Lie 2-algebra models

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1. Introduction

- Current formulations of string theory **background dependent**
- Ways out:
  - **IKKT** (string theory)
  - **BFSS** (M-theory)
- Nice picture: background + dynamics **emergent**
  based on symplectic manifolds and Poisson/Lie algebras
- **However**, M-theory suggests:
  2-plectic manifolds, Lie 2-algebras
- **Question**: Background-independent formulation with Lie 2-algebras?
- Also: Our ideas are only good if they generalize as much as possible.

2. IKKT model

- Regularization of Green-Schwarz action of superstring in Schild gauge:

$$S=\int \mathrm{d}^2\sigma\sqrt{g}\alpha\left(\tfrac{1}{4}\{X_\mu,X_\nu\}^2-\tfrac{\mathrm{i}}{2}\bar{\psi}\Gamma^\mu\{X_\mu,\psi\}\right)+\beta\sqrt{g}\ .$$

- $\{-,-\}\rightarrow[-,-]$, $\int\rightarrow$ tr. This yields a 0d matrix model:

$$S_{\rm IKKT}=\alpha\,\mathrm{tr}\,\left(-\tfrac{1}{4}\,[A_\mu,A_\nu]^2-\tfrac{1}{2}\,\bar{\psi}\Gamma^\mu[A_\mu,\psi]+\beta\mathbb{1}\right)\ .$$

- Alternatively: full dimensional reduction of 10d SYM $\nabla_\mu\rightarrow A_\mu$, $\int\rightarrow -$:

$$S=\int \mathrm{d}^{10}x\quad \mathrm{tr}\,\left(F_{\mu\nu}F^{\mu\nu}-\tfrac{\mathrm{i}}{2}\bar{\psi}\Gamma^\mu\nabla_\mu\psi\right)$$

- Allow for deformation terms

$$S_{\rm def}=S_{\rm IKKT}+\ \mathrm{tr}\,\left(-\tfrac{1}{2}\sum_\mu m_{1,\mu}^2A_\mu A_\mu+\tfrac{\mathrm{i}}{2}m_2\bar{\psi}\psi+c_{\mu\nu\kappa}A^\mu[A^\nu,A^\kappa]\right)\ ,$$

- Suggested as a background independent definition of type IIB superstring theory.

3. Quantization of symplectic manifolds

- Classical Phase space:
  - Symplectic manifold (M, ω)
  - Associative alg. of observables $\mathcal{C}^\infty(M)$.
  - Additional Poisson (Lie) algebra structure:

$$\iota_{X_f}\omega = \mathrm{d}f \ , \quad \{f, g\} := \iota_{X_f}\iota_{X_g}\omega \ .$$

- Quantization:
  - Quantum Hilbert space $\mathcal{H}$
  - Associative alg. of observables $\mathsf{End}\,(\mathcal{H})$
  - Lie alg. structure: $[A, B]$
  - quantization map $\hat{-} : \mathcal{C}^\infty(M) \to \mathsf{End}\,(\mathcal{H})$
  - $\hat{f}\hat{g} = \widehat{fg} + \mathcal{O}(\hbar)$
  - $[\hat{f}, \hat{g}] = -\mathrm{i}\hbar\widehat{\{f, g\}} + \mathcal{O}(\hbar^2)$
  - Quantization: Lie algebra homomorphism up to $\mathcal{O}(\hbar)$

- Examples: $\mathbb{R}^2$, $\omega = \mathrm{d}x^1 \wedge \mathrm{d}x^2$, $\{f, g\} = \varepsilon^{ab}(\partial_a f)(\partial_b g)$
  Moyal space: $[\hat{x}^1, \hat{x}^2] = -\mathrm{i}\hbar\mathbb{1}$

- Examples: S^2, $\omega = \sin\theta \mathrm{d}\theta \wedge \mathrm{d}\phi$, $\{f, g\} = \frac{\varepsilon^{ab}}{\sin\theta}(\partial_a f)(\partial_b g)$
  Fuzzy sphere: $S^2 \hookrightarrow \mathbb{R}^3$: $[\hat{x}^i, \hat{x}^j] = -\mathrm{i}\hbar\varepsilon^{ijk}\hat{x}^k$.

4. Emergence of spacetime and dynamics

- EOMs of IKKT model:

$$[A^\mu, [A_\mu, A_\nu]] - m_{1,\nu}^2 A_\nu + c_{\nu\mu\kappa}[A^\mu, A^\kappa] = 0 \tag{4.1}$$

- Solutions:
  - For $m_{1,\nu} = c_{\mu\nu\kappa} = 0$:
    $[\hat{x}^i, \hat{x}^j] = -\mathrm{i}\hbar\mathbb{1}$: Moyal space and products thereof.
  - For $m_{1,\nu} = 0$, $c_{ijk} \sim \varepsilon_{ijk}$:
    $[\hat{x}^i, \hat{x}^j] = -\mathrm{i}\hbar\varepsilon^{ijk}\hat{x}^k$, Fuzzy sphere

- Dynamics:
  - Linear perturbations around vacuum solution: $A_\mu \to X_\mu + a_\mu$.
  - For IKKT, this yields NC Yang Mills with eom: $[X^\mu, [X_\mu, a_\mu]] = 0$ etc.

5. M-theory "lift"

- Follow IKKT route: Consider membrane action in Schild-type gauge:

$$S = T_{\rm M2} \int {\rm d}^3\sigma \{X^M, X^N, X^K\}^2 \ , \quad M, N, K = 0, \ldots, 10 \ .$$

- How to regularize this: Nambu-Poisson structure → 3-Lie algebra → differential crossed module → Lie 2-algebra
- Also: B-field of string theory becomes C-field of M-theory
  or: 2-forms (symplect) become 3-forms (2-plectic)
- Finally: D3: $\mathbb{R}^2 \times \mathbb{R}^2 \rightarrow$ M5: $\mathbb{R}^3 \times \mathbb{R}^3$
  D1-D3: fuzzy S^2, M2-M5: fuzzy S^3
- Lie 2-algebras (and even more so L_∞-algebras) all over the place in string/M-theory

6. Strong homotopy Lie algebras and Lie 2-algebras

- These are generalizations of (differential graded) Lie algebras
- Graded vector space $L = \oplus_{\ell \leq 0} L_p$, $\ell \in L_p \Rightarrow |\ell_p| = p$
- Multilinear, totally skew-symmeric brackets $\mu_i : L^{\otimes i} \rightarrow L$ of degree $2 - i$
- Homotopy Jacobi identities:

$$\sum_{i+j=n} \sum_{\sigma} \pm \mu_{j+1}(\mu_i(x_{\sigma(1)}, \cdots, x_{\sigma(i)}), x_{\sigma(i+1)}, \cdots, x_{\sigma(i+j)}) = 0$$

(Semistrict) Lie 2-algebras are 2-term L_∞-algebras

- $L = L_{-1} \oplus L_0 = V \oplus W$
- Non-trivial:

$$\mu_1 : V \rightarrow W \ , \quad \mu_2 : W \wedge W \rightarrow W \ , \quad \mu_2 : V \wedge W \rightarrow V \ , \quad \mu_3 : W \wedge W \wedge W \rightarrow V \ .$$

- Homotopy Jacobi rules, e.g.

$$\mu_1(\mu_2(w, v)) = \mu_2(w, \mu_1(v)) \ , \quad \mu_2(\mu_1(v_1), v_2) = \mu_2(v_1, \mu_1(v_2)) \ ,$$
$$\mu_1(\mu_3(w_1, w_2, w_3)) = -\mu_2(\mu_2(w_1, w_2), w_3) - \mu_2(\mu_2(w_3, w_1), w_2) - \mu_2(\mu_2(w_2, w_3), w_1) \ ,$$

- Example: $\mathbb{R} \rightarrow \mathfrak{g}$, $\mu_1(r) = 0$, $\mu_2(g_1, g_2) = [g_1, g_2]$ $\mu_3(g_1, g_2, g_3) = {\rm tr}\,(g_1[g_2, g_3])$
- Example: $\mu_3 = 0$ $\mathfrak{h} \rightarrow \mathfrak{g}$ **differential crossed module**.

7. Metric Lie 2-algebras

- "Gauge trafos" in IKKT: $g \in \mathsf{U}(N)$, $A_\mu \to g^{-1}A_\mu g$, $\mathrm{tr}\,(A_\mu A_\nu)$ etc. invariant.
- Lie 2-algebra automorphism:
  $\Psi_{-1} : V \to V'$, $\Psi_0 : W \to W'$, $\Psi_2 : W \times W \to V'$ with

$$\begin{aligned} \Psi_0\left(\mu_2(w_1,w_2)\right) &= \mu_2\left(\Psi_0(w_1),\Psi_0(w_2)\right) + \mu_1(\Psi_2(w_1,w_2))\ , \\ \Psi_{-1}\left(\mu_2(w,v)\right) &= \mu_2(\Psi_0(w),\Psi_{-1}(v)) + \Psi_2(w,\mu_1(v))\ , \\ \mu_3(\Psi_0(w_1),\Psi_0(w_2),\Psi_0(w_3)) &= \Psi_{-1}\left(\mu_3(w_1,w_2,w_3)\right) - [\Psi_2(w_1,\mu_2(w_2,w_3)) \\ &\quad + \mu_2\left(\Psi_0(w_1),\Psi_2(w_2,w_3)\right) + \text{cyclic}\,(w_1,w_2,w_3)]\,. \end{aligned}$$

- Need "metric" invariant under these. There are essentially two candidates:
- Kontsevich's suggestion:

$$\langle x_1,x_2\rangle_\infty = (-1)^{\tilde{x}_1+\tilde{x}_2}\langle x_2,x_1\rangle_\infty\ .$$
$$\langle \mu_n(x_1,\dots,x_n),x_0\rangle_\infty = (-1)^{n+\tilde{x}_0(\tilde{x}_1+\dots+\tilde{x}_n)}\langle \mu_n(x_0,\dots,x_{n-1}),x_n\rangle_\infty\ ,$$

  Very restricted: $\langle \mu_1(v),w\rangle_\infty = \langle \mu_2(v_1,w),v_2\rangle_\infty = \langle \mu_3(w_1,w_2,w_3),v\rangle_\infty = 0$
  However: seems to suffice for most of our purposes.

- Modified inner product on differential crossed module:

$$\langle v_1,v_2\rangle_0 = \langle v_2,v_1\rangle_0\ ,\quad \langle w_1,w_2\rangle_0 = \langle w_2,w_1\rangle_0\ ,\quad \langle v,w\rangle_0 = \langle w,v\rangle_0 = 0\ ,$$
$$\langle \mu_2(w_1,x_1),x_2\rangle_0 + \langle x_1,\mu_2(w_1,x_2)\rangle_0 = 0$$

  More natural from the perspective of quantization.
  Necessary for linking to M2-brane models.

8. Quantized 2-plectic manifolds

- Two paths: Nambu-Poisson and 2-plectic. First is problematic.
- 2-plectic manifold (M,ϖ) comes with a Lie 2-algebra:
  - Hamiltonian one-forms: $\iota_{X_\alpha}\omega = \mathrm{d}\alpha$
  - $L = \mathcal{C}^\infty(M) \oplus \Omega^1_{\mathrm{Ham}}(M)$
  - $\pi_1 = \mathrm{d}$, $\pi_2(\alpha,\beta) = \iota_{X_\alpha}\iota_{X_\beta}\varpi$, $\pi_3(\alpha,\beta,\gamma) = \iota_{X_\alpha}\iota_{X_\beta}\iota_{X_\gamma}\varpi$.
  - Example: Heisenberg Lie 2-algebra of $\mathbb{R}^3$:
    $\xi_i = \frac{1}{2}\varepsilon_{ijk}x^j\mathrm{d}x^k$, $\pi_2(\xi_i,\xi_j) = \varepsilon_{ijk}\mathrm{d}x^k$ and $\pi_3(\xi_i,\xi_j,\xi_k) = -\varepsilon_{ijk}$.
  - Compatible with loop space picture

- Associative product on L unknown, problems ... loop space etc.
- Quantization: Lie 2-algebra homomorphism up to $\mathcal{O}(\hbar)$. Restrict to:

$$\mu_1(\hat{X}) = -\mathrm{i}\hbar\, \widehat{\pi_1(X)} + \mathcal{O}(\hbar)\ , \quad \mu_2(\hat{X},\hat{Y}) = -\mathrm{i}\hbar\, \widehat{\pi_2(X,Y)} + \mathcal{O}(\hbar^2)\ ,$$
$$\mu_3(\hat{X},\hat{Y},\hat{Z}) = -\mathrm{i}\hbar\, \widehat{\pi_3(X,Y,Z)} + \mathcal{O}(\hbar^2)\ .$$

9. Lie 2-algebra models

- Construct "actional functionals" using metric Lie 2-algebras
- Example: Kontsevich's ∞-metric:

$$\begin{aligned} S_\infty :=& \tfrac{1}{2} m_{ab}\langle X^a, X^b\rangle_\infty + \tfrac{1}{3} c_{abc}\langle X^a, \mu_2(X^b,X^c)\rangle_\infty + \tfrac{1}{4}\langle \mu_2(X^a,X^b), \mu_2(X^a,X^b)\rangle_\infty \\ =& \tfrac{1}{2} m_{ab}\langle v^a, v^b\rangle_\infty + \tfrac{1}{2} m_{ab}\langle w^a, w^b\rangle_\infty + \tfrac{1}{3} c_{abc}\langle w^a, \mu_2(w^b,w^c)\rangle_\infty + \\ & + \tfrac{1}{4}\langle \mu_2(w^a,w^b), \mu_2(w^a,w^b)\rangle_\infty\ . \end{aligned}$$

- Solutions:
  - $m = c = 0$: quantized $\mathbb{R}^3$
  - $m \neq 0$ or $c \neq 0$: quantized S^3
  - This is exactly how the solutions appear in the IKKT model.
- For the Lie 2-algebra $* \to \mathfrak{u}(N)$, we recover the IKKT model with deformations!
- Quantization of solution also nicely reduces from 2-plectic to symplectic

10. Inhomogeneous Lie 2-algebra models

- Recall: IKKT obtained from SYM reduced to 0d.
- What happens for M2-brane models?
- Lie 2-algebra from "3-Lie algebras are special differential crossed modules"
- Allowing fields $X \in V$ and $Y \in W$ (and superpartners $\Psi \in X$)

$$\begin{aligned} S_{\rm M2} =& \tfrac{1}{6}\varepsilon^{ijk}\langle Y^i, \mu_2(Y^j,Y^k)\rangle - \tfrac{1}{2}\langle \mu_2(Y^i,X^a), \mu_2(Y^i,X^a)\rangle + \tfrac{\mathrm{i}}{2}\langle \bar{\Psi}, \mu_2(\Gamma^i Y^i, \Psi)\rangle \\ & - \tfrac{\mathrm{i}}{4}\langle \bar{\Psi}, \mu_2(\mu_2^*(X^a,X^b), \Gamma_{ab}\Psi)\rangle - \tfrac{1}{12}\langle \mu_2(\mu_2^*(X^a,X^b),X^c), \mu_2(\mu_2^*(X^a,X^b),X^c)\rangle \end{aligned}$$

- For L a differential crossed module with $\mathsf{t} = 0$: (generalized) BLG model.

11. Background field expansion

- Higher gauge theory with one- and two-forms A, B,

$$\begin{aligned}\mathcal{F} :=& \mathrm{d}A + \tfrac{1}{2}\mu_2(A,A) - \mu_1(B) = 0 \ ,\\ \mathcal{H} :=& \mathrm{d}B + \mu_2(A,B) + \tfrac{1}{6}\mu_3(A,A,A) \ .\end{aligned}$$

- Consider BF-theory:

$$S_{\mathrm{BF}} = \int_{\mathbb{R}^3} \langle \lambda_1, \mathcal{F}\rangle + \langle \lambda_0, H\rangle \ , \tag{11.1}$$

- Dimensionally reduce: $A \to Y_i$, $B \to X_{ij}$

$$\begin{aligned}S_{0d} =& \varepsilon^{ijk}\langle \lambda_i, \mu_2(Y_j,Y_k) - \tfrac{1}{2}\mu_1(X_{jk})\rangle + 2\mathrm{i}\hbar\langle \lambda_i, \widehat{\mathrm{d}x^i}\rangle\\ &+ \varepsilon^{ijk}\langle \lambda, \tfrac{1}{2}\mu_2(Y_i,X_{jk}) + \tfrac{1}{6}\mu_3(Y_i,Y_j,Y_k)\rangle - \mathrm{i}\hbar\langle \lambda, \mathbb{1}\rangle \ .\end{aligned}$$

- Solutions: Heisenberg 2-algebra of $\mathbb{R}^3$.
  Moreover: $\mathrm{d}x^i \wedge \pi_2(\xi_i, \alpha) = \mathrm{d}\alpha$, $\xi_i \to \partial_i$.

- Expand around background: $Y_i = \hat{\xi}_i + \hat{A}_i$ and $X_{ij} = 0 + \hat{B}_{ij}$:

$$\begin{aligned}\hat{\mathcal{F}}_{ij} &= \mu_2(\hat{\xi}_i, \hat{A}_j) - \mu_2(\hat{\xi}_j, \hat{A}_i) + \mu_2(\hat{A}_i, \hat{A}_j) - \mu_1(\hat{B}_{ij}) \ ,\\ \hat{H}_{ijk} &= \tfrac{1}{2}\mu_2(\hat{\xi}_{[i}, \hat{B}_{jk]}) + \tfrac{1}{2}\mu_2(\hat{A}_{[i}, \hat{B}_{jk]}) + \tfrac{1}{6}\mu_3(\hat{\xi}_i + \hat{A}_i, \hat{\xi}_j + \hat{A}_j, \hat{\xi}_k + \hat{A}_k) - \mathrm{i}\hbar\hat{\mathbb{1}}\end{aligned}$$

- This is BF-theory on quantum $\mathbb{R}^3$.

12. Summary

- Started a study of 0d Lie 2-algebra models
- Structure easier than higher gauge theory
- Recovered essentially everything known classically about the IKKT model
- Models comprise IKKT-model and reductions of M2-brane models
- Quantization via one-forms reasonable

Future directions

- Supersymmetry
- Schwinger-Dyson equations for Wilson loops and Lie 2-algebras
- Stefano's results: BFSS vs Lie 2-algebra models
- Extend quantization of 2-plectic manifolds