otimes Cl} + \frac{i}{2} (\bar{\Psi}, \Gamma^\mu \nabla_\mu \Psi)_{\mathcal{A}} \\ & + \frac{i}{4} (\bar{\Psi}, [X, X, \Psi])_{\mathcal{A}} - \frac{1}{12} ([X, X, X], [X, X, X])_{\mathcal{A} \otimes Cl}\end{aligned}$$

## Further results:

- The model is classically conformal and seems rather unique.
- If  $\mathcal{N} = 8$  SUSY not anomalous  $\Rightarrow$  vanishing  $\beta$ -function
- The model is parity invariant.
- Under some assumptions: reduction mechanism M2  $\rightarrow$  D2.

(Mukhi, Papageorgakis, 0803.3218)

- Recast into the ABJM version, it yields integrable spin chain.

- Review part
  - The Nahm equation or D1-D3 branes
  - The Basu-Harvey equation or M2-M5 branes
  - Stacks of flat M2-branes: The BLG model
- ▶ Superspace formulations of BLG-like models
  - Manifestly  $\mathcal{N} = 2$  supersymmetric formulation
  - Manifestly  $\mathcal{N} = 4$  supersymmetric formulation
- Generalized 3-Lie algebras and BLG-like models
  - The structure of generalized 3-Lie algebras
  - The unifying picture by Figueroa-O'Farrill et al.
  - Representations on  $*$ -algebras
- The framework of strong homotopy Lie algebras
  - $L_\infty$  algebras and homotopy Maurer-Cartan (hMC) equations
  - The Nahm and the Basu-Harvey equations as hMC equations
  - The BLG equation as hMC equations

# Manifestly $\mathcal{N} = 2$ SUSY Formulation

There is a manifestly  $\mathcal{N} = 2$  SUSY formulation, allowing for various deformations.

**Approach:** Take  $\mathcal{N} = 1$ , 4d superspace  $\mathbb{R}^{1,3|4}$  and reduce to 3d.

Field content of the theory:

- The matter fields  $X^I$ ,  $\Psi$  are encoded in four chiral multiplets:

$$\Phi^i(y) = \phi^i(y) + \sqrt{2}\theta\psi^i(y) + \theta^2 F^i(y) ,$$

- The gauge potential  $A_\mu$  is contained in a vector superfield:

$$\begin{aligned} V(x) = & -\theta^\alpha\bar{\theta}^{\dot{\alpha}}(\sigma_{\alpha\dot{\alpha}}^\mu A_\mu(x) + i\varepsilon_{\alpha\dot{\alpha}}\sigma(x)) \\ & + i\theta^2(\bar{\theta}\bar{\lambda}(x)) - i\bar{\theta}^2(\theta\lambda(x)) + \frac{1}{2}\theta^2\bar{\theta}^2 D(x) , \end{aligned}$$

$\mathcal{N} = 2$  superspace formulation of BLG (Cherkis, CS, 0807.0808)

$$\begin{aligned} \mathcal{L} = & \int d^4\theta \kappa (i(V, (\bar{D}_\alpha D^\alpha V))_{\mathfrak{g}} + \frac{2}{3}(V, \{(\bar{D}^\alpha V), (D_\alpha V)\})_{\mathfrak{g}}) \\ & + (\bar{\Phi}_i, e^{2iV} \cdot \Phi^i)_{\mathcal{A}} + \alpha \left( \int d^2\theta \varepsilon_{ijkl} ([\Phi^i, \Phi^j, \Phi^k], \Phi^l)_{\mathcal{A}} + c.c. \right) \end{aligned}$$

# Manifestly $\mathcal{N} = 2$ SUSY Formulation

There is a manifestly  $\mathcal{N} = 2$  SUSY formulation, allowing for various deformations.

$$\mathcal{L} = \int d^4\theta \kappa \left( i(V, (\bar{D}_\alpha D^\alpha V))_{\mathfrak{g}} + \frac{2}{3}(V, \{(\bar{D}^\alpha V), (D_\alpha V)\})_{\mathfrak{g}} \right) \\ + (\bar{\Phi}_i, e^{2iV} \cdot \Phi^i)_{\mathcal{A}} + \alpha \left( \int d^2\theta \varepsilon_{ijkl} ([\Phi^i, \Phi^j, \Phi^k], \Phi^l)_{\mathcal{A}} + c.c. \right)$$

## Observations:

- Superfield description of **BLG** analogous to that of **SYM**.
- This Lagrangian is **not manifestly gauge invariant**.
- There are various  $\mathcal{N} = 2$  deformations.
- Deforming by a **Yang-Mills term** breaks conformal invariance, but might lead to new **dualities**.

# Manifestly $\mathcal{N} = 2$ SUSY Formulation

Perturbative calculations are much simpler using supergraphs.

Super Feynman rules for Chern-Simons theory with matter are easily derived, see e.g. [S.J. Gates, H. Nishino, PLB281 \(1992\) 72](#)

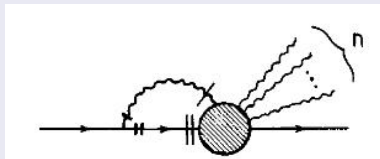
Gluon propagator (Landau gauge):  $\frac{i\bar{D}^\alpha D_\alpha(p, \theta^1)}{p^2} \delta^4(\theta^1 - \theta^2)$

Matter propagator:  $\frac{1}{p^2} \delta^4(\theta^1 - \theta^2)$

Vertices: from action, insert  $-\frac{1}{4}\bar{D}^2 / -\frac{1}{4}D^2$  for  $\Phi / \bar{\Phi}$   
add the usual **loop integrals**, **symmetry factors**, ...

## Vanishing lemma

The following diagrams essentially vanish due to  $D^3 = \bar{D}^3 = 0$ :

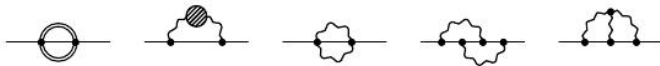


# Manifestly $\mathcal{N} = 2$ SUSY Formulation

Perturbative calculations are much simpler using supergraphs.

**Example:** Field strength renormalization.

In components: [Gaiotto, Yin, JHEP 08 \(2007\) 056](#)



With **supergraphs**, only the 3rd diagram survives due to the lemma.

# Manifestly $\mathcal{N} = 4$ Supersymmetric Formulation

In projective superspace, one can make  $\mathcal{N} = 4$  SUSY in the BLG model manifest.

**Field content** of the BLG model in projective superspace:

- Matter  $X^I, \Psi$ :  $4 \mathcal{N} = 2$  chiral mps.  $\Rightarrow 2 \mathcal{N} = 4$  hypermps.
- Gauge  $A_\mu$ :  $\mathcal{N} = 2$  vector multiplet  $\Rightarrow \mathcal{N} = 4$  tropical multiplet

Supersymmetric **action**: (Cherkis, Dotsenko, CS, 0812.3127)

$$\int \mu \kappa \left( i(\mathcal{V}, (\bar{\mathcal{D}}_\alpha \mathcal{D}^\alpha \mathcal{V}))_{\mathfrak{g}} + \frac{2}{3}(\mathcal{V}, \{(\bar{\mathcal{D}}^\alpha \mathcal{V}), (\mathcal{D}_\alpha \mathcal{V})\})_{\mathfrak{g}} \right) + (\bar{\eta}_k, e^{2i\mathcal{V}} \cdot \eta_k)_{\mathcal{A}}$$

**Observations:**

- Chern-Simons term completely reduces to  $\mathcal{N} = 1$  form.
- The complex linear superfield  $\Sigma$  in the hypermultiplet
$$\eta_k = \bar{\Phi} \frac{1}{\zeta^2} + \bar{\Sigma} \frac{1}{\zeta} + X - \zeta \Sigma + \zeta^2 \Phi$$
can be dualized to a chiral multiplet.
- To compute the interaction terms, one would have to solve a **Riemann-Hilbert problem**. However, its **symmetries** tell us that this is the BLG model.



- Review part
  - The Nahm equation or D1-D3 branes
  - The Basu-Harvey equation or M2-M5 branes
  - Stacks of flat M2-branes: The BLG model
- Superspace formulations of BLG-like models
  - Manifestly  $\mathcal{N} = 2$  supersymmetric formulation
  - Manifestly  $\mathcal{N} = 4$  supersymmetric formulation
- ▶ Generalized 3-Lie algebras and BLG-like models
  - The structure of generalized 3-Lie algebras
  - The unifying picture by Figueroa-O'Farrill et al.
  - Representations on  $*$ -algebras
- The framework of strong homotopy Lie algebras
  - $L_\infty$  algebras and homotopy Maurer-Cartan (hMC) equations
  - The Nahm and the Basu-Harvey equations as hMC equations
  - The BLG equation as hMC equations

# Extending The Structure of A 3-Lie Algebra

The notion of a 3-Lie algebra is too restrictive and one has to find a generalized notion.

**Problem:** Given a three-algebra  $\mathcal{A}$ , if its bilinear form  $(\cdot, \cdot)_{\mathcal{A}}$  is positive definite, then  $\mathcal{A}$  is  $A_4$  or a direct sum thereof.

$A_4$  supposedly describes a stack of 2 M2-branes, not enough.

Mukhi, Papageorgakis, 0803.3218

**Possible extensions:**

- (1) Assume, 3-Lie algebras appear accidentally  $\Rightarrow$  ABJM model
- (2) Give up positive definiteness of  $(\cdot, \cdot)_{\mathcal{A}} \Rightarrow$  ghosts
- (3) Relax conditions on 3-Lie algebras

Guideline: Demand **gauge invariance** of the  $\mathcal{N} = 2$  Lagrangian

$$\mathcal{L} = \int d^4\theta \kappa (i(V, (\bar{D}_\alpha D^\alpha V))_{\mathfrak{g}} + \frac{2}{3}(V, \{(\bar{D}^\alpha V), (D_\alpha V)\})_{\mathfrak{g}}) \\ + (\bar{\Phi}_i, e^{2iV} \cdot \Phi^i)_{\mathcal{A}} + \alpha \left( \int d^2\theta \varepsilon_{ijkl} ([\Phi^i, \Phi^j, \Phi^k], \Phi^l)_{\mathcal{A}} + c.c. \right)$$

# Admissible 3-Algebraic Structures

Imposing gauge invariance in the  $\mathcal{N} = 2$  BLG-like model leads to more freedom.

Demanding **gauge invariance** in above theory yields the condition:

$$\begin{aligned}([A, B, C], D)_A &= -([B, A, C], D)_A \\ &= -([A, B, D], C)_A = ([C, D, A], B)_A\end{aligned}$$

Cherkis, CS, 0807.0808

## Generalized metric 3-Lie algebras

Same as a 3-Lie algebra, but relax  $([A, B, C], D)_A$  from totally antisymmetric to the above symmetry properties.

# Hermitian 3-Lie Algebras

Another generalization of 3-Lie algebras are the Hermitian ones yielding  $\mathcal{N} = 6$  SUSY.

Alternatively to our way of extending 3-Lie algebras:

Reduce supersymmetry to  $\mathcal{N} = 6$ , i.e. assume the following:

$$\delta\phi^i = \sqrt{2}\bar{\varepsilon}^{ij}\bar{\psi}_j ,$$

$$\delta\bar{\psi}_i = -i\sqrt{2}\sigma^\mu\varepsilon_{ij}\nabla_\mu\phi^j + [\phi^j, \phi^k; \bar{\phi}_j]\varepsilon_{ik} + [\phi^j, \phi^k; \bar{\phi}_i]\varepsilon_{jk} ,$$

$$\delta A_\mu = -i\varepsilon_{ij}\sigma_\mu\phi^i \wedge \psi^j + i\bar{\varepsilon}^{ij}\sigma_\mu\bar{\phi}_i \wedge \bar{\psi}_j .$$

where  $\varepsilon^{ij}$  is in the **6** of  $SU(4)$ . Closure of this algebra implies:

$$[A, B; C] = -[B, A; C] \quad ([A, B; C], D) = (B, [C, D; A]) .$$

$$[[C, D; E], A; B] - [[C, A; B], D; E] - [C, [D, A; B]; E] + [C, D; [E, B; A]] = 0 .$$

An associated Lie algebra  $\mathfrak{g}_A := \text{im}(D_{A\otimes A})$  is induced by

$$D_{A\wedge B}(C) := [C, A; B] , \quad A, B, C \in \mathcal{A}$$

This leads to the ABJM model, a Chern-Simons-matter theory.

Aharony, Bergman, Jafferis, Maldacena, 0806.1218

Bagger, Lambert, 0807.0163

# Equivalence to Gauge Theories

The above generalizations of 3-Lie algebras can be recast into Lie algebra language.

Recall the BLG Lagrangian:

$$\begin{aligned}\mathcal{L}_{\text{BLG}} = & + \frac{1}{2} \varepsilon^{\mu\nu\kappa} ((A_\mu, \partial_\nu A_\kappa)_{\mathfrak{g}} + \frac{1}{3} (A_\mu, [A_\nu, A_\kappa])_{\mathfrak{g}}) \\ & - \frac{1}{2} (\nabla_\mu X, \nabla^\mu X)_{\mathcal{A} \otimes Cl} + \frac{i}{2} (\bar{\Psi}, \Gamma^\mu \nabla_\mu \Psi)_{\mathcal{A}} \\ & + \frac{i}{4} (\bar{\Psi}, [X, X, \Psi])_{\mathcal{A}} - \frac{1}{12} ([X, X, X], [X, X, X])_{\mathcal{A} \otimes Cl}\end{aligned}$$

Up to potential terms, this is an ordinary gauge theory.

What is the relationship between Lie algebras and 3-Lie algebras?

Unifying picture:

(Medeiros, Figueroa-O'Farrill, Mendez-Escobar, Ritter, 0809.1086)

Generalized 3-Lie algebras  $\leftrightarrow (\mathfrak{g}, V)$   $\mathfrak{g}$ : real Lie algebra

$V$ : faithful orthogonal  $\mathfrak{g}$ -mod.

similar statement for Hermitian 3-Lie algebras.

# Current Situation:

It is not clear, if 3-Lie algebras are necessary at all.

## Observations:

- 3-Lie algebras too restrictive, only one example:  $A_4$ .
- Generalizations lead to less than  $\mathcal{N} = 8$  supersymmetry.
- All models can be rewritten as gauge theories.

⇒ We need more input from physics.

Particularly important here: **AdS/CFT correspondence**

We need some kind of  $N \rightarrow \infty$  limit, so let's look at representations of (generalized) 3-Lie algebras in terms of matrix algebras.

# Classifications of $\ast$ -Algebra Representations of 3-Algebras

Representations on matrix algebras, which are useful for  $N \rightarrow \infty$ , can be constructed.

Representation of metric 3-algebras on  $\ast$ -algebras:

Take a  $\ast$ - or **matrix algebra** equipped with a trace form. Construct a 3-bracket on this algebra from matrix products and the involution and use the Hilbert-Schmidt scalar product  $(A, B) = \text{tr}(A^\dagger B)$ .

Classification of all such representations in the real and hermitian case using MuPad done in [Cherkis, Dotsenko, CS, 0812.3127](#)

# Classifications of $\ast$ -Algebra Representations of 3-Algebras

Representations on matrix algebras, which are useful for  $N \rightarrow \infty$ , can be constructed.

The **Real case**.  $[A, B, C] :=$

$$\text{I} : \alpha([A^\dagger, B], C) + [[A, B^\dagger], C] + [[A, B], C^\dagger] - [[A^\dagger, B^\dagger], C^\dagger)$$

$$\text{II} : \alpha([A, B^\dagger], C) + [[A^\dagger, B], C])$$

$$\text{III} : \alpha(AB^\dagger - BA^\dagger)C + \beta C(A^\dagger B - B^\dagger A)$$

$$\text{IV} : \alpha([A, B], C) + [[A^\dagger, B^\dagger], C] + [[A^\dagger, B], C^\dagger] + [[A, B^\dagger], C^\dagger]) \\ + \beta([A, B], C^\dagger) + [[A^\dagger, B], C] + [[A, B^\dagger], C] + [[A^\dagger, B^\dagger], C^\dagger]) .$$

The class of examples  $\mathcal{C}_{2d}$ ,

$$[\gamma_a, \gamma_b, \gamma_c] := [[\gamma_a, \gamma_b]\gamma_{ch}, \gamma_c] ,$$

is contained in **III**, with  $\alpha = \beta = -1$  and the  $\ast$ -algebra is the algebra of  $d \times d$  matrices.



# Classifications of $\ast$ -Algebra Representations of 3-Algebras

Representations on matrix algebras, which are useful for  $N \rightarrow \infty$ , can be constructed.

The **Hermitian case**.  $[A, B, C] :=$

$$I_\alpha : A, B, C \mapsto \alpha(AC^\dagger B - BC^\dagger A) .$$

This is precisely the Hermitian 3-Lie algebra used in **Bagger**, **Lambert**, **0807.0163** to obtain the ABJM model in 3-algebra form.

# BLG-like Models with Generalized 3-Algebras

The manifestly supersymmetric actions from above can be used with any such 3-algebra.

Recall the  $\mathcal{N} = 2$  superfield formulation of the BLG model:

$$\begin{aligned} \mathcal{L} = & \int d^4\theta \kappa \left( i(V, (\bar{D}_\alpha D^\alpha V))_{\mathfrak{g}} + \frac{2}{3}(V, \{(\bar{D}^\alpha V), (D_\alpha V)\})_{\mathfrak{g}} \right) \\ & + (\bar{\Phi}_i, e^{2iV} \cdot \Phi^i)_{\mathcal{A}} + \alpha \left( \int d^2\theta \varepsilon_{ijkl} ([\Phi^i, \Phi^j, \Phi^k], \Phi^l)_{\mathcal{A}} + c.c. \right) \end{aligned}$$

as well as the  $\mathcal{N} = 4$  superfield formulation:

$$\int \mu \kappa \left( i(\mathcal{V}, (\bar{\mathcal{D}}_\alpha \mathcal{D}^\alpha \mathcal{V}))_{\mathfrak{g}} + \frac{2}{3}(\mathcal{V}, \{(\bar{\mathcal{D}}^\alpha \mathcal{V}), (\mathcal{D}_\alpha \mathcal{V})\})_{\mathfrak{g}} \right) + (\bar{\eta}_k, e^{2i\mathcal{V}} \cdot \eta_k)_{\mathcal{A}}$$

In both cases,  $\mathcal{A}$  can also be a **generalized** or a **Hermitian 3-Lie alg.**

- Review part
  - The Nahm equation or D1-D3 branes
  - The Basu-Harvey equation or M2-M5 branes
  - Stacks of flat M2-branes: The BLG model
- Superspace formulations of BLG-like models
  - Manifestly  $\mathcal{N} = 2$  supersymmetric formulation
  - Manifestly  $\mathcal{N} = 4$  supersymmetric formulation
- Generalized 3-Lie algebras and BLG-like models
  - The structure of generalized 3-Lie algebras
  - The unifying picture by Figueroa-O'Farrill et al.
  - Representations on  $*$ -algebras
- ▶ The framework of strong homotopy Lie algebras
  - $L_\infty$  algebras and homotopy Maurer-Cartan (hMC) equations
  - The Nahm and the Basu-Harvey equations as hMC equations
  - The BLG equation as hMC equations

# $L_\infty$ -algebras and Homotopy Maurer-Cartan Equations

The eom of the BLG model can be reformulated as homotopy Maurer-Cartan equations.

## $L_\infty$ - or strong homotopy Lie algebras

- Introduced by **Stasheff** (1992) “only way to extend Lie algebras”
- appear in string FT, top. conf. FT, Morse theory

**Definition:**

$R$ -module  $\mathcal{L}$ , with family of  $R$ -multilinear maps  $\mu_n : \mathcal{L}^{\times n} \rightarrow \mathcal{L}$  s.t.:

$$\mu_n(x_{\sigma(1)} \dots x_{\sigma(n)}) = \epsilon(\sigma) \mu_n(x_1 \dots x_n)$$

**Homotopy Jacobi-type identity:**

$$\sum_{i=1}^n \sum_{\sigma \in Sh(i, n-i)} (-1)^{i(n+1)} \epsilon(\sigma) \mu_{n-i+1}(\mu_i(x_{\sigma(1)}, \dots, x_{\sigma(i)}), x_{\sigma(i+1)}, \dots, x_{\sigma(n)}) = 0$$

There is also a **graded** version, then  $\mu_n$  is of degree  $2 - n$ .

Note:  $\mu_1$  is a differential,  $\mu_1, \mu_2 \neq 0 \rightarrow$  diff. (grad.) Lie algebra

Interestingly,  $n$ -Lie algebras are (ungraded)  $L_\infty$ -algebras  
(Hanlon, Wachs 1995, Dzhumadil'daev, math/0202043)

# $L_\infty$ -algebras and Homotopy Maurer-Cartan Equations

The eom of the BLG model can be reformulated as homotopy Maurer-Cartan equations.

Equations employing  $L_\infty$  algebras:

homotopy Maurer-Cartan equation

Given a (graded)  $L_\infty$ -algebra  $\mathcal{L} = \bigoplus_i \mathcal{L}_i$ ,

$$\sum_{\ell \geq 0} \frac{(-1)^{\ell(\ell+1)/2}}{\ell!} \mu_\ell(\varphi^{\otimes \ell}) = 0, \quad \varphi \in \mathcal{L}$$

is invariant under the gauge transformations

$$\delta\varphi = - \sum_{\ell \geq 1} \frac{(-1)^{\ell(\ell-1)/2}}{(\ell-1)!} \mu_\ell(\alpha \otimes \varphi^{\ell-1}), \quad \alpha \in \mathcal{L}_0$$

Andrei Losev: “All classical equations of motion are of hMC form.”

If only  $\mu_1, \mu_2 \neq 0$ , then hMC are ordinary Maurer-Cartan eqs.

The following examples are all developed in

C. I. Lazaroiu, D. McNamee, CS and A. Zejmak, 0901.3905

# The Nahm Equation as hMC Equations

Both the Nahm and the Basu-Harvey equations can trivially be put into hMC form.

**Example (1):** The Nahm equation

$$\nabla_s X^i + \varepsilon^{ijk} [X^j, X^k] = 0$$

with gauge algebra  $\mathfrak{su}(N)$ . Using  $X = \sigma_i X^i$ , rewrite as

$$d_A X + [X, X] ds = 0$$

**Vector space** for the  $L_\infty$ -algebra:  $\mathcal{L} := \Omega^\bullet(\mathbb{R}, Cl(\mathbb{R}^3)) \otimes \mathfrak{su}(N)$

**Grading arises from**  $\widetilde{ds} = 1, \widetilde{\sigma}_i = 1$

Higher products reproducing the Nahm equation:

$$\mu_1(X) := dX, \quad \mu_2(A, X) := [A, X], \quad \mu_2(X, X) := [X, X] ds$$

Higher products taking care of gauge transformations:

$$\mu_1(\lambda) := d\lambda, \quad \mu_2(\lambda, A) := [\lambda, A], \quad \mu_2(\lambda, X) := [\lambda, X]$$

This reproduces both the **eom** and **gauge symmetry** correctly.

# The Nahm Equation as hMC, Higher Jacobi Identities

Most of the higher Jacobi identities are automatically satisfied.

Homotopy Jacobi identity:

$$\sum_{i=1}^n \sum_{\sigma \in Sh(i, n-i)} (-1)^{i(n+1)} \epsilon(\sigma) \mu_{n-i+1}(\mu_i(x_{\sigma(1)}, \dots, x_{\sigma(i)}), x_{\sigma(i+1)}, \dots, x_{\sigma(n)}) = 0$$

Higher products responsible for equations of motion:

$$\mu_1(X) := dX, \quad \mu_2(A, X) := [A, X], \quad \mu_2(X, X) := [X, X]ds$$

These satisfy the higher Jacobi identities trivially:

$$\begin{array}{ccc}
 \Omega^0(\mathbb{R}^1) \otimes Cl_1(\mathbb{R}^3) \otimes \mathfrak{su}(N) & & \Omega^1(\mathbb{R}^1) \otimes \mathfrak{su}(N) \\
 \downarrow \mu_1(X) & \downarrow \mu_2(X, X) \quad \downarrow \mu_2(A, X) & \\
 & \Omega^1(\mathbb{R}^1) \otimes Cl_1(\mathbb{R}^3) \otimes \mathfrak{su}(N) & 
 \end{array}$$

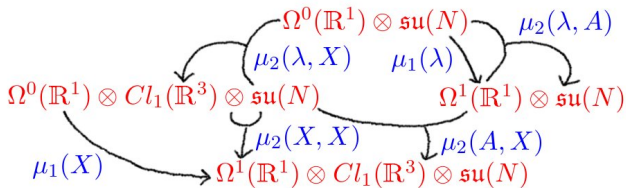
# The Nahm Equation as hMC, Higher Jacobi Identities

Most of the higher Jacobi identities are automatically satisfied.

All higher products:

$$\mu_1(X) := dX, \quad \mu_2(A, X) := [A, X], \quad \mu_2(X, X) := [X, X]ds$$

$$\mu_1(\lambda) := d\lambda, \quad \mu_2(\lambda, A) := [\lambda, A], \quad \mu_2(\lambda, X) := [\lambda, X]$$



The hom. Jacobi identities define the following higher products:

$$\begin{aligned} \mu_1(\mu_2(\lambda, X)) \quad \& \quad \mu_2(\mu_1(\lambda), X) & \Rightarrow \quad \mu_2(\lambda, \mu_1(X)) \\ \mu_2(\mu_2(\lambda, A), X) \quad \& \quad \mu_2(\mu_1(\lambda, X), A) & \Rightarrow \quad \mu_2(\lambda, \mu_2(A, X)) \\ \mu_2(\mu_2(\lambda, X), X) & \Rightarrow \quad \mu_2(\lambda, \mu_2(X, X)) \end{aligned}$$



# The Basu-Harvey Equation as hMC Equations

Both the Nahm and the Basu-Harvey equations can trivially be put into hMC form.

**Example (2):** The Basu-Harvey equation

$$\nabla_s X^i + \varepsilon^{ijkl} [X^j, X^k, X^l] = 0$$

with 3-Lie algebra  $\mathcal{A}$  and associated gauge algebra  $\mathfrak{g}_{\mathcal{A}}$ . Rewrite:

$$d_A X + \gamma_5 [X, X, X] ds = 0 \quad , \quad X := \gamma_i X^i$$

**Vector space** for the  $L_\infty$ -algebra:  $\mathcal{L} := \Omega^\bullet(\mathbb{R}, Cl(\mathbb{R}^4)) \otimes (\mathcal{A} \oplus \mathfrak{g}_{\mathcal{A}})$

**Grading arises from**  $\widetilde{ds} = 1, \widetilde{\gamma}_i = 1$  Higher products:

$$\mu_1(X) := dX \quad \mu_2(A, X) := [A, X] \quad \mu_3(X, X) := \gamma_5 [X, X, X] ds$$

Higher products taking care of gauge transformations:

$$\mu_1(\lambda) := d\lambda \quad \mu_2(\lambda, A) := [\lambda, A] \quad \mu_2(\lambda, X) := [\lambda, X]$$

Higher Jacobi identities require us to define further products. This reproduces **eom** and **gauge symmetry** correctly.

# The Super Yang-Mills Equations as hMC Equations

Because of their second-order nature, the hMC form of the SYM equations is more subtle.

**Example (3):** The (bosonic part of the) super Yang-Mills eqns:

$$\nabla_\mu F^{\mu\nu} = [X^i, \nabla^\nu X^i] \quad \nabla_\mu \nabla^\mu X^i = [[X^i, X^j], X^j]$$

gauge algebra  $\mathcal{A} = \mathfrak{su}(N)$ .

**New:** differential operators of **second order**. Rewrite  $X := X^i \gamma_i$ :

$$*d_A * d_A A \gamma_{\text{ch}} = \text{tr}_{Cl}([X, d_A X]) \gamma_{\text{ch}} \quad (\Delta_A X) \omega = \gamma_{\text{ch}} [X, \gamma_{\text{ch}} [X, X]] \omega$$

**Vector space** for the  $L_\infty$ -algebra:  $\mathcal{L} := \Omega^\bullet(\mathbb{R}^{1,p}) \otimes Cl(\mathbb{R}^{9-p}) \otimes \mathcal{A}$

**Grading arises from**  $\widetilde{dx}^\mu = 1$ ,  $\widetilde{\gamma}_i = 1$  Higher products, 1st eq.:

$$\begin{aligned} \mu_1(A) &:= -(*d * dA) \gamma & \mu_2(A, A) &:= (*[A, *dA] + *d * [A, A]) \gamma, \\ \mu_3(A, A, A) &:= (*[A, *[A, A]]) \gamma & \mu_2(X, X) &:= \text{tr}_C([X, dX]) \gamma \\ \mu_3(X, A, X) &:= \text{tr}_C([X, [A, X]]) \gamma. \end{aligned}$$

Higher products, 2nd eq.:

$$\begin{aligned} \mu_1(X) &= -\Delta X \omega & \mu_3(X, X, X) &= -6(\gamma[X, \gamma[X, X]]) \omega \\ \mu_2(A, X) &= -([A_\mu, \partial^\mu X] \omega + \partial_\mu [A^\mu, X]) \omega & \mu_3(A, A, X) &= [A_\mu, [A^\mu, X]] \omega, \end{aligned}$$

Gauge sym., higher Jacobi identities, SUSY  $\Rightarrow$  higher products.

# The BLG Equations as hMC Equations

The eom of the BLG model can be reformulated as homotopy Maurer-Cartan equations.

**Example (4):** The (bosonic part of the) BLG eqns:

$$\begin{aligned}\nabla_\mu \nabla^\mu X + \frac{1}{2} \Gamma[X, X, \Gamma[X, X, X]] &= 0 \\ [\nabla_\mu, \nabla_\nu] + \varepsilon_{\mu\nu\kappa} (\text{tr}_{Cl}(X \wedge (\nabla^\kappa X))) &= 0\end{aligned}$$

Start with 3-Lie algebra  $\mathcal{A}$  and introduce the module

$$\mathcal{L} := \Omega^\bullet(\mathbb{R}^3) \otimes_{\mathbb{C}} Cl_8 \otimes (\mathcal{A} \oplus \mathfrak{g}_{\mathcal{A}})$$

define gradings:

$$\text{deg}(\Omega^0(\mathbb{R}^3) \otimes_{\mathbb{C}} Cl_{8,0} \otimes \mathfrak{g}_{\mathcal{A}}) = 0$$

$$\text{deg}(\Omega^1(\mathbb{R}^3) \otimes_{\mathbb{C}} Cl_{8,0} \otimes \mathfrak{g}_{\mathcal{A}}) = \text{deg}(\Omega^0(\mathbb{R}^3) \otimes_{\mathbb{C}} Cl_{8,1} \otimes \mathcal{A}) = 1$$

$$\text{deg}(\Omega^2(\mathbb{R}^3) \otimes_{\mathbb{C}} Cl_{8,0} \otimes \mathfrak{g}_{\mathcal{A}}) = \text{deg}(\Omega^3(\mathbb{R}^3) \otimes_{\mathbb{C}} Cl_{8,1} \otimes \mathcal{A}) = 2$$

The fields will live in the following subspaces:

$$A \in \Omega^1(\mathbb{R}^3) \otimes Cl_{8,0} \otimes \mathfrak{g}_{\mathcal{A}} \quad X \in \Omega^0(\mathbb{R}^3) \otimes Cl_{8,1} \otimes \mathcal{A}$$

$$\lambda \in \Omega^0(\mathbb{R}^3) \otimes Cl_{8,0} \otimes \mathfrak{g}_{\mathcal{A}}$$

# The BLG Equations as hMC Equations

The eom of the BLG model can be reformulated as homotopy Maurer-Cartan equations.

BLG equations of motion (bosonic part):

$$\begin{aligned}\nabla_\mu \nabla^\mu X + \frac{1}{2} \Gamma[X, X, \Gamma[X, X, X]] &= 0 \\ [\nabla_\mu, \nabla_\nu] + \varepsilon_{\mu\nu\kappa} (\text{tr}_{Cl}(X \wedge (\nabla^\kappa X))) &= 0\end{aligned}$$

Define the following brackets:

$$\begin{aligned}\mu_1(A) &:= dA & \mu_2(A, A) &:= [[A \wedge A]] , \\ \mu_2(X, X) &:= *\tau(X \wedge dX) & \mu_3(A, X, X) &:= *\tau(X \wedge [A, X]) \\ \mu_1(X) &:= \Delta X \omega & \mu_2(A, X) &:= \partial_\mu [A^\mu, X] \omega + [A_\mu, \partial^\mu X] \omega \\ \mu_3(A, A, X) &:= [A_\mu, [A^\mu, X]] \omega & \mu_5(X^{\otimes 5}) &:= \Gamma[X, X, \Gamma[X, X, X]]\end{aligned}$$

further brackets consistently from gauge symmetry, supersymmetry and homotopy Jacobi identities.

The hMC equations  $\sum_{\ell \geq 0} \frac{(-1)^{\ell(\ell+1)/2}}{\ell!} \mu_\ell(\varphi^{\otimes \ell}) = 0$  reproduce the BLG model together with its gauge invariance. (**SUSY extension**)

# Conclusions

## Summary and Outlook.

### Past work:

- Identification of **extended 3-algebraic structures**
- **Classification** of categorical matrix representations
- **Manifestly  $\mathcal{N} = 2$  and  $\mathcal{N} = 4$  supersymmetric formulations** of BLG-like models
- Identification of  **$L_\infty$ -algebra structure**
- BLG eoms rewritten as **homotopy Maurer-Cartan equations**

### Future directions:

- Are  **$L_\infty$ -algebras** useful here? Extendable? Classifications?
- What 3-algebras yield **integrable spin chains?**  **$\beta$ -deformations?**
- Extend SUSY models by **Yang-Mills** term, analyze
- Lift the **Nahm/Fourier-Mukai transform** to M-theory
- Ultimately: find **analogous models for M5 branes**

# On Stacks of M2-Branes

Christian Sämann



*School of Mathematics, Trinity College Dublin*

Irish Quantum Field Theory Meeting, May, 15th 2009