On the Phase Diagram of Fuzzy Scalar Field Theory

Christian Sämann

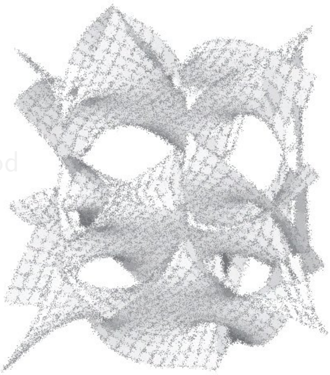
Dublin Institute for Advanced Studies

Bayrischzell Workshop 2007

Based on:

- hep-th/0705.????, Denjoe O'Connor and CS.

Outline



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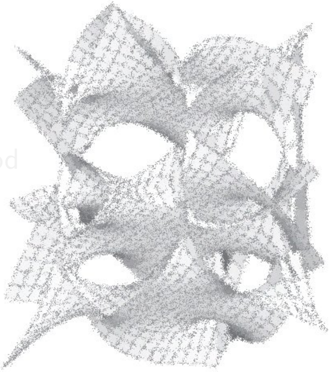

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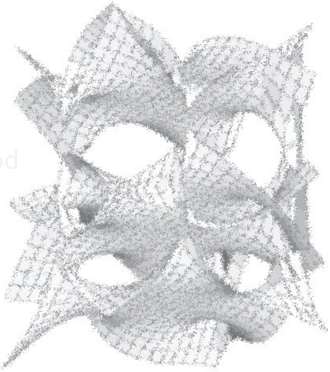
2. The Fuzzy Sphere and Related Spaces
3. Fuzzy Scalar Field Theory
4. Known Results
5. Perturbative Expansion
6. Large N Limit & Saddle Point Method
7. Modification of the Model
8. Conclusions



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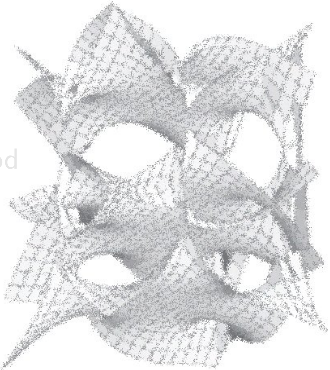

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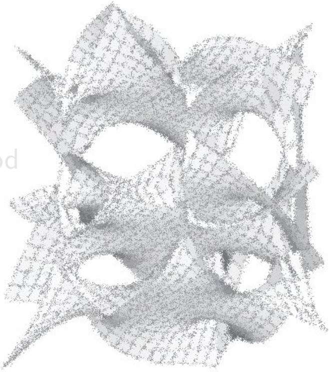

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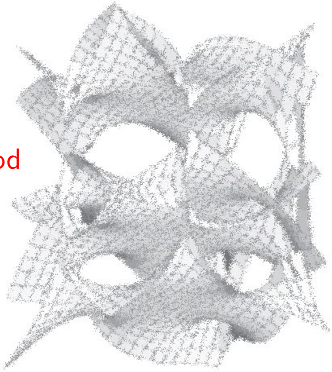

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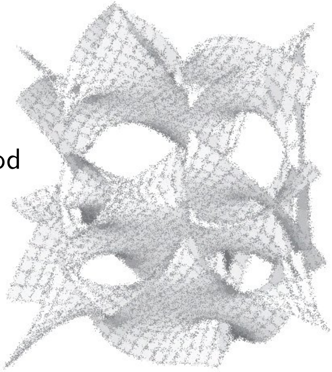

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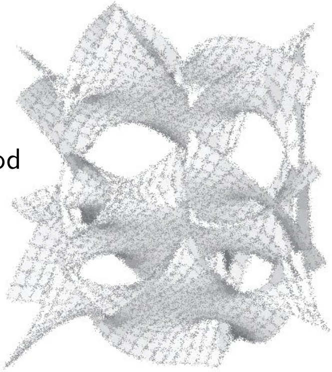

Fuzzy Geometry as a Regulator

Studying fuzzy geometry is well motivated.

Planck-Scale Structure of Spacetime

Smooth structure of spacetime probably not to arbitrary scales.
The most prominent modifications: SUSY and Noncommutativity.
Fuzzy Geometry: NC on compact symplectic Riemannian spaces arise naturally in string theory

Regularization of Field Theories

Field theories on fuzzy spaces: finite-dimensional matrix models.
QFTs are finite and path integrals well-defined.
Advantages over lattice approach:
Isometries preserved, no fermion doubling, analytical handle
Numerical Simulations are easily done

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Taking the commutative limit does not reproduce scalar ϕ^4-theory.
UV/IR-mixing distorts the picture.

[Vaidya, Chu, Madore, Steinacker]

Modifications of the naïve model, however, could cure this problem.

[Dolan, O'Connor, Presnajder]

To do:
Obtain an analytical handle on fuzzy ϕ^4-theory, in particular its phase diagram and study the effect of the proposed modifications.

(Gauge theory on the fuzzy sphere has recently been solved

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Quantization of the sphere:

As usually, do not quantize space itself, but algebra of functions.

Basis: Spherical harmonics Y_{lm} with $l = 0, .., \infty, m = -l, ..., l$.

Quantization: Truncate angular momentum $l \leq L$
Multiplication will not close any more: $Y_{l_1 ...} Y_{l_2 ...} = Y_{l_1 + l_2 ...} + ...$
However, deforming the product to the star product

$$[x^i \overset{\star}{,} x^j] \sim \mathrm{i}\varepsilon^{ijk} x^k \ ,$$

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Consider irreducible representation ρ of $\mathrm{SU}(n)$, extended to $\mathrm{U}(n)$:

$n^2 - n$ "simple" raising and lowering operators $E^{\pm}_{\vec{\alpha}_i}$.
a_i: nontrivial actions $(E^{-}_{\vec{\alpha}_i})^{a_i}|\mu\rangle \neq 0 = (E^{-}_{\vec{\alpha}_i})^{a_i+1}|\mu\rangle$
n Cartan generators, isotropy group of $|\mu\rangle$: $H \supset U(1)^{\times n}$

One-To-One Correspondence:

Coherent States $|p\rangle \in \rho \leftrightarrow p \in$ Coset $\mathrm{U}(n)/(U(m_1) \times \ldots \times U(m_k))$

$\Rightarrow$ Fuzzy Flag (Super) Manifolds [S. Murray, CS, hep-th/0611328]

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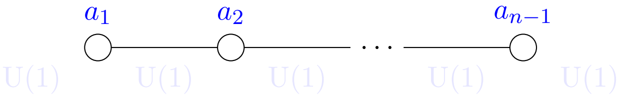

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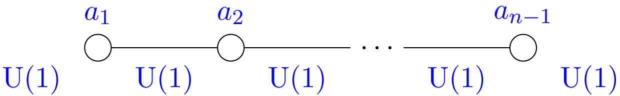

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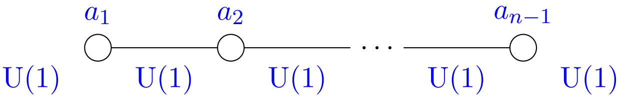

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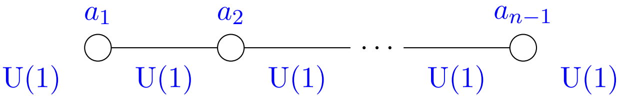

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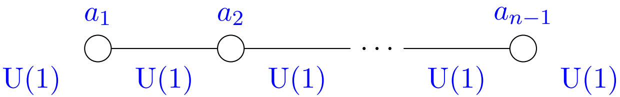

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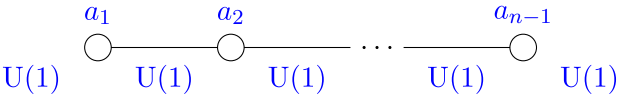

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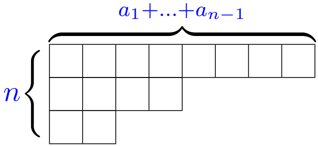
Young diagrams yield a Fock space construction of NC functions on fuzzy geometries.

Representation $\rho = (a_1, \ldots, a_{n-1})$ corresponds to Young diagram



Fuzzy sphere: $\rho = (a_1) = L$ of $\mathrm{SU}(2)$:

$$\cong \quad \mathrm{span}(\ \hat{a}^\dagger_{\alpha_1} \ldots \hat{a}^\dagger_{\alpha_L}|0\rangle\ ) \ , \quad \hat{\mathcal{L}}_i = \hat{a}^\dagger_\alpha \sigma^i_{\alpha\beta} \hat{a}_\beta$$

Isometry-preserving quantization of functions on S^2 via the rule:

$$f(p) = \langle p|\hat{f}|p\rangle \ ,$$

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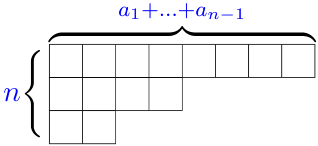

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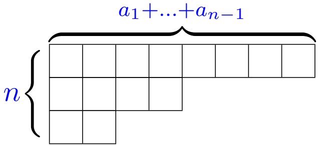

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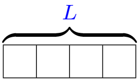

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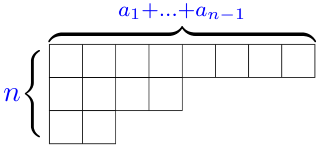

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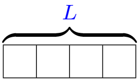

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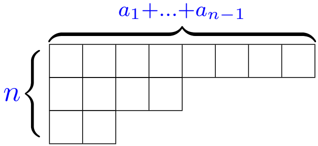

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$$\hat{f} \in \overbrace{\square\square\square\square}^{L} \otimes \overbrace{\square\square\square\square}^{L} \;\cong\; \mathrm{span}(\ \hat{a}^\dagger_{\alpha_1} \ldots \hat{a}^\dagger_{\alpha_L}|0\rangle\langle 0|\hat{a}_{\beta_1} \ldots \hat{a}_{\beta_L}\ )$$

The Fuzzy Sphere

(b) The truncated coordinate ring is mapped to an L-particle Hilbert space.

$$S^2 \cong \mathbb{C}P^1$$

The spherical harmonics $Y_{lm}, l \leq L$ can be written in terms of homogeneous coordinates z_α $(x^i \sim \bar{z}_\alpha \sigma^i_{\alpha\beta} z_\beta)$ on $\mathbb{C}P^1$ in terms of

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with $\alpha_i, \beta_i = 1, 2$ due to the Hopf fibration

$$0 \rightarrow \mathrm{U}(1) \rightarrow S^3 \rightarrow \mathbb{C}P^1 \rightarrow 0 \ .$$

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$$S^2 \cong (\mathbb{C}P^1, \omega)$$

Take the line bundle $\mathscr{L} := \mathcal{O}(1)$ as the quantum line bundel.
Toeplitz quantization (Geometric quantization):

$$T^{(L)}: \quad C^\infty(M) \to \mathrm{End}\left(\Gamma(M, \mathscr{L}^{\otimes L})\right) .$$

The set of sections $\Gamma(\mathbb{C}P^1, \mathscr{L}^{\otimes L})$ is spanned by

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The quantized algebra of functions is thus spanned by

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(c) Geometric Quantization yields the same result.

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$$\hat{a}^\dagger_{\alpha_1}\dots\hat{a}^\dagger_{\alpha_L}|0\rangle\langle 0|\hat{a}_{\beta_1}\dots\hat{a}_{\beta_L} \quad \cdot \quad \hat{a}^\dagger_{\gamma_1}\dots\hat{a}^\dagger_{\gamma_L}|0\rangle\langle 0|\hat{a}_{\delta_1}\dots\hat{a}_{\delta_L}$$

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With $f\star g := \mu(\mathcal{F} f\otimes g)$, the twist element reads as

$$\mathcal{F} = \left(\frac{1}{L!}\frac{\partial}{\partial \bar{z}^{\alpha_1}}\dots\frac{\partial}{\partial \bar{z}^{\alpha_L}}\right)\otimes\left(\frac{1}{L!}\frac{\partial}{\partial z^{\alpha_1}}\dots\frac{\partial}{\partial z^{\alpha_L}}\right)$$

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Fuzzy Scalar Field Theory: Definition.

Scalar field theory on the fuzzy sphere is a finite hermitian matrix model

Quantized algebra of functions on the fuzzy sphere:

$$\text{span}(\ \hat{a}^\dagger_{\alpha_1} \dots \hat{a}^\dagger_{\alpha_L} |0\rangle\langle 0| \hat{a}_{\beta_1} \dots \hat{a}_{\beta_L}\ ) \cong \text{Mat}(L+1)$$

Integration and Laplacian on the fuzzy sphere:

$$\int_{S^2} \mathrm{d}A\ f \ \to\ \frac{4\pi R^2}{N} \mathrm{tr}\,(\hat{f}) \qquad \mathcal{L}_i = \mathrm{i}\varepsilon_{ijk} x^j \partial_k \to [L_i, \cdot]\ , \quad \Delta \to C_2$$

The action of real scalar field theory on the fuzzy sphere:

$$S\ \sim\ \mathrm{tr}\,\left(a[L_i,\Phi][L_i,\Phi] + b\,\Phi^2 + c\,\Phi^4\right)$$

We define the partition function

$$Z\ =\ \int \mathrm{d}\mu_D(\Phi)\ \mathrm{e}^{-\,\mathrm{tr}\,\left(a[L_i,\Phi][L_i,\Phi]+b\,\Phi^2+c\,\Phi^4\right)}$$

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Fuzzy Scalar Field Theory vs. HMMs

Fuzzy scalar field theory is significantly harder than matrix models usually considered.

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First example: One-Hermitian Matrix Model

$$Z \ = \ \int \mathrm{d}\mu_D(\Phi)\ \mathrm{e}^{-\,\mathrm{tr}\,\left(b\,\Phi^2+c\,\Phi^4\right)}$$

Solution: splitting $\Phi = \Omega\Lambda\Omega^\dagger$, $\Lambda \ = \ \mathrm{diag}(\lambda_1,\dots,\lambda_N)$ as well as $\int \mathrm{d}\mu_D(\Phi) \ = \ \int \prod_{i=1}^N \mathrm{d}\lambda_i \Delta^2(\Lambda) \int \mathrm{d}\mu_H(\Omega)$ yields

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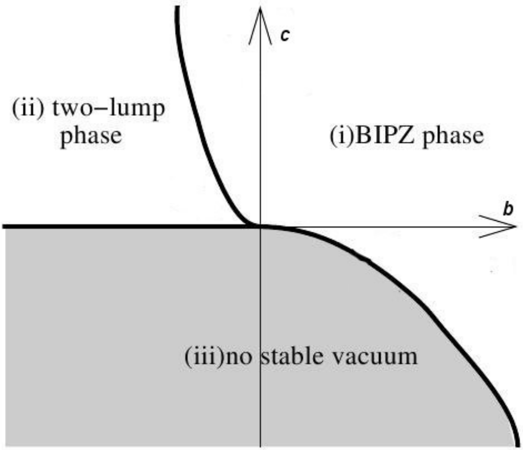
Known Results: Matrix Model Phase Diagram

The matrix model phase diagram suggests two phases.

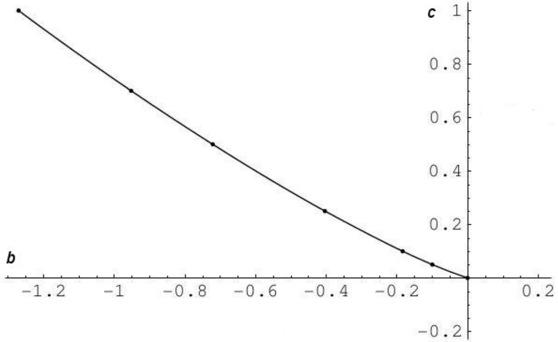
$$Z = \int \mathrm{d}\mu_D(\Phi)\ \mathrm{e}^{-\,\mathrm{tr}\,\left(b\,\Phi^2 + c\,\Phi^4\right)}\ , \qquad 0 < \frac{\mathrm{d}x}{\mathrm{d}\lambda} < \infty$$



Known Results: Phase Diagram for ϕ^4-Theory on $\mathbb{R}^2$

The (lattice) model has two different phases.

$$Z = \int \mathscr{D}\phi \; \mathrm{e}^{-\int \mathrm{d}^2x \frac{1}{2}(\nabla\phi)^2 + b\phi^2 + c\phi^4} \;, \quad \langle\phi\rangle = 0 \;, \quad \frac{c}{b} = \text{const.}$$

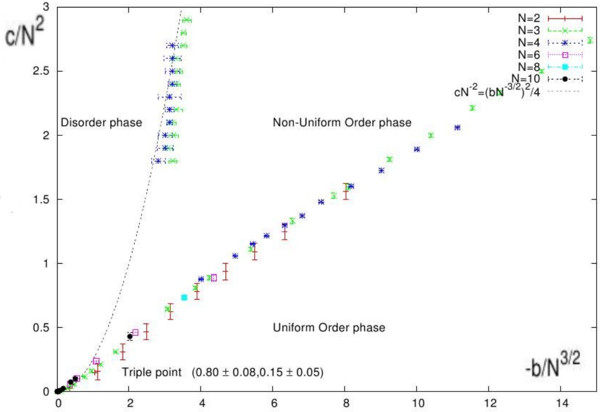


proof of existence: [Glimm, Jaffe, Spencer, 1974/1975]
exact shape numerically: [Loinaz, Willey, hep-lat/9712008],
confirmed by [Lee, hep-th/9811117]

Known Results: Numerical Simulations

The phase diagram suggests a combination of the phases.

$$Z = \int \mathrm{d}\mu_D(\Phi)\ \mathrm{e}^{-\beta\,\mathrm{tr}\left(a[L_i,\Phi][L_i,\Phi]+b\,\Phi^2+c\,\Phi^4\right)}\ ,\qquad \chi = \frac{\partial}{\partial\beta^2}\log(Z)$$



[Flores, O'Connor, Martin, hep-th/0601012,
Panero, hep-th/0608202]

Known Results: ϕ^4-Theory on $\mathbb{R}^2_\theta$

The fuzzy model corresponds actually to regularized NC ϕ^4 theory.

- In ϕ^4-theory on $\mathbb{R}^2_\theta$:
  New phase predicted [Gubser, Sondhi, hep-th/0006119], analytically confirmed [Chen, Wu, hep-th/0110134] and numerically confirmed [Ambjorn, Catterall, hep-th/0209106].
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Symmetry arguments yield simplifications.

Fuzzy scalar field theory: $Z=\int \mathrm{d}\mu_D(\Phi)\ \mathrm{e}^{-\,\mathrm{tr}\,\left(a[L_i,\Phi][L_i,\Phi]+b\,\Phi^2+c\,\Phi^4\right)}$

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Symmetries:

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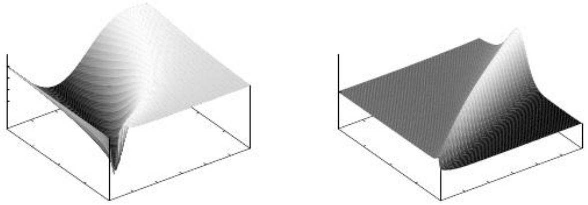
Idea: Treat the kinetic term perturbatively.

Motivation:

- Hopping parameter expansion successfully used on the lattice.
- Specific heat up to $\mathcal{O}(a^8)$ for $L = 1$:



- Group theoretical considerations allow everything else to be treated exactly.

Perturbative expansion: Principles

The angular variables can be integrated out in the perturbative series.

Introduce $K_{ab} := \mathrm{tr}\,([L_i,\tau_a][L_i,\tau_b])$, $\Phi^a = \mathrm{tr}\,(\tau^a\Omega\Lambda\Omega^\dagger)$. Then:

$$\mathrm{e}^{a\Phi^a K_{ab}\Phi^b} = 1 + a\Phi^a K_{ab}\Phi^b + \frac{a^2}{2}\Phi^a K_{ab}\Phi^b\,\Phi^c K_{cd}\Phi^d + \ldots$$

To integrate over $\mathrm{d}\mu_H(\Omega)$ we need to compute terms like

$$\int \mathrm{d}\mu_H(\Omega) K_{ab}\,\mathrm{tr}\,(\tau^a\Omega\Lambda\Omega^\dagger)\,\mathrm{tr}\,(\tau^b\Omega\Lambda\Omega^\dagger)$$

Recall: $\displaystyle\int \mathrm{d}\mu_H(\Omega)\;\; [\rho(\Omega)]_{ij}\;[\rho^\dagger(\Omega)]_{kl} = \frac{1}{\dim(\rho)}\delta_{il}\delta_{jk}$

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Thus: $\displaystyle\int \mathrm{d}\mu_H(\Omega) K_{ab}\Phi^a\Phi^b = K_{ab}\sum_\rho \tfrac{1}{\dim(\rho)}\,\mathrm{tr}_\rho(\tau^a\otimes\tau^b)\,\mathrm{tr}_\rho(\Lambda\otimes\Lambda)$

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Perturbative expansion: Principles

The angular variables can be integrated out in the perturbative series.

Introduce $K_{ab} := \mathrm{tr}\,([L_i, \tau_a][L_i, \tau_b])$, $\Phi^a = \mathrm{tr}\,(\tau^a \Omega \Lambda \Omega^\dagger)$. Then:

$$e^{a\Phi^a K_{ab} \Phi^b} = 1 + a\Phi^a K_{ab} \Phi^b + \frac{a^2}{2} \Phi^a K_{ab} \Phi^b \, \Phi^c K_{cd} \Phi^d + \ldots$$

To integrate over $\mathrm{d}\mu_H(\Omega)$ we need to compute terms like

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Perturbative expansion: Results

Up to $\mathcal{O}(a^2)$, the perturbative expansion is easily doable.

After some group theory and algebra, we obtain:

$$\int \mathrm{d}\mu_H(\Omega) K_{ab}\Phi^a\Phi^b = \frac{1}{2} N \sum_{i>j} (\lambda_i - \lambda_j)^2$$

$$\int \mathrm{d}\mu_H(\Omega) K_{ab}\Phi^a\Phi^b K_{cd}\Phi^c\Phi^d = \frac{(2\,\mathrm{tr}\,K^2 + (\mathrm{tr}\,K)^2)}{N^2(N^4 - 10N^2 + 9)}(\alpha_1 A_1 + \alpha_2 A_2)$$
$$+ \frac{1}{N(-36 + N^2(-7 + N^2)^2)}(\beta_1 A_1 + \beta_2 A_2) K^\neg K \ ,$$

where

$$A_1 = \sum_{i>j} (\lambda_i - \lambda_j)^4 \quad \text{and} \quad A_2 = \left(\sum_{i>j} (\lambda_i - \lambda_j)^2 \right)^2$$

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Large N expansion

The large N expansion is rewritten in terms of a multi-matrix model.

In the large N limit, we have:

$$\int \mathrm{d}\mu_H(\Omega) K_{ab}\Phi^a\Phi^b = \frac{1}{2} N \sum_{i>j} (\lambda_i - \lambda_j)^2$$

$$\int \mathrm{d}\mu_H(\Omega) (K_{ab}\Phi^a\Phi^b)^2 = -\frac{N^2}{2} \sum_{i>j} (\lambda_i - \lambda_j)^4 + \frac{N^2}{4} \left(\sum_{i>j} (\lambda_i - \lambda_j)^2 \right)^2$$

After re-exponentiating the terms (still exact to $\mathcal{O}(a^2)$)

$$S = \sum_i \left(b\lambda_i^2 + c\lambda_i^4 \right) + \sum_{i>j} \Big(-\tfrac{a}{2} N (\lambda_i - \lambda_j)^2 + \tfrac{a^2}{4} N^2 (\lambda_i - \lambda_j)^4 - 2 \ln |\lambda_i - \lambda_j| \Big)$$

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Saddle point approximation

The saddle point approximation gives a rought picture of what is going on.

Rewrite: $\lambda_i \to \lambda(\frac{i}{N}) = \lambda(x)$, $0 < x < 1$, $\sum_{i=0}^{N} \to N \int_0^1 \mathrm{d}x$

Rescale: $a = N^{\theta_a}\tilde{a},\ b = N^{\theta_b}\tilde{b},\ c = N^{\theta_c}\tilde{c},\ \lambda(x) = N^{\theta_\lambda}\tilde{\lambda}(x)$

Partition function: $Z = \int \mathscr{D}\lambda\ \exp(-N^2\tilde{S})$

Action:

$$\tilde{S} = \int_0^1 \mathrm{d}x \Big(\tilde{b}\tilde{\lambda}^2(x) + \tilde{c}\tilde{\lambda}^4(x) + \int_0^1 \mathrm{d}y \Big(-\frac{\tilde{a}}{4}(\tilde{\lambda}(x)-\tilde{\lambda}(y))^2 + \frac{\tilde{a}^2}{8}(\tilde{\lambda}(x)-\tilde{\lambda}(y))^4 - \ln|\tilde{\lambda}(x)-\tilde{\lambda}(y)| \Big)\Big)$$

Saddle point solution (one symmetric cut $[-\delta, \delta]$): $u(\tilde{\lambda}) =$

$$\left(4\tilde{b} - \tilde{a} + 12\pi\tilde{a}^2 c_2 + 4\left(\tilde{c} + \frac{\pi\tilde{a}^2}{2}\right)\delta^2 + 8\left(\tilde{c} + \frac{\pi\tilde{a}^2}{2}\right)\tilde{\lambda}^2\right)\sqrt{\delta^2 - \tilde{\lambda}^2}$$

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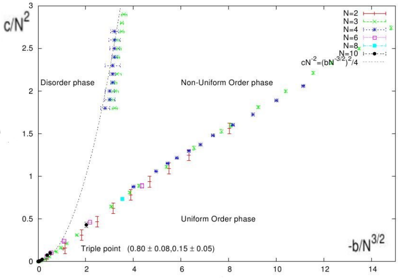
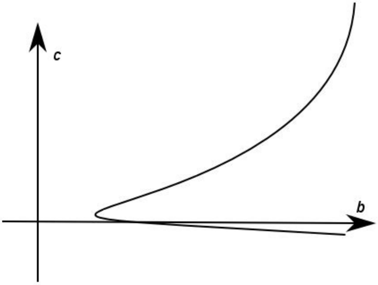
Saddle point approximation

New phase lies within the region of the two-cut solution.

The boundary of the region of validity of the one-cut solution is consistent with the data.



Numerics, fuzzy ϕ^4-theory



Analytical results, multi-trace MM

$\Rightarrow$ "New phase" must already be a feature of the matrix model.

Modification of the model

The proposed modification of the model moves the triple point in the right direction.

Modify the action, assuming momentum-dependent wave function regularization $\mathcal{Z}_L(C_2) \approx 1 + \kappa C_2$:

$$\tilde{S} = \operatorname{tr}\left(a\Phi(C_2 + \kappa C_2 C_2)\Phi + b\,\Phi^2 + c\,\Phi^4\right)$$

This implies the following modification in our analysis

$$K_{ab} \to \check{K}_{ab} := K_{ab} + \kappa K_{ac}K_{cb} \quad \text{and} \quad \check{a} = a(1 + \tfrac{2}{3}\kappa(N^2 - 1))$$

Rescaling of κ to keep highest order term yields $\tilde{\check{a}} = a(1 + \frac{2}{3}\tilde{\kappa})$.

$\Rightarrow$ The triple point moves off to infinity for increasing $\tilde{\kappa}$.

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We achieved the following:

- formulated a generalized character expansion technique
- reformulated fuzzy ϕ^4-theory as multi-trace matrix model
- preliminary analysis of the approximation looks promising

Future directions:

- Examine all possible one- and two-cut solutions; this should yield a (full) explanation of the phase diagram
- Examine modifications of the model for extended solutions
- Extract more information from the expansion
- Study other models (other fuzzy spaces) with this technique
- Relation to $c > 1$ string theories?

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