Multisymplectic Geometry – Applications to String theory

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Notes for Talk given at Workshop on Multisymplectic Geometry
KU Leuven - 11.9.2020

Abstract

In this talk, I will discuss a number of reasons why people working in string theory are interested in multisymplectic geometry. I will present some previous work related to string groups and Courant algebroids, but I will also focus on relevant open questions.

1. Outline

- Motivation
- Quantization of S^3
- String Lie 2-algebra
- Vinogradov algebroids
- A few questions...

Note:

- Not quite sure what I should talk about (I haven't used the term 'multisymplectic' in a while in a paper), but then much of my work is still connected. Hopefully you get some pleasure out of the appearances of multisymplectic geometry.
- References complicated, ask me
- very personally biased perspective

2. Motivation

String theory:

- Theories of strings moving through $M^{1,9}$
- Also contains Dp-branes, p-dimensional objects moving through $M^{1,9}$
- strings and Dp-branes can end on Dp-branes

- Roughly: we live on a D3-brane, and endpoints of strings are photons
- QFT is the "shadow" of string theory
- World-lines of point particles couple to 1-form, world volume of string couple to 2-form potential (connective structure of gerbe). Non-degenerate curvature: 2-plectic geometry.
- SUSY: higher curvature forms, higher n-plectic geometries.

However: # of different string theories ≥ 5.
M-theory:

- Theory of M2- and M5-branes moving through $M^{1,10}$
- M2-branes worldvolume couple to 3-form potential: 3-plectic geometry.
- M2-branes can end on M5-branes with a 1d boundary ("self-dual string")
- These strings can lead again to 2-plectic geometry
- String theory is shadow of M-theory:
  - $M^{1,10} = S^1 \times M^{1,9}$
  - M2 wrapping S^1: string
  - M2 transverse to S^1: D2-brane
  - M5 wrapping S^1: D4-brane

Understanding of String Theory is okish; need to understand M-theory.
In particular:

- Multiple Dp-branes give rise to a gauge theory, a physical theory of a connection on a principal bundle on the submanifold described by the Dp-brane
- Multiple M2-branes form Chern–Simons-matter theory
- Multiple M5-branes: (2,0)-theory, "holy grail of string theory"

Simple/important configuration: D1-branes ending on D4-brane (see e.g. [1, Section 1] and [2, Section 2]) !!sketch!!

- each point on D1-brane polarizes into fuzzy/geometrically quantized sphere S^2
- Chern class of prequantum line bundle is # of D1-branes
- $\mathsf{End}(\mathcal{H})$ is interesting gauge algebra

Lift to M-theory: M2-branes ending on M5-brane !!sketch!!

- each point on M2-brane polarizes into fuzzy S^3?
- Need to know!

3. Quantization of S^3

Quantization:

- Map functions f to endomorphisms $\hat{f}$ on Hilbert space
- such that $[\hat{f},\hat{g}] = \mathrm{i}\hbar\widehat{\{f,g\}}$

Recall $S^2 \subset \mathbb{R}^3$:

- $\{x^i, x^j\} = \varepsilon^{ijk}x^k \to [\hat{x}^i, \hat{x}^j] = \mathrm{i}\hbar\varepsilon^{ijk}\hat{x}^k$.

1st attempt: quantize functions on $S^3 \subset \mathbb{R}^4$ (cf. [3] and references therein)

- $\{x^\mu, x^\nu, x^\kappa\} = \varepsilon^{\mu\nu\kappa\lambda}x^\lambda$: Nambu–Poisson structure
- Doesn't quantize nicely
- $\mathbb{R}^3 \times \mathbb{R}^3$ does not work as expected

2nd attempt: qunatize multisymplectic observables (cf. Antonio's and Leyli's talks, [4, 5])

- $\omega = \mathrm{vol}_{S^3}$ is 2-plectic
- Hamiltonian 1-forms $\alpha \in \Omega^1_H(S^3)$: $\mathrm{d}\alpha = \iota_{X_\alpha}\omega$
- Lie 2-algebra or 2-term L_∞-algebra $\Pi_{S^3,\omega}$:
    - $C^\infty(S^3) \to \Omega^1_H(S^3)$
    - $\mu_1(f) = \mathrm{d}f$, $\mu_2(\alpha,\beta) = \iota_{X_\alpha}\iota_{X_\beta}\omega$, $\mu_3(\alpha,\beta,\gamma) = \iota_{X_\alpha}\iota_{X_\beta}\iota_{X_\gamma}$
- We "categorified," and this is a recurring theme in going from QFT to string theory and from string theory to M-theory.
    - Potentials/connections gain a form degree
    - principal bundles become gerbes
    - Lie algebras become Lie 2-algebras

Quantization (in principle, cf. [6])

- Prequantum gerbe $\mathscr{G}$ with DD class ω
- Prequantum Hilbert space
  - Sections (i.e. morphisms from trivial gerbe into $\mathscr{G}$): bundle gerbe modules
  - Hermitian vector bundle with connection
  - One sees higher-rank analogues of $\Pi_{S^3,\omega}$ arise.
- But: polarization, explicit constructions hard, may need diffferent approach...

Altogether: the multisymplectic observables $\Pi_{S^3,\omega}$ seem correct approach, even if difficult.

4. String Lie 2-algebra

Recall D1-branes on D3-branes: (S^2, ω)

- $\omega = c_1$ of principal bundle over S^2
- For $c_1 = 1$, Hopf fibration $S^1 \to S^3 \to S^2$
- S^3 carries group structure: $\mathsf{SU}(2)$.
- "First interesting" gauge Lie group.

For M2-branes on M5-branes: (S^3, ω), cf. e.g. [7, section 2]

- $\omega = dd$ of abelian gerbe over S^3 (central groupoid extension)
- For $dd = 1$, "categorified Hopf fibration:" $\mathsf{BU}(1) \to \mathscr{G} \to S^3$
- S^3 carries 2-group structure: $\mathsf{String}(3)$
- Recall: Whitehead tower:
  $$\ldots \to \mathsf{String}(n) \to \mathsf{Spin}(n) \to \mathsf{Spin}(n) \to \mathsf{SO}(n) \to \mathsf{O}(n)$$
  (Each arrow kills one homotopy group π_i, starting with π_0 for orientability.)
- Indeed important as a gauge group in description of M5-branes

String Lie 2-algebra $\mathfrak{string}(\mathfrak{g})$ (by higher Lie differentiation):

- $\mathbb{R} \to \mathfrak{g}$, $\mathfrak{g}$ quadratic Lie algebra
- $\mu_1 = 0$, $\mu_2(x_1, x_2) = [x_1, x_2]$, $\mu_3(x_1, x_2, x_3) = (x_1, [x_2, x_3])$.

Observation: (Rogers, Baez [8]):

- The string Lie 2-algebra $\mathfrak{string}(3)$ is the subalgebra of $\Pi_{S^3,\omega}$ for left-invariant Hamiltonian 1-forms.

5. Vinogradov algebroids

String background: Riemannian manifold with B-field/gerbe: $g_{\mu\nu}$, $B_{\mu\nu}$. Symmetries:

- Diffeos + 1-form gauge transformations: $B \mapsto \tilde{B} = B + \mathrm{d}\Lambda$
- described by sections of Courant algebroid $TM \oplus T^*M$
- Generalized Geometry

More powerful perspective: Vinogradov algebroids (cf. e.g. [9, Section 3.3])

- $\mathcal{V}_n(M) = T^*[n]T[1]M$, coordinates: $\overset{0}{x^\mu}, \overset{1}{\xi^\mu}, \overset{n-1}{\zeta_\mu}, \overset{n}{p_\mu}$
- Canonical symplectic form $\omega = \mathrm{d}x^\mu \wedge \mathrm{d}p_\mu + \mathrm{d}\xi^\mu \wedge \mathrm{d}\zeta_\mu$
- Hamiltonian: $\mathcal{Q} = \xi^\mu p_\mu$, Vector field $Q = \xi^\mu \frac{\partial}{\partial x^\mu} + p_\mu \frac{\partial}{\partial \zeta_\mu}$
- $\{\mathcal{Q}, \mathcal{Q}\} = 0 \Leftrightarrow Q^2 = 0$.
- Can "twist" by closed n-form ϖ (multisymplectic!):

$$\mathcal{Q} = \xi^\mu p_\mu + \tfrac{1}{n!}\varpi_{\mu_1\dots\mu_n}\xi^{\mu_1}\dots\xi^{\mu_n}$$

- Every symplectic L_∞-algebroid comes with associated Lie n-algebra:

$$\begin{array}{rccccc} \mathsf{L}(M) := & C^\infty_0(M) \to & C^\infty_1(M) \to & \dots \to & C^\infty_{n-2}(M) \to & C^\infty_{n-1}(M) \\ = & \mathsf{L}_{n-1}(M) \to & \mathsf{L}_{n-2}(M) \to & \dots \to & \mathsf{L}_1(M) \to & \mathsf{L}_0(M) \quad , \end{array}$$

  with brackets

$$\begin{aligned} \mu_1(\ell) &= Q\ell - \delta\ell \ , \\ \mu_2(\ell_1,\ell_2) &= \tfrac{1}{2}\big(\{\delta\ell_1,\ell_2\} \pm \{\delta\ell_2,\ell_1\}\big) \ , \\ \mu_3(\ell_1,\ell_2,\ell_3) &= -\tfrac{1}{12}\big(\{\{\delta\ell_1,\ell_2\},\ell_3\} \pm \dots\big) \ , \end{aligned}$$

  where we abbreviated

$$\delta(\ell) = \begin{cases} Q\ell & \ell \in C^\infty_0(M) \ , \\ 0 & \text{else} \ . \end{cases}$$

- Note: On $\mathcal{V}_n(M)$, $\mathsf{L}_0(M) = \mathfrak{X}(M) \oplus \wedge^{n-1}(M)$
  L describes symmetries (diffeos+gauge) of gravity + n-form potential
- $\mathcal{V}_2(M)$ is exact Courant algebroid $E = TM \oplus T^*M$

- $\mathcal{V}_3(M)$ and higher for M-theory.

E.g. Symplectic manifold (M, ϖ)

- Vinogradov algebroid $\mathcal{V}_1(M)$, twisted by ϖ
- Consider subset $(X_f, f) \in \mathsf{L}_0(M)$
- These form sub Lie algebra of $\mathsf{L}(M)$, which is Poisson algebra

E.g. 2-plectic manifold (M, ϖ) (Rogers, [10])

- Vinogradov algebroid $\mathcal{V}_2(M)$, twisted by ϖ
- Consider subset $(X_\alpha, \alpha) \in \mathsf{L}_0(M)$
- These form sub Lie 2-algebra of $\mathsf{L}(M)$, which is $\Pi_{M,\varpi}$

This generalizes and higher products given by evident formula: multisymplectic observables are a sub L_∞-algebra of the associated L_∞-algebra of symmetries of a twisted Vinogradov algebroid (cf. also [11, Theorem 4.10]).

6. A few questions...

- Multisymplectic observables don't multiply. Is this really a problem?
- $\mathfrak{string}$ is an extension of a Lie algebra by a cocycle. $\mathfrak{m5brane}$ is an extension of that by a further cocycle. Multisymplectic geometry and observables with two forms of different degree?
- Quantization of S^3 directly, no Hilbert space?
- Multisymplectic geometry sees more than symplectic geometry on loop space? Transgress, minimal model, differences to Poisson algebra on loop space?

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