M2-brane Models

Christian Sämann

School of Mathematical and Computer Sciences
Heriot-Watt University, Edinburgh

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Based on:
- Basu-Harvey
- Bagger-Lambert, Gustavsson, ABJM
- my own work with various collaborators

Motivation

Find an effective description for M2-branes.

Lightning review of branes:

- D-branes appear in string theory as objects that open strings can end on. They correspond to BPS solutions in supergravity.
- IIA: D0, D2, D4, D6, D8, IIB: D(-1), D1, D3, D5, D7, D9
- Dp-brane: spatially p-dimensional object.
- Turn off gravity: we obtain a supersymmetric gauge theory.
- D-branes stacked together increases rank of gauge group.
- They can intersect and sometimes end on each other.
- Two different perspectives of the same configuration: duality.
- IIA string theory/IIA SUGRA: limit of M theory/11d SUGRA.
- In 11d, BPS solutions are M2- and M5-branes.

Question: What is the effective description for M2-branes?

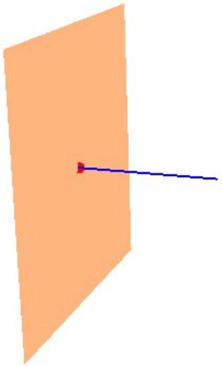
Outline

- BPS configurations: The Basu-Harvey equation
- A new gauge structure: 3-Lie algebras
- $\mathcal{N} = 8$ supersymmetry: The BLG Model
- Arbitrarily many M2-branes: The ABJM Model
- Test: Superconformality
- Noncommutativity from M2-brane models
- Relations to M5-brane models
- Outlook

D1-D3-Branes and the Nahm Equation

D1-branes ending on D3-branes can be described by the Nahm equation.

dim	0	1	2	3	...	6
$D1$	$\times$					$\times$
$D3$	$\times$	$\times$	$\times$	$\times$		



k D1-branes ending on D3-branes:

A Monopole appears.

$X^i \in \mathfrak{u}(k)$: transverse fluctuations

Nahm equation: $(s = x^6)$

$$\frac{\mathrm{d}}{\mathrm{d}s} X^i + \varepsilon^{ijk}[X^j, X^k] = 0$$

Note $\mathsf{SO}(3)$-invariance.

Solution: $X^i = r(s) G^i$ with

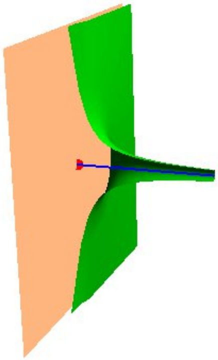
$$r(s) = \frac{1}{s}\ , \quad G^i = \varepsilon^{ijk}[G^j, G^k]$$

Nahm, Diaconescu, Tsimpis

D1-D3-Branes and the Nahm Equation

The D1-branes end on the D3-branes by forming a fuzzy funnel.

dim	0	1	2	3	...	6
$D1$	$\times$					$\times$
$D3$	$\times$	$\times$	$\times$	$\times$		



Solution: $X^i = r(s) G^i$

$$r(s) = \frac{1}{s} \; , \quad G^i = \varepsilon^{ijk}[G^j, G^k]$$

The D1-branes form a fuzzy funnel:

G^i form irrep of $\mathfrak{su}(2)$:
coordinates on fuzzy sphere S^2_F

D1-worldvolume polarizes: $2d \to 4d$

Myers

Lifting D1-D3-Branes to M2-M5-Branes

The lift to M-theory is performed by a T-duality and an M-theory lift

IIB	0	1	2	3	4	5	6
$D1$	$\times$						$\times$
$D3$	$\times$	$\times$	$\times$	$\times$			

T-dualize along x^5:

IIA	0	1	2	3	4	5	6
$D2$	$\times$					$\times$	$\times$
$D4$	$\times$	$\times$	$\times$	$\times$		$\times$	

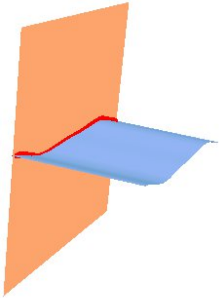
Interpret x^4 as M-theory direction:

M	0	1	2	3	4	5	6
$M2$	$\times$					$\times$	$\times$
$M5$	$\times$	$\times$	$\times$	$\times$	$\times$	$\times$	

The Basu-Harvey Lift of the Nahm Equation

M2-branes ending on M5-branes yield a Nahm equation with a cubic term.

M	0	1	2	3	4	5	6
$M2$	×					×	×
$M5$	×	×	×	×	×	×	



Basu, Harvey, hep-th/0412310

A Self-Dual String appears.

Substitute SO(3)-inv. Nahm eqn.

$$\frac{\mathrm{d}}{\mathrm{d}s}X^i + \varepsilon^{ijk}[X^j, X^k] = 0$$

by the SO(4)-invariant equation

$$\frac{\mathrm{d}}{\mathrm{d}s}X^\mu + \varepsilon^{\mu\nu\rho\sigma}[X^\nu, X^\rho, X^\sigma] = 0$$

Solution: $X^\mu = r(s)G^\mu$ with

$$r(s) = \frac{1}{\sqrt{s}}\ , \quad G^\mu = \varepsilon^{\mu\nu\rho\sigma}[G^\nu, G^\rho, G^\sigma]$$

The Basu-Harvey Lift of the Nahm Equation

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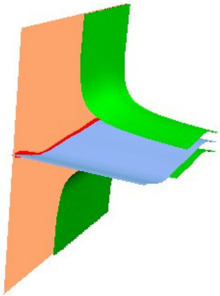
$M2$	$\times$					$\times$	$\times$
$M5$	$\times$	$\times$	$\times$	$\times$	$\times$	$\times$	



Solution: $X^\mu = r(s)G^\mu$

$$r(s) = \frac{1}{\sqrt{s}}\ ,\ \ G^\mu = \varepsilon^{\mu\nu\rho\sigma}[G^\nu, G^\rho, G^\sigma]$$

The M2-branes form a fuzzy funnel:

G^μ form a rep of $\mathfrak{so}(4)$:
coordinates on fuzzy sphere S^3_F

M2-worldvolume polarizes: $3d \to 6d$

- What is this triple bracket?
- What is a quantized S^3?

What is the algebra behind the triple bracket?

In analogy with Lie algebras, we can introduce 3-Lie algebras.

BH: $$\frac{\mathrm{d}}{\mathrm{d}s}X^\mu + [A_s, X^\mu] + \varepsilon^{\mu\nu\rho\sigma}[X^\nu, X^\rho, X^\sigma] = 0\ , \quad X^\mu \in \mathcal{A}$$

3-Lie algebra

Obviously: $\mathcal{A}$ is a vector space, $[\cdot,\cdot,\cdot]$ trilinear+antisymmetric.

Demand a "3-Jacobi identity," the fundamental identity:

$$\begin{aligned}[A, B, [C, D, E]] =& [[A, B, C], D, E] + [C, [A, B, D], E] \\ &+ [C, D, [A, B, E]]\end{aligned}$$

Filippov (1985)

Gauge transformations from Lie algebra of inner derivations:

$$D : \mathcal{A} \wedge \mathcal{A} \to \mathrm{Der}(\mathcal{A}) =: \mathfrak{g}_\mathcal{A} \quad D(A, B) \rhd C := [A, B, C]$$

Commutator of inner dervs. closes due to fundamental identity.

What is the algebra behind the triple bracket?

In analogy with Lie algebras, we can introduce 3-Lie algebras.

To write down an action, i.e. gauge invariant terms,
we need an invariant pairing on $\mathcal{A}$:

$$(\cdot,\cdot) : \mathcal{A} \otimes \mathcal{A} \to \mathbb{C}$$

Invariance under gauge transformations:

$$([A,B,C],D) + (C,[A,B,D]) = 0$$

On $\mathrm{Der}(\mathcal{A})$, there are now two pairings $(\!(\cdot,\cdot)\!)$:

1. The usual Killing form
2. A pairing induced from the pairing on $\mathcal{A}$:

$$(\!(D(A,B),D(C,D))\!) = (D,[A,B,C])$$

Key to constructing a maximally SUSY model later: use the latter.

Examples of 3-Lie algebras
In analogy with Lie algebras, we can introduce 3-Lie algebras.

Examples:

Lie algebra	3-Lie algebra
Heisenberg-algebra: $[\tau_a, \tau_b] = \varepsilon_{ab}\mathbb{1}, \quad [\mathbb{1}, \cdot] = 0$	Nambu-Heisenberg 3-Lie Algebra: $[\tau_i, \tau_j, \tau_k] = \varepsilon_{ijk}\mathbb{1}, \quad [\mathbb{1}, \cdot, \cdot] = 0$
$\mathfrak{su}(2) \simeq \mathbb{R}^3$: $[\tau_i, \tau_j] = \varepsilon_{ijk}\tau_k$	$A_4 \simeq \mathbb{R}^4$: $[\tau_\mu, \tau_\nu, \tau_\kappa] = \varepsilon_{\mu\nu\kappa\lambda}\tau_\lambda$

Focus on A_4

- The associated Lie algebra is $\mathfrak{g}_{A_4} = \mathfrak{so}(4) \cong \mathfrak{su}(2) \times \mathfrak{su}(2)$.
- Its bilinear pairing $(\!(\cdot, \cdot)\!)$ has split signature:

$$(\!(D(\tau_a, \tau_b), D(\tau_c, \tau_d))\!) = \varepsilon_{abcd}$$

Approaching the Effective Description of M2-Branes

Spacetime symmetries and BPS equations give helpful constraints on the description.

A stack of flat M2-branes in $\mathbb{R}^{1,10}$ should be effectively described by a conformal field theory with the following constraints:

Spacetime symmetries: $\mathsf{SO}(1,10) \to \mathsf{SO}(1,2) \times \mathsf{SO}(8)$
extended by $\mathcal{N} = 8$ SUSY.

Field content: X^I, $I = 1, \ldots, 8$, and superpartners Ψ_α

Assumption

Take BPS/SUSY transformations from Basu-Harvey equation and therefore the matter fields take values in a metric 3-Lie algebra.

$$\delta X = \mathrm{i}\bar{\varepsilon}\Gamma^I\Psi \qquad \delta\Psi = \partial_\mu X^I \Gamma_I \Gamma^\mu \varepsilon - \tfrac{1}{6}\Gamma_{IJK}[X^I, X^J, X^K]\varepsilon$$

Recipe: Try to close SUSY algebra. Constraints yield equations of motion for matter fields.

The Bagger-Lambert-Gustavsson Model

This model is an unconventional supersymmetric Chern-Simons matter theory.

BLG found that for SUSY, we need to introduce gauge symmetry.
$\Rightarrow$ Field content: $X^I \in \mathcal{A}$, $\Psi \in \mathcal{A}$ and gauge potential $A_\mu \in \mathfrak{g}_\mathcal{A}$.

The Bagger-Lambert-Gustavsson model

$$\begin{aligned}\mathcal{L}_{\rm BLG} \;=\; & +\tfrac{k}{4\pi}\varepsilon^{\mu\nu\kappa}\left(\langle\!\langle A_\mu, \partial_\nu A_\kappa\rangle\!\rangle + \tfrac{1}{3}\langle\!\langle A_\mu, [A_\nu, A_\kappa]\rangle\!\rangle\right)\\ & - \tfrac{1}{2}(\nabla_\mu X^I, \nabla^\mu X^I)_{Cl} + \tfrac{\mathrm{i}}{2}(\bar{\Psi}, \Gamma^\mu \nabla_\mu \Psi)\\ & + \tfrac{\mathrm{i}}{4}(\bar{\Psi}, \Gamma_{IJ}[X^I, X^J, \Psi]) - \tfrac{1}{12}([X^I, X^J, X^K], [X^I, X^J, X^K])\end{aligned}$$

This model is invariant under the supersymmetry transformations:

$$\delta X = \mathrm{i}\bar{\varepsilon}\Gamma^I\Psi\;, \qquad \delta\Psi \;=\; \nabla_\mu X^I \Gamma_I \Gamma^\mu \varepsilon - \tfrac{1}{6}\Gamma_{IJK}[X^I, X^J, X^K]\varepsilon\;,$$

$$\delta A_\mu = \mathrm{i}\bar{\varepsilon}\Gamma_\mu\Gamma_I(D(X^I, \Psi))$$

Consistency checks

The BLG model passes a number of consistency checks.

$$\mathcal{L}_{\rm BLG} = +\tfrac{k}{4\pi}\varepsilon^{\mu\nu\kappa}\left(\langle\!\langle A_\mu, \partial_\nu A_\kappa\rangle\!\rangle + \tfrac{1}{3}\langle\!\langle A_\mu, [A_\nu, A_\kappa]\rangle\!\rangle\right)$$
$$- \tfrac{1}{2}(\nabla_\mu X^I, \nabla^\mu X^I)_{Cl} + \tfrac{\mathrm{i}}{2}(\bar\Psi, \Gamma^\mu\nabla_\mu\Psi)$$
$$+ \tfrac{\mathrm{i}}{4}(\bar\Psi, \Gamma_{IJ}[X^I, X^J, \Psi]) - \tfrac{1}{12}([X^I, X^J, X^K], [X^I, X^J, X^K])$$

Further results:

- The model is classically conformal and seems rather unique.
- If $\mathcal{N}=8$ SUSY not anomalous $\Rightarrow$ vanishing β-function
- The model is parity invariant.
- Under some assumptions: reduction mechanism M2→D2.

(Mukhi, Papageorgakis,0803.3218)

- $k=2$: moduli space matches 2 M2-branes at tip of $\mathbb{R}^8/\mathbb{Z}_2$.

Problem: The only 3-Lie algebra with pos. def. metric is A_4

Real and Hermitian 3-Algebras

There are two natural generalizations of 3-Lie algebras.

Way out: sacrifice (manifest) SUSY

Real 3-Algebras ($\mathcal{N}=2$)

Almost the same as 3-Lie algebras: triple bracket only antisymmetric in first two slots. S. Cherkis, CS, 0807.0808

Hermitian 3-Algebras ($\mathcal{N}=6$)

Start from a complex vector space $\mathcal{A}$. Bracket $[\,\cdot\,,\,\cdot\,;\,\cdot\,]$ satisfies

$$[A,B;C]=-[B,A;C],\ \ [\alpha A,B;C]:=\alpha[A,B;C],\ \ [A,B;\alpha C]:=\alpha^*[A,B;C]$$
$$[[C,D;E],A;B]-[[C,A;B],D;E]-[C,[D,A;B];E]+[C,D;[E,B;A]]=0$$

Bagger, Lambert, 0807.0163

Representation: $[A,B;C]:=AC^\dagger B-BC^\dagger A$
Aharony, Bergman, Jafferis, Maldacena, 0806.1218

All M2-brane constructions usually generalize to these two types.

ABJM action

- Re-arrange 8 real into 4 complex scalars: $\mathsf{SO}(8) \to \mathsf{SU}(4)$.
- Action:

$$S = \int \mathrm{d}^3x \, \mathrm{tr} \Big[- \nabla_\mu \bar{\phi}_A \nabla^\mu \phi^A - \mathrm{i}\bar{\psi}^A \gamma^\mu \nabla_\mu \psi_A$$
$$\frac{k}{4\pi} \varepsilon^{\mu\nu\lambda} \Big(A^R_\mu \partial_\nu A^R_\lambda + \tfrac{2}{3} A^R_\mu A^R_\nu A^R_\lambda - A^L_\mu \partial_\nu A^L_\lambda - \tfrac{2}{3} A^L_\mu A^L_\nu A^L_\lambda \Big)$$
$$+ \frac{4\pi^2}{3k^2} \Big(\phi^A \bar{\phi}_A \phi^B \bar{\phi}_B \phi^C \bar{\phi}_C + \bar{\phi}_A \phi^A \bar{\phi}_B \phi^B \bar{\phi}_C \phi^C$$
$$+ 4\phi^A \bar{\phi}_B \phi^C \bar{\phi}_A \phi^B \bar{\phi}_C - 6\phi^A \bar{\phi}_B \phi^B \bar{\phi}_A \phi^C \bar{\phi}_C \Big) + V_{ferm} \Big] .$$

- Model can be engineered in string theory.
- This model reproduces $N^{3/2}$-scaling.

  Drukker, Marino, Putrov, 1007.3837.
- Has an integrable spin chain. Minahan, Zarmbo, 0806.3951.

Recovering SYM Features: Marginal Deformations

The BLG model shares features with $\mathcal{N}=4$ SYM. What about marginal deformations?

Observation: BLG/ABJM seems similar to $\mathcal{N}=4$ SYM
($\rightarrow$ integrable spin chains).

$\mathcal{N}=4$ SYM admits (exactly) marginal deformations:

$$\begin{aligned}\mathcal{W} &= \varepsilon_{ijk}\,\mathrm{tr}\left([\Phi^i,\Phi^j]_\beta \Phi^k\right)\\ [\Phi^i,\Phi^j]_\beta &:= \mathrm{e}^{\mathrm{i}\beta}\Phi^i\Phi^j - \mathrm{e}^{-\mathrm{i}\beta}\Phi^j\Phi^i\end{aligned}$$

R. G. Leigh and M.J. Strassler, Nucl. Phys. B 447 (1995).

Conformality for β-deformed SYM to all orders in perturbation theory: S. Ananth, S. Kovacs, H. Shimada, JHEP 01 (2007) 046.

Such deformations correspond to deformations of $AdS_5 \times S^5$.

Similar deformations for $AdS_4 \times S^7$ in the literature.
What about BLG/ABJM and their deformations on quantum level?

Manifestly $\mathcal{N}=2$ SUSY Formulation

There is a manifestly $\mathcal{N}=2$ SUSY formulation, allowing for various deformations.

Approach: Take $\mathcal{N}=1$, 4d superspace $\mathbb{R}^{1,3|4}$ and reduce to 3d.

Field content of the theory:

- The matter fields X^I, Ψ are encoded in four chiral multiplets:

$$\Phi^i(y) = \phi^i(y) + \sqrt{2}\theta\psi^i(y) + \theta^2 F^i(y) \ ,$$

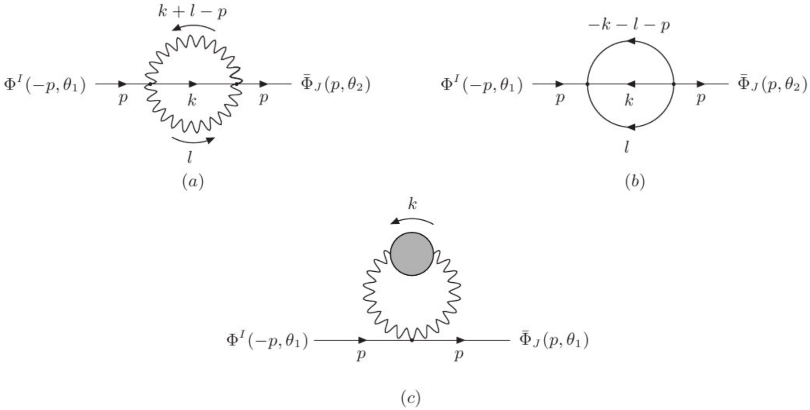
- The gauge potential A_μ is contained in a vector superfield:

$$\begin{aligned} V(x) \ &= \ -\theta^\alpha\bar\theta^{\dot\alpha}(\sigma^\mu_{\alpha\dot\alpha}A_\mu(x) + \mathrm{i}\varepsilon_{\alpha\dot\alpha}\sigma(x)) \\ &\qquad + \mathrm{i}\theta^2(\bar\theta\bar\lambda(x)) - \mathrm{i}\bar\theta^2(\theta\lambda(x)) + \tfrac{1}{2}\theta^2\bar\theta^2 D(x) \ , \end{aligned}$$

$\mathcal{N}=2$ superspace formulation of BLG (Cherkis, CS, 0807.0808)

$$\begin{aligned} \mathcal{L} = \int \mathrm{d}^4\theta \ \kappa \left(\mathrm{i}(\!(V,(\bar D_\alpha D^\alpha V))\!) + \tfrac{2}{3}(\!(V,\{(\bar D^\alpha V),(D_\alpha V)\})\!)\right) \\ + (\bar\Phi_i, \mathrm{e}^{2iV}\cdot\Phi^i) + \alpha\left(\int \mathrm{d}^2\theta \ \varepsilon_{ijkl}([\Phi^i,\Phi^j,\Phi^k],\Phi^l) + c.c.\right) \end{aligned}$$

Contributing diagrams (only 2-pt contributions are divergent):



Potential flow of the couplings due to anomalous dimensions.

Results: The β-function for multitrace deformations

The BLG model is conformally invariant at two loops.

Example of a deformation:

$$\mathcal{W} = \Big[R^{(1)}_{ijkl}(\Phi^l,[\Phi^i,\Phi^j,\Phi^k]) + R^{(2)}_{ijkl}(\Phi^i,\Phi^j)(\Phi^k,\Phi^l)\Big]$$

Total anomalous dimension:

$$\begin{aligned}\gamma_i{}^j \;=\; \frac{1}{8\pi^2\kappa^2}\Big\{&\big[k_2 + k_1^2 + \tfrac{1}{12}(2k_2 + N_f k_3)\big]\delta_i{}^j \\ &+\; 8\kappa^2\Big[R^{(1)}_{iklm}\big(-c_3 R^{jklm}_{(1)} + 2c_2 R^{jmlk}_{(1)} + 2c_1 R^{jmlk}_{(2)}\big) \\ &\qquad + R^{(2)}_{iklm}\big(d\,R^{jklm}_{(2)} + 2R^{jmlk}_{(2)} + 2c_1 R^{jmlk}_{(1)}\big)\Big]\Big\}\end{aligned}$$

Quick test: BLG. $R^{(2)}_{ijkl} = 0$, $\mathcal{A} = A_4$, therefore $R^{(1)}_{ijkl} = \lambda\varepsilon_{ijkl}$ and

$$d = 4 \quad k_1 = 0 \quad k_2 = -3 \quad k_3 = 6 \quad c_1 = 0 \quad c_2 = c_3 = -6$$

The β-function reads as (the phase does not flow)

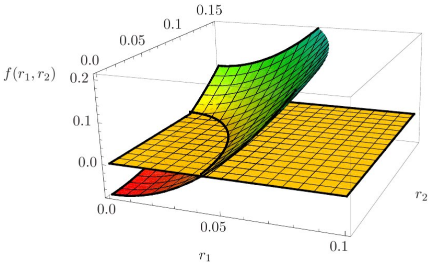
$$\beta^{(1)}_{ijkl} = -\tfrac{3}{4\pi^2\kappa^2}\big[1 - (4!\kappa)^2|\lambda|^2\big]R^{(1)}_{ijkl} \quad \text{so} \quad |\lambda| = \frac{1}{4!\kappa}$$

For simplicity, we take $\mathcal{A} = A_4$ and the superpotential

$$R^{(1)}_{ijkl} = \frac{\lambda_1}{\kappa}\varepsilon_{ijkl} \quad \text{and} \quad R^{(2)}_{ijkl} = \frac{\lambda_2}{\kappa}\delta_{ij}\delta_{kl}~, \qquad \lambda_i = r_i \mathrm{e}^{\varphi_i}$$

The β-function at two loops reads as (phases do not flow)

$$\beta^{(\ell)}_{ijkl} = \frac{f(r_1,r_2)}{\kappa^2} R^{(\ell)}_{ijkl} \qquad f(r_1,r_2) := -\frac{3}{4\pi^2}\left[1 - 96\left(6r_1^2 + r_2^2\right)\right]$$



BLG: $r_1 = \frac{1}{24}, r_2 = 0$

points on ellipse:
IR fixed points

Recover β-deformations
Akerblom&CS&Wolf 0906.1705

We have a paper factory:
Take your favourite phenomenon in $\mathcal{N}=4$ Super Yang-Mills and translate it to the ABJM/BLG models.

Motivation:

- Fuzzy S^3-funnel appearing in M2-M5-configurations.
- M5-brane perspective: Turning on 3-form background,

$$C = \theta \mathrm{d}x^0 \wedge \mathrm{d}x^1 \wedge \mathrm{d}x^2 + \theta' \mathrm{d}x^0 \wedge \mathrm{d}x^1 \wedge \mathrm{d}x^2 \ ,$$

  one gets interesting noncommutative deformations:
  - Noncommutative loop space
    Kawamoto, Sasakura and Bergshoeff et al. (2000)
  - $[x^0, x^1, x^2] = \theta$ and $[x^3, x^4, x^5] = \theta'$ Chu, Smith (2009)
- Non-associative structures from strings in H-field backgrounds
  Blumenhagen, Deser, Lüst, Plauschinn, Rennecke (2010/11)
- Baez et al.: Phase space of bosonic string is 2-plectic

2-plectic Manifolds

Certain 2-plectic manifolds naturally come with a prequantum gerbe.

Symplectic manifold (M, ω) with $\omega \in H^2(M, \mathbb{Z})$:
$\Rightarrow$ Prequantum line bundle with connection ∇, $F_\nabla = 2\pi\mathrm{i}\omega$.

2-plectic manifold (M, ϖ) with $\varpi \in H^3(M, \mathbb{Z})$:
$\Rightarrow$ Prequant. abelian gerbe with connect. struct. incl. $H = 2\pi\mathrm{i}\varpi$.

- First idea: Categorify Hawkins' approach (2-groupoids, etc.)
  (work in progress, cf. Freed, Baez, Rogers ...)
- Second idea: Transgression gives again symplectic manifolds.

The Symplectic Loop Space of a 2-plectic Manifold.

A 2-plectic manifold has a symplectic structure on its loop space.

Consider the following double fibration:

$$\begin{array}{ccc} & \mathcal{L}M \times S^1 & \\ {}^{ev}\swarrow & & \searrow^{pr} \\ M & & \mathcal{L}M \end{array}$$

Transgression

$$\mathcal{T} : H^{k+1}(M) \to H^k(\mathcal{L}M)~, \quad \mathcal{T} = pr! \circ ev^*$$

$$(\mathcal{T}\omega)_x(v_1(\tau), \dots, v_k(\tau)) := \int_{S^1} \mathrm{d}\tau\, \omega(v_1(\tau), \dots, v_k(\tau), \dot{x}(\tau))$$

- Transgression is a chain map.
- Maps 2-plectic structures to symplectic structures.
- Maps abelian gerbes to line bundles.
- Previously successfully applied: Lift ADHMN construction to M-theory. CS, Papageorgakis&CS, Palmer&CS

Explicitly, this works as follows:

We start from $\mathbb{R}^3$ with 2-plectic form $\varpi = \varepsilon_{ijk}\mathrm{d}x^i \wedge \mathrm{d}x^j \wedge \mathrm{d}x^k$.

Transgression yields a symplectic form on loop space $\mathcal{L}\mathbb{R}^3$:

$$\omega = \oint \mathrm{d}\tau \oint \mathrm{d}\sigma\ \varepsilon_{ijk}\dot{x}^k(\tau)\delta(\tau-\sigma)\ \delta x^i(\tau) \wedge \delta x^j(\sigma)$$

Kernel of ω:

$$\iota_X(\mathcal{T}\varpi) = 0 \quad \text{for} \quad X = \oint \mathrm{d}\rho\ \dot{x}^i(\rho)\,\frac{\delta}{\delta x^i(\rho)}$$

This vector field generates reparameterizations of the loops in $\mathcal{L}\mathbb{R}^3$.

We can therefore invert ω and obtain the Poisson bracket

$$\{f,g\} := \oint \mathrm{d}\tau \ \oint \mathrm{d}\rho\,\delta(\tau-\rho)\,\theta^{ijk}\,\frac{\dot{x}_k(\rho)}{|\dot{x}(\rho)|^2}\,\left(\frac{\delta}{\delta x^i(\tau)}f\right)\,\left(\frac{\delta}{\delta x^j(\rho)}g\right)$$

Noncommutative loop space

We recover a previously found result on noncommutative loop spaces.

This leads to the following noncommutativity on loop space:

$$[x^i(\tau), x^j(\sigma)] = \theta^{ijk} \frac{\dot{x}_k(\tau)}{|\dot{x}(\tau)|^2} \delta(\tau - \sigma) + \mathcal{O}(\theta^2)$$

CS&Szabo, 1211.0395

Note:

- This result agrees with that of Kawamoto, Sasakura and Bergshoeff et al. (2000)
- It is also compatible with one-form quantization of Baez et al.

What are these 3-Lie algebras really?

3-Lie algebras can be regarded as special cases of gauge algebras of non-abelian gerbes.

- The machinery of 3-Lie algebras seems slightly awkward.
- Just switch to matrices as in ABJM?
- Strong homotopy algebras might be a guess...
  C. I. Lazaroiu, D. McNamee, CS and A. Zejak, 0901.3905
- Nahm-Transform/Integrability ($\rightarrow$ my talk on Friday):
  M2-branes and M5-branes have similar gauge structures.

Best guess for M5-brane models:

use non-abelian gerbes/categorified principal bundles

Categorifying Gauge Groups

A Lie 2-group is a Lie groupoid with extra structure.

Warning: Categorification neither unique nor straightforward.

Lie 2-group

A Lie 2-group is a

- monoidal category, morph. invertible, obj. weakly invertible.
- Lie groupoid + product $\otimes$ obeying weakly the group axioms.

Simplification: use strict Lie 2-groups $\xleftrightarrow{1:1}$ Lie crossed modules

Lie crossed modules

Pair of Lie groups (G, H), written as $(\mathsf{H} \xrightarrow{\mathsf{t}} \mathsf{G})$ with:

- left automorphism action $\rhd : \mathsf{G} \times \mathsf{H} \to \mathsf{H}$
- group homomorphism $\mathsf{t} : \mathsf{H} \to \mathsf{G}$ such that

$$\mathsf{t}(g \rhd h) = g\mathsf{t}(h)g^{-1} \quad \text{and} \quad \mathsf{t}(h_1) \rhd h_2 = h_1 h_2 h_1^{-1}$$

Also: strict Lie 2-algebras $\xleftrightarrow{1:1}$ differential crossed modules

Examples of Lie Crossed Modules

Lie crossed modules come in a large variety.

Lie crossed modules

Pair of Lie groups (G, H), written as $(\mathsf{H} \xrightarrow{\mathsf{t}} \mathsf{G})$ with:

- left automorphism action $\rhd \colon \mathsf{G} \times \mathsf{H} \to \mathsf{H}$
- group homomorphism $\mathsf{t} : \mathsf{H} \to \mathsf{G}$

$$\mathsf{t}(g \rhd h) = g\mathsf{t}(h)g^{-1} \quad \text{and} \quad \mathsf{t}(h_1) \rhd h_2 = h_1 h_2 h_1^{-1}$$

Simplest examples:

- Lie group G, Lie crossed module: $(1 \xrightarrow{\mathsf{t}} G)$.
- Abelian Lie group G, Lie crossed module: $BG = (G \xrightarrow{\mathsf{t}} 1)$.

More involved:

- Automorphism 2-group of Lie group G: $(G \xrightarrow{\mathsf{t}} \mathsf{Aut}(G))$

Differential Crossed Modules from 3-Algebras

3-algebras are merely special classes of differential crossed modules.

Recall the definition of a 3-algebra $\mathcal{A}$:

- $[\cdot,\cdot,\cdot] : \mathcal{A}^{\otimes 3} \to \mathcal{A}$
- Fundamental identity says that $[a, b, \cdot] \in \mathsf{Der}(\mathcal{A})$, $a, b \in \mathcal{A}$.

Theorem

$$\text{3-algebras} \overset{1:1}{\longleftrightarrow} \begin{array}{l} \text{metric Lie algebras } \mathfrak{g} \cong \mathsf{Der}(\mathcal{A}) \\ \text{faithful orthog. representations } V \cong \mathcal{A} \end{array}$$

J Figueroa-O'Farrill et al., 0809.1086

Observations

- $V \xrightarrow{\mathsf{t}} \mathfrak{g}$ is a simple differential crossed modules
- M2- and M5-brane models have the same gauge structure.
- Via Faulkner construction, all DCMs come with $[\cdot,\cdot,\cdot]$
- Application of this to M2- and M5-models looks promising.

S Palmer & CS, 1203.5757

Status of M5-brane models

Progress is even being made in constructing M5-brane models.

Although the situation for M5-branes was assumed to be more hopeless than that for M2-branes, progress is being made:

- Lambert, Papageorgakis, 1007.2982: Non-abelian tensor field equations based on 3-Lie algebras.
- Chu, 1108.5131: Non-abelian tensor gauge fields, no supersymmetry, non-local fields.
- Samtleben, Sezgin, Wimmer, 1108.4060: from tensor hierarchies $\mathcal{N} = (1, 0)$ supersymmetry, no reduction to super Yang-Mills theory.
- CS, Wolf, 1205.3108, ...: Manifestly $\mathcal{N} = (2, 0)$ superconf. field equations from twistor space. ($\rightarrow$ my talk on Friday)
- ... and many more!

Summary:

- ✓ M2-brane models exist and are interesting.
- ✓ Models pass many consistency checks
- ✓ Models are very similar to $\mathcal{N}=4$ super Yang-Mills theory
- ✓ Quantum geometries from loop spaces.
- ✓ Arising gauge structures suggests to use categorification.
- ✓ Construction of M5-brane models on its way.
- ✓ A better understanding of M-theory around the corner?

M2-brane Models

Christian Sämann

School of Mathematical and Computer Sciences
Heriot-Watt University, Edinburgh

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