Perturbative Quantum Field Theory, Colour–Kinematics Duality, and Homotopy Algebras

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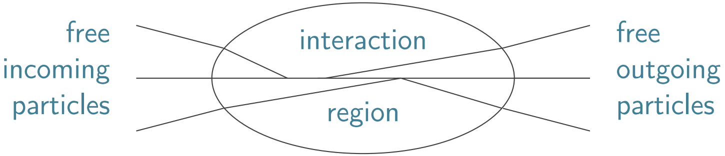
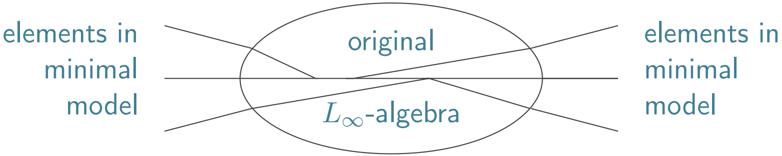
- arXiv: [1809.09899], 2007.13803, 2102.11390, 2106.00108
- in particular arXiv: 2205.?????

An observation

Tree-level Scattering Amplitudes in perturbative QFT:



Practically identical to comput. of Minimal Model of L_∞-algebra:



Both are linked by the Batalin–Vilkovisky Formalism

Scattering amplitude structures $\leftrightarrow$ homotopy algebraic structures

Outline

- From Lie algebras to L_∞-algebras
- Everything is homotopy Maurer–Cartan theory
- Homotopy algebras and related technology
- Examples of applications in QFT
- Color–kinematics duality from homotopy $BV^{\blacksquare}$-algebras
- How the double copy works homotopy algebraically
- Comments on the loop level

From Lie algebras to L_∞-algebras

Symmetries and beyond...

From Lie algebras to L_∞-algebras

Lie algebra $\mathfrak{g}$:

- Vector space $\mathfrak{g}$
- Lie bracket $[-,-]$ such that
  - $[a,b] = -[b,a]$
  - $[a,[b,c]] = [[a,b],c] + [b,[a,c]]$
- Basis τ_α defines structure constants $f^\alpha_{\beta\gamma}$ with $[\tau_\beta, \tau_\gamma] = f^\alpha_{\beta\gamma}\tau_\alpha$

Dually (Chevalley–Eilenberg algebra)

- Dual vector space $\mathfrak{g}[1]^*$ (all elements have degree 1)
- Basis ξ^α (of degree 1) are coordinate functions on $\mathfrak{g}[1]$.
- Most general vector field Q of degree 1 on $\mathfrak{g}[1]$:

$$Q = -\tfrac{1}{2} f^\alpha_{\beta\gamma}\xi^\beta\xi^\gamma \frac{\partial}{\partial\xi^\alpha}$$

- $Q^2 = 0$ is equivalent to Jacobi identity

L_∞-algebras as duals to general dgcas

Chevalley–Eilenberg algebra of a Lie algebra:

- Dual vector space $\mathfrak{g}[1]^*$ with basis ξ^α
- Homological vector field $Q - \frac{1}{2} f^\alpha_{\beta\gamma} \xi^\beta \xi^\gamma \frac{\partial}{\partial \xi^\alpha}$ on $\mathfrak{g}[1]$
- $Q^2 = 0$ is equivalent to Jacobi identity
- $(C^\infty(\mathfrak{g}[1]), Q)$ forms a differential graded commutative algebra

Chevalley–Eilenberg algebra of an L_∞-algebra:

- Dual graded vector space $\mathfrak{L}[1]^*$ with basis ξ^α
- Homological vector field Q of degree 1 on $\mathfrak{L}[1]$ with $Q^2 = 0$:

$$Q = \sum_{k=1}^{\infty} \pm f^\alpha_{\beta_1 \dots \beta_k} \xi^{\beta_1} \dots \xi^{\beta_k} \frac{\partial}{\partial \xi^\alpha}$$

- New "brackets" on $\mathfrak{L}$:

$$\mu_k(\tau_{\beta_1}, \dots \tau_{\beta_k}) := f^\alpha_{\beta_1 \dots \beta_k} \tau_\alpha$$

- $Q^2 = 0$ is equivalent to homotopy Jacobi identity

Dually: L_∞-algebra in bracket picture:

- Graded vector space: $\mathfrak{L} = \cdots \oplus \mathfrak{L}_{-2} \oplus \mathfrak{L}_{-1} \oplus \mathfrak{L}_0 \oplus \mathfrak{L}_1 \oplus \ldots$
- Graded totally antisymmetric multilinear "brackets"

$$\mu_i : \wedge^i \mathfrak{L} \to \mathfrak{L} \ , \quad |\mu_i| = 2 - i$$

- Satisfying higher/homotopy Jacobi identity:

$$\sum_{i+j=n} \sum_{\sigma \in \mathrm{Sh}(i,n-i)} \pm \mu_{i+1}(\mu_j(\ell_{\sigma(1)}, \ldots, \ell_{\sigma(j)}), \ell_{\sigma(j+1)}, \ldots, \ell_{\sigma(n)}) = 0$$

  - $\mu_1(\mu_1(\ell)) = 0$: μ_1 is a differential turning $\mathfrak{L}$ into a complex

$$\ldots \xrightarrow{\mu_1} \mathfrak{L}_{-2} \xrightarrow{\mu_1} \mathfrak{L}_{-1} \xrightarrow{\mu_1} \mathfrak{L}_0 \xrightarrow{\mu_1} \mathfrak{L}_1 \xrightarrow{\mu_1} \mathfrak{L}_2 \xrightarrow{\mu_1} \ldots$$

  - $\mu_1(\mu_2(\ell_1, \ell_2)) = \mu_2(\mu_1(\ell_1), \ell_2) \pm \mu_2(\ell_1, \mu_1(\ell_2))$:
    μ_1 is a derivation with respect to μ_2
  - $\mu_2(\mu_2(\ell_1, \ell_2), \ell_3) + \text{cycl.} = \pm \mu_1(\mu_3(\ell_1, \ell_2, \ell_3))$:
    Jacobi identity violated in a controlled way, by cocycle

Special cases: L_∞-algebras

- Lie algebras ($\mathfrak{L} = \mathfrak{L}_0$)
- graded Lie algebras ($\mu_i = 0$ for $i \neq 2$)
- differential graded Lie algebras ($\mu_i = 0$ for $i > 2$)
- categorified Lie algebras ($\mathfrak{L}_i = *$ for $i > 0$)
- Lie algebra $\mathfrak{g}$, representation $\rho : \mathfrak{g} \mapsto \mathsf{End}(V)$

$$\mathfrak{L}_0 = \mathfrak{g}\ , \quad \mathfrak{L}_{-1} = V \qquad \begin{aligned} \mu_2(\ell_1,\ell_2) &= [\ell_1,\ell_2] \\ \mu_2(\ell_1,v) &= \rho(\ell_1)v \end{aligned}$$

- complexes, e.g. de Rham complex ($\mu_i = 0$ for $i > 1$)

L_∞-algebras are generalizations of dg Lie algebras.

L_∞-algebras: Inner products

Inner product on Lie algebra $\mathfrak{g}$: $\langle -,-\rangle : \mathfrak{g}\times\mathfrak{g}\to\mathbb{R}$

- positive definite/non-degenerate
- symmetric
- bilinear
- satisfying cyclic relation:

$$\langle \ell_1,[\ell_2,\ell_3]\rangle = \langle \ell_2,[\ell_3,\ell_1]\rangle$$

Inner product dually: sympl. form ω on $\mathfrak{g}[1]$, $|\omega|=-1$, $\mathcal{L}_Q\omega=0$.

Induces: Cyclic structure on L_∞-algebra $\mathfrak{L}$: $\langle -,-\rangle : \mathfrak{L}\times\mathfrak{L}\to\mathbb{R}$

- non-degenerate
- graded symmetric
- bilinear
- satisfying cyclic relation:

$$\langle \ell_1,\mu_i(\ell_2,\ldots,\ell_{1+i})\rangle = \pm\langle \ell_2,\mu_i(\ell_3,\ldots,\ell_{1+i},\ell_1)\rangle$$

Homotopy Maurer–Cartan Theory

"One ring to rule them all ..."

Maurer–Cartan equation for differential graded Lie algebra, $(\mathfrak{g}, \mathrm{d})$:

$$\mathrm{d}a + \tfrac{1}{2}[a,a] = 0\ , \quad a \in \mathfrak{g}\ .$$

L_∞-algebras are generalizations of dg Lie algebras.

Homotopy Maurer–Cartan equation:

$$f := \mu_1(a) + \tfrac{1}{2}\mu_2(a,a) + \tfrac{1}{3!}\mu_3(a,a,a) + \cdots = 0\ , \quad a \in \mathfrak{L}_1$$

Nomenclature: a: gauge potential f: curvature

Bianchi identity:

$$\mu_1(f) - \mu_2(f,a) + \tfrac{1}{2}\mu_3(f,a,a) - \tfrac{1}{3!}\mu_4(f,a,a,a) + \cdots = 0\ .$$

Homotopy Maurer–Cartan Action:

$$S_{\mathrm{MC}}[a] := \sum_{i\geq 1} \frac{1}{(i+1)!}\langle a, \mu_i(a,\ldots,a)\rangle_{\mathfrak{L}}\ .$$

All perturbative field theories are hMC theory

Recall the BV-formalism:

- Resolve the quotient space of observables:
  - Introduce ghosts to resolve gauge redundancy ("BRST")
  - Introduce anti-fields to resolve equations of motion
  - Differential Q_{BV} encodes gauge sym.+EOM
- anti-bracket/symplectic form defines pairing
- BV-field space $\mathfrak{L}[1]$ is a symplectic graded vector space
- Defines a dgca, the dual data of an L_∞-algebra $\mathfrak{L}$.
- Can show: both original and BV-action are hMC-theories.

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$\cdots$	$\mathfrak{L}_0$	$\mathfrak{L}_1$	$\mathfrak{L}_2$	$\mathfrak{L}_3$	$\cdots$
$\cdots$	gauge transf.	physical fields	equations of motion	Noether identities	$\cdots$

Interestingly: action for closed string field theory also hMC.

Example: Yang–Mills theory

Complex:

$$\underbrace{\Omega^0(M,\mathfrak{g})}_{\mathfrak{L}_0} \xrightarrow{\mu_1 := \mathrm{d}} \underbrace{\Omega^1(M,\mathfrak{g})}_{\mathfrak{L}_1} \xrightarrow{\mu_1 := \mathrm{d}\star\mathrm{d}} \underbrace{\Omega^{d-1}(M,\mathfrak{g})}_{\mathfrak{L}_2} \xrightarrow{\mu_1 := \mathrm{d}} \underbrace{\Omega^d(M,\mathfrak{g})}_{\mathfrak{L}_3}$$

Higher products:

$$\mu_1(c_1) := \mathrm{d}c_1 \ , \quad \mu_1(A_1) := \mathrm{d}\star\mathrm{d}A_1 \ , \quad \mu_1(A_1^+) := \mathrm{d}A_1^+ \ ,$$
$$\mu_2(c_1,c_2) := [c_1,c_2] \ , \quad \mu_2(c_1,A_1) := [c_1,A_1] \ , \quad \mu_2(c_1,A_2^+) := [c_1,A_2^+] \ ,$$
$$\mu_2(c_1,c_2^+) := [c_1,c_2^+] \ , \quad \mu_2(A_1,A_2^+) := [A_1,A_2^+] \ ,$$
$$\mu_2(A_1,A_2) := \mathrm{d}\star[A_1,A_2] + [A_1,\star\mathrm{d}A_2] + [A_2,\star\mathrm{d}A_1] \ ,$$
$$\mu_3(A_1,A_2,A_3) := [A_1,\star[A_2,A_3]] + [A_2,\star[A_3,A_1]] + [A_3,\star[A_1,A_2]]$$

Homotopy Maurer–Cartan action with $a = A$ is Yang–Mills action!
hMC action with $a = c_0 + A + A^+ + c^+$ is BV action!

Homotopy algebras and applications to physics

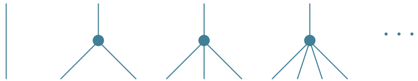
Abstract nonsense orders everything.

Homotopy algebras

L_∞-algebras are homotopy Lie algebras.

Operads: powerful tool for describing algebras

- "Things" with n incoming and one outgoing legs



- Rules for composition
- Each algebra: algebra over an operad, e.g. $\mathcal{A}ss$, $\mathcal{C}om$, $\mathcal{L}ie$
- Koszul duality e.g. $\mathcal{A}ss^! = \mathcal{A}ss$, $\mathcal{L}ie^! = \mathcal{C}om$
- Homotopy algebra: take dual of diff. grad. Koszul dual algebra
- Examples:
  - L_∞- or homotopy Lie algebras: dual to dgCom algebras.
  - A_∞-algebras: dual to dgAss algebras
  - C_∞-algebras: dual to dgLie algebras

Benefits of abstract perspective

Strong and useful mathematical results:

- Notions of structures such as tensor products
- Notions of morphisms and equivalences
- Strictification theorem:
  Any homotopy $\mathcal{O}$-algebra $\cong$ a dg$\mathcal{O}$-algebra.
- Minimal model theorem:
  Any homot. $\mathcal{O}$-algebra $\cong$ a homot. $\mathcal{O}$-algebra on its cohomlg.
- Homotopy Perturbation Lemma/Homotopy Transfer:
  A recursive construction prescription of equivalent homotopy algebras, starting from equivalent complexes.

All have interesting consequences for/applications to field theory.

Applications: Tensor products

Color-stripping, e.g. in Yang–Mills theory

- Separate kinematics from color in Yang–Mills amplitudes
- Homotopy algebraic perspective: factorization

$$\mathfrak{L}^{\text{Yang–Mills}} = \mathfrak{g} \otimes \mathfrak{C}^{\text{Yang–Mills}}$$

where $\mathfrak{C}^{\text{Yang–Mills}}$ is a C_∞-algebra.

- Homotopy algebraic generalization of the tensor product

$$\mathcal{L}ie \otimes \mathcal{C}om = \mathcal{L}ie$$

Applications: Strictification

Rendering a field theory cubic

- Simpler to argue about field theories with only cubic vertices
- Homotopy algebraic perspective: strictification

$$\mathfrak{L} \cong \mathfrak{L}^{\rm st} ,$$

  guaranteed to exist by strictification theorem
- Construction algorithm available
- Used in particular in the context of color–kinematics duality

Homotopy Transfer

Recall: L_∞-algebras have underlying differential complex

$$\dots \xrightarrow{\mu_1} \mathfrak{L}_{-2} \xrightarrow{\mu_1} \mathfrak{L}_{-1} \xrightarrow{\mu_1} \mathfrak{L}_0 \xrightarrow{\mu_1} \mathfrak{L}_1 \xrightarrow{\mu_1} \mathfrak{L}_2 \xrightarrow{\mu_1} \dots$$

Homotopy Transfer:

- Start from two quasi-isomorphic complexes $\mathfrak{L} \cong \mathfrak{L}'$
  I.e.: have chain map $\phi : \mathfrak{L} \to \mathfrak{L}'$ inducing $H^\bullet_{\mu_1}(\mathfrak{L}) \cong H^\bullet_{\mu'_1}(\mathfrak{L}')$
- Consider higher products μ_i, $i > 1$ on $\mathfrak{L}$ as perturbation
- Recursive prescription of how this induces brackets μ'_i on $\mathfrak{L}'$.

Applications:

- Construct/show semi-classical equivalence of field theories
- For $\mathfrak{L}' = H^\bullet_{\mu_1}(\mathfrak{L})$: recover minimal model and tree-level Feynman diagram expansion
- Introducing another perturbation $\mathrm{i}\hbar\Delta^*$ creating field/anti-field pairs yields loop-level Feynman diagram expansion
- Recursive character underlies Berends–Giele recursion relation, which exists for all field theories at loop level

Homotopy algebraic perspective on field theory

Feynman diagram expansion $\leftrightarrow$ computing minimal model

Conclusions:

- Easily explainable to mathematicians
- Study scattering amplitudes through homotopy algebraic lense
- L_∞-algebras: put action and amplitudes on equal footing
- Structure of amplitudes $\leftrightarrow$ homotopy algebraic structure

Color–kinematics duality and double copy

- Consider a field theory with symmetry Lie algebra $\mathfrak{g}$
- The theory is CK-dual if its tree-level amplitudes can be parameterized by cubic Feynman diagrams such that the kinematical numerators have the same algebraic properties as the $\mathfrak{g}$- or color-factors
- Proved e.g. for Yang–Mills theory

Bern, Carrasco, Johansson, 2008, 2010

- Consider a CK-dual field theory, amplitudes in CK-dual form
- Replace color factors with second copy of kinematical nums.
- Result is the double copy of the original theory
- Double copy of Yang–Mills is $\sim \mathcal{N} = 0$ supergravity

More details: Leron's and other talks

Homotopy algebraic perspective on

Colour–Kinematics Duality and the Double Copy

Abstract nonsense clarifies everything.

Color–kinematics duality and homotopy algebras

- The theory is CK-dual if its tree-level amplitudes can be parameterized by cubic Feynman diagrams such that the kinematical numerators have the same algebraic properties as the $\mathfrak{g}$- or color-factors

Homotopy algebraic perspective:

- Cubic Feynman diagrams: strictification
- Kinematic numerators vs. color factors: factorization

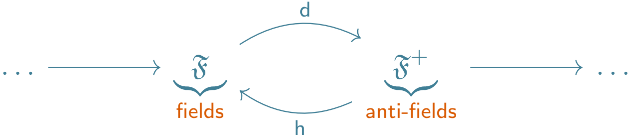
$$\mathfrak{L}^{\mathsf{YM,st}} = \mathfrak{g} \otimes \mathfrak{C}^{\mathsf{YM,st}}$$

  where $\mathfrak{C}^{\mathsf{YM,st}}$ is differential graded commutative algebra
- Kinematical factors map pairs of fields to fields
- Besides the degree 0 maps μ_2: fields$\times$fields $\rightarrow$ anti-fields new map of degree -1, related to μ_2

Towards $BV^{\blacksquare}$-algebras

Motivation:

- Brackets $\sim \mu_2$ of degree -1: anti-bracket in BV-algebra
- Prominence of d'Alembertian $\Box$ in constructions
- Structure of complex



suggests

$$\mathsf{dh} + \mathsf{hd} = [\mathsf{d}, \mathsf{h}] = \Box$$

- Also: open string BRST Hilbert space $[Q_{\mathsf{BRST}}, b_0] = L_0$

We need a $BV^{\blacksquare}$-algebra.

CK-duality from $BV^{\blacksquare}$-algebra

Content: also: Akman (1997)

- complex $(\mathfrak{B}, \mathsf{d})$
- Hodge triple $(\mathsf{d}, \mathsf{h}, \Box)$ with $\mathsf{dh} + \mathsf{hd} = [\mathsf{d}, \mathsf{h}] = \Box$
- graded commutative product $- \cdot -$ with d derivation
- second product $\{-,-\}$ of degree -1,

$$\{v_1, v_2\} := \mathsf{h}(v_1 \cdot v_2) - \mathsf{h}(v_1) \cdot v_2 - (-1)^{|v_1|} v_1 \cdot \mathsf{h}(v_2)$$

$$\{v_1, v_2 \cdot v_3\} = \{v_1, v_2\} \cdot v_3 + (-1)^{(|v_1|-1)|v_2|} v_2 \cdot \{v_1, v_3\}$$

- some additional, technical stuff

Interpretation:

- Factorize strictified (cubic) field theory $\mathfrak{L}^{\mathsf{st}} = \mathfrak{g} \otimes \mathfrak{C}^{\mathsf{st}}$
- Theory is CK-dual, if $\mathfrak{C}^{\mathsf{st}}$ can be extended to $BV^{\blacksquare}$-algebra $\mathfrak{B}^{\mathsf{st}}$
- $(\mathfrak{B}^{\mathsf{st}}, \mathsf{d}, - \cdot -)$ form dgCom algebra $\mathfrak{C}^{\mathsf{st}}$ in $\mathfrak{B}^{\mathsf{st}}$
- $(\mathfrak{B}^{\mathsf{st}}_1, \{-,-\})$ is kinematical Lie algebra

Conclusion

A field theory in manifestly CK-dual form can be color-stripped to dgCom-algebra, which can be enhanced to $BV^{\blacksquare}$-algebra.

Examples:

- Biadjoint scalar field theory
- Chern–Simons theory

General CK-duality

More general field theory, e.g. ordinary Yang–Mills action:

- Split off color: $\mathfrak{L} = \mathfrak{g} \otimes \mathfrak{C}$
- $\mathfrak{C}$ to manifestly CK-dual $\mathfrak{C}^{\mathsf{st}}$ is strictification
- Inversely: $\mathfrak{C}$ can be extended to $BV^{\blacksquare}_{\infty}$-algebra $\mathfrak{B}$

Field theory with L_∞-algebra $\mathfrak{L} = \mathfrak{g} \otimes \mathfrak{C}$ is CK-dual
$\Leftrightarrow$ $\mathfrak{C}$ extends to $BV^{\blacksquare}_{\infty}$-algebra.

see also Reiterer (2019)

Even more generally (e.g. $\mathcal{N} = 0$ supergravity):

- Can regard $\mathfrak{g}$ as $BV^{\blacksquare}_{\infty}$-algebra
- Define a generalized tensor product between $BV^{\blacksquare}_{\infty}$-algebras
- CK-duality: factorization into $BV^{\blacksquare}_{\infty}$-algebras

Advantages

- Clean mathematical definition of CK-duality
- Definition of kinematical algebra
- Interpretation of steps of rendering action manifestly CK-dual:
  - "Blowing up vertices": strictification
  - "Tolotti–Weinzierl terms": particular products
  - ⋮
- Evidence for computational advantages
- Involves the whole BV-complex: ghosts, anti-fields, etc.
- Full set of manifestly CK-dual Feynman rules
- Explicit link to string field theory
- More general notion of CK-duality/double copy

Double Copy

Inspiration from string theory:

- Naive $\mathscr{H}_{\text{closed}} := \mathscr{H}_{\text{open}} \otimes \mathscr{H}_{\text{open}}$ doubles domain ($\Rightarrow$ DFT)
- Restrict: states of form $|p,\ldots\rangle \otimes |p,\ldots\rangle$ ("section condition")
- Further constraint: $(b_0 - \tilde{b}_0)|\phi\rangle = 0$ and $(L_0 - \tilde{L}_0)|\phi\rangle = 0$
- Identification $Q_{\text{BRST}} = \mathsf{d}$, $b_0 = \mathsf{h}$, and $L_0 = \Box$

Field theories under consideration are "special"

- Allow for Hodge triple $\mathsf{d} = \mathsf{m}_1$, $\mathsf{h} = [1]$, $\blacksquare = \Box$
- Allows for a generalized tensor product $BV_\infty^\blacksquare \otimes BV_\infty^\blacksquare = \mathcal{L}ie_\infty$
- has $(\mathsf{h} - \tilde{\mathsf{h}})v = 0$ and $(\Box - \tilde{\Box})v = 0$ implemented

Double copy: generalized tensor product between $BV_\infty^\blacksquare$-algebras

Comments on the loop level

What about loop level?

- Our constructions are based on strictification
- Physically: semi-classical equivalence
  (i.e. equivalent tree-level S-matrix)
- Actions related by field redefinitions, integrating in/out fields
- But: field redefinitions+path integral measure $\rightarrow$ Jacobians
- Some harmless: $\rightarrow 1$ in dimensional regularization
- But: these seem not sufficient for our purposes
- Missing: counterterms from the Jacobians, unitarity broken
- Question: Is this a problem for our intents and purposes?
- Note: existence of counterterms given, they are under control
- Note: we are close(r) to the string theory origin of CK-duality

Summary

- Perturbative Quantum Field Theory using Homotopy Algebras
- Bridge between both: BV-formalism
- Many useful and powerful tools from Mathematics
- Unifying perspective on actions and amplitudes
- Any structure on amplitudes:
  homotopy algebraic refinement of L_∞-algebras
- Important example: CK-duality via $BV_\infty^\blacksquare$-algebras
- Theoretical advantage evident
- Evidence for computational advantage, but to be demonstrated

Thank You!