

Towards an M5-brane model I: A 6d superconformal field theory

Christian Sämann



*School of Mathematical and Computer Sciences
Heriot-Watt University, Edinburgh*

Workshop on Higher Gauge Theory, Leeds, 1.3.2018

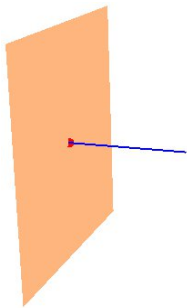
Based on:

- CS & L Schmidt, [arXiv:1705.02353](https://arxiv.org/abs/1705.02353)
- CS & L Schmidt, [arXiv:1712.06623](https://arxiv.org/abs/1712.06623)

Motivation: Dynamics of multiple M5-branes

2/35

To understand M-theory, an effective description of M5-branes would be very useful.

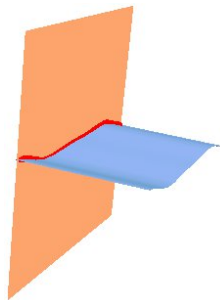


D-branes

- D-branes **interact** via strings.
- Effective description: theory of **endpoints**
- Parallel transport of these: **Gauge theory**
- Study string theory **via gauge theory**

M5-branes

- M5-branes **interact** via M2-branes.
- Eff. description: theory of **self-dual strings**
- Parallel transport: **Higher gauge theory**
- Long sought $(2, 0)$ -theory a **HGT?**



Outline

- The (2,0)-Theory: What we know and want
- Arguments against existence of classical M5-brane model
- Higher Gauge Theory: Lightning review
- Guidance from BPS self-dual strings
- The 6d superconformal field theory
- Consistency checks
- Problems and potential solutions

The (2,0)-Theory: What we know and want

Pre-history:

- **Conformal QFTs**: particularly interesting and important
- Conformal algebra on $\mathbb{R}^{p,q}$: $\mathfrak{so}(p+1, q+1)$
- Supersymmetric extensions only for $p+q \leq 6$ **Nahm, 1978**
- Examples for $p+q \leq 4$ known for long time
- Belief: $p+q = 4$ **maximum** for interacting QFTs

String theory:

Witten, 1995

- **Type IIB** superstring theory on $\mathbb{R}^{1,5} \times K_3$
- Moduli space has orbifold singularities of **ADE-type**
- At singularities: volume of $S^2 \hookrightarrow K_3$ vanishes
- D3-branes wrapping $S^2 \hookrightarrow K_3$ become massless strings
- B -field self-dual: **self-dual strings**, **SUGRA decouples**
- \Rightarrow **$(2,0)$ -theory**, a six-dimensional $\mathcal{N} = (2,0)$ SCFT

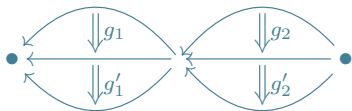
More on the (2,0)-theory

- Also appears in M-theory **Witten, Strominger 1995/1996**
 - **self-dual strings**: boundaries of M2- between M5-branes
 - become **massless**, if M5-branes approach each other
 - description of **stacks of parallel M5-branes**
- Field content: $\mathcal{N} = (2, 0)$ **tensor multiplet** **Nahm 1978**
 - a **self-dual 3-form field strength**
 - five (Goldstone) **scalars**
 - **fermionic partners**
- Observables: **Wilson surfaces**, i.e. parallel transport of strings
- Belief: **No Lagrangian description**
- As important as $\mathcal{N} = 4$ **super Yang-Mills** for string theory
- Huge interest in string theory: **AGT, AdS₇-CFT₆, S-duality, ...**
- Mathematics: **Geom. Langlands, Khovanov Homology, ...**

- A **successful M5-brane model** should have the following properties:
- Contain an **interacting**, self-dual 2-form gauge potential
 - Based on a **sound mathematical foundation**: **higher bundles**
 - **Field content** of the $(2,0)$ -theory, $\mathcal{N} = (1,0)$ supersymmetric
 - **Gauge structure** natural, match some **expectations** (ADE, ...)
 - Non-trivial coupling, **interacting field theory**
 - Possible restriction to **free $\mathcal{N} = (2,0)$ tensor multiplet**
 - contains the **non-abelian self-dual string soliton** as BPS state
 - **Reduction to 4d SYM theory with ADE gauge algebras**
 - and to **3d Chern–Simons-matter models** with discrete coupling
 - match expected **moduli space** of $(2,0)$ -theory

Arguments **against existence** of classical M5-brane model

Non-abelian parallel transport of strings problematic:



Consistency of parallel transport requires:

$$(g'_1 g'_2)(g_1 g_2) = (g'_1 g_1)(g'_2 g_2)$$

This renders group G abelian.

Eckmann and Hilton, 1962
Physicists 80'ies and 90'ies

Way out: 2-categories, Higher Gauge Theory.

Two operations \circ and \otimes satisfying Interchange Law:

$$(g'_1 \otimes g'_2) \circ (g_1 \otimes g_2) = (g'_1 \circ g_1) \otimes (g'_2 \circ g_2) .$$

Standard **objection** beyond the previous no-go theorem:

- theory at conformal fixed points \Rightarrow **no dimensionful parameter**
- fixed points are isolated \Rightarrow **no dimensionless parameter**
- “**No parameters** \Rightarrow **no classical limit** \Rightarrow **no Lagrangian.**”
string theory folklore
- Furthermore: **no continuous deformations** of free theory
Bekaert, Henneaux, Sevrin (1999)

Answers:

- Same arguments for **M2-brane** Schwarz, 2004
- There, integer parameters arose from **orbifold** $\mathbb{R}^8/\mathbb{Z}_k$
- **Same should happen for M5-branes**

Note:

- **Analogy with M2-brane models:** only $\mathcal{N} = (1, 0)$ local SUSY

Final common objection: Dimensional reduction is unclear.

- (2,0)-theory should reduce to $\mathcal{N} = 2$ SYM theory in 5d
- Reduction on $\mathbb{R}^{1,4} \times S^1$, radius R yields volume form $2\pi R d^5x$
- Conformal invariance of $F \wedge *F$ requires volume form $\frac{1}{R} d^5x$

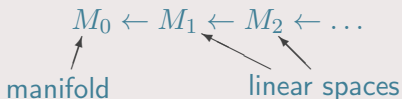
Our solution:

- Reduction to $\mathcal{N} = 2$ SYM in 4d works fine
- Can dimensionally oxidize to 5d SYM afterwards (?)

Higher Gauge Theory: Lightning review

N-manifolds, NQ-manifold

- \mathbb{N}_0 -graded manifold with coordinates of degree $0, 1, 2, \dots$



- **NQ-manifold**: vector field Q of degree 1, $Q^2 = 0$
- **Physicists**: think ghost numbers, BRST charge, SFT
- Functions on (M, Q) form **differential graded algebra**

Examples:

- **Tangent algebroid** $T[1]M$, $C^\infty(T[1]M) \cong \Omega^\bullet(M)$, $Q = d$
- **Lie algebra** $\mathfrak{g}[1]$, coordinates ξ^α of degree 1:

$$Q = -\frac{1}{2} f_{\beta\gamma}^\alpha \xi^\beta \xi^\gamma \frac{\partial}{\partial \xi^\alpha} \quad , \quad \text{Jacobi identity} \Leftrightarrow Q^2 = 0$$

- Idea: **Cartan**, More: **Strobl et al.**, **Sati**, **Schreiber**, **Stasheff**
- Local gauge theory: **differential forms** and **Lie algebras**
- Unify both in **differential graded algebras** from **Weil algebra**:

$$W(\mathfrak{g}) := C^\infty(T[1]\mathfrak{g}[1]) = C^\infty(\mathfrak{g}[1] \oplus \mathfrak{g}[2]), \quad \sigma : \mathfrak{g}^*[1] \xrightarrow{\cong} \mathfrak{g}^*[2]$$

$$Q|_{C^\infty(\mathfrak{g}[1])} = Q_{CE} + \sigma, \quad Q_{CE}\sigma = -\sigma Q_{CE}$$

- **Potentials/curvatures/Bianchi identities** from **dga-morphisms**

$$(A, F) : W(\mathfrak{g}) \longrightarrow W(M) = \Omega^\bullet(M)$$

$$\xi^\alpha \longmapsto A^\alpha$$

$$(\sigma\xi^\alpha) = Q\xi^\alpha + \frac{1}{2}f_{\beta\gamma}^\alpha \xi^\beta \xi^\gamma \longmapsto F^\alpha = (dA + \frac{1}{2}[A, A])^\alpha$$

$$Q(\sigma\xi^\alpha) = -f_{\beta\gamma}^\alpha (\sigma\xi^\alpha) \xi^\beta \longmapsto (\nabla F)^\alpha = 0$$

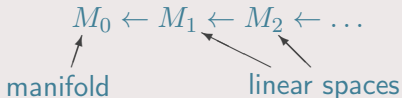
- **Gauge transformations**: **homotopies** between dga-morphisms
- **Topological invariants**: **invariant polynomials** on \mathfrak{g} in $W(\mathfrak{g})$

⇒ **General notion of gauge theory** from pairs of dgas

Back to

NQ -manifolds

- \mathbb{N}_0 -graded manifold with coordinates of degree $0, 1, 2, \dots$



- NQ -manifold: vector field Q of degree 1, $Q^2 = 0$
- Functions on (M, Q) form differential graded algebra

- Lie n -algebra or n -term L_∞ -algebra:

$$* \leftarrow M_1 \leftarrow M_2 \leftarrow \dots \leftarrow M_n \leftarrow * \leftarrow * \leftarrow \dots$$

We shall be interested in Lie 2- and Lie 3-algebras.

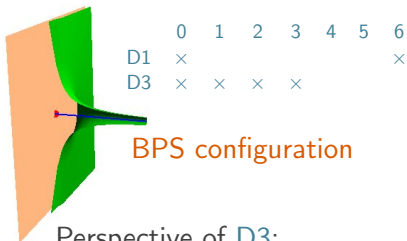
- **Graded vector space:** $* \leftarrow W[1] \leftarrow V[2] \leftarrow * \leftarrow \dots$
- **Coordinates:**
 - w^a of degree 1 on $W[1]$
 - v^i of degree 2 on $V[2]$
- Most general vector field Q of degree 1:

$$Q = -m_i^a v^i \frac{\partial}{\partial w^a} - \frac{1}{2} m_{ab}^c w^a w^b \frac{\partial}{\partial w^c} - m_{ai}^j w^a v^i \frac{\partial}{\partial v^j} - \frac{1}{3!} m_{abc}^i w^a w^b w^c \frac{\partial}{\partial v^i}$$

- Induces “brackets”/“higher products”:
 - $\mu_1(\tau_i) = m_i^a \tau_a$
 - $\mu_2(\tau_a, \tau_b) = m_{ab}^c \tau_c$, $\mu_2(\tau_a, \tau_i) = m_{ai}^j \tau_j$
 - $\mu_3(\tau_a, \tau_b, \tau_c) = m_{abc}^i \tau_i$
- $Q^2 = 0 \Leftrightarrow$ **Homotopy Jacobi identities**, e.g.
 - $\mu_1(\mu_1(-)) = 0$: μ_1 is a differential
 - $\mu_2(x, \mu_2(y, z)) + \text{cycl.} = \mu_1(\mu_3(x, y, z))$: **Jacobiator**
- Analogously: **Lie 3-algebras**

Which higher Lie algebra to take?

Guidance from **BPS self-dual strings**



BPS configuration

Perspective of D3:

Bogomolny monopole eqn.

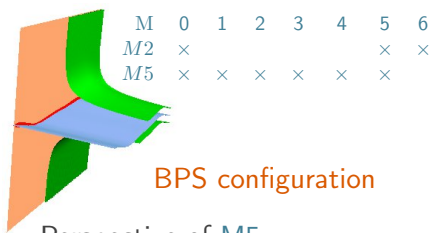
$$F = \nabla^2 = *\nabla\Phi \text{ on } \mathbb{R}^3$$

↕ Nahm transform ↕

Perspective of D1:

Nahm eqn.

$$\frac{d}{dx^6} X^i + \varepsilon^{ijk} [X^j, X^k] = 0$$



BPS configuration

Perspective of M5:

Abelian Self-dual string eqn.

$$H := dB = *d\Phi \text{ on } \mathbb{R}^4$$

↕ genlzd. Nahm transform (?) ↕

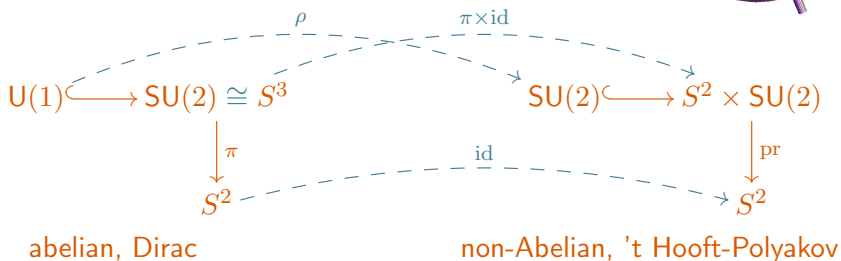
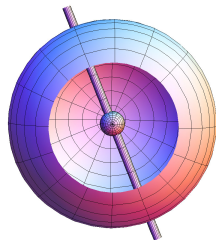
Perspective of M2:

Hoppe-Basu-Harvey eqn. (??)

$$\frac{d}{dx^6} X^\mu + \varepsilon^{\mu\nu\rho\sigma} [X^\nu, X^\rho, X^\sigma] = 0$$

Monopoles

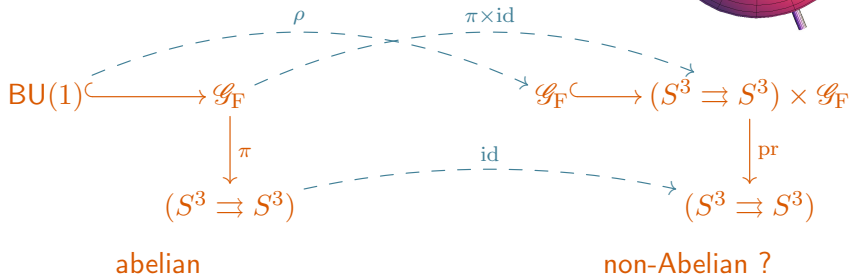
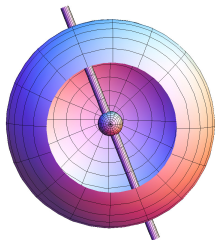
- Solution to **Bogomolny equation** $F = *\nabla\phi$
- Abelian: singular on \mathbb{R}^3 , **Dirac strings**
- Principal bundle over S^2
- Non-Abelian: non-singular on \mathbb{R}^3



\Rightarrow Choose $SU(2)$, as trivialization possible.

Self-Dual Strings

- Abelian: singular on \mathbb{R}^4 , Dirac strings
- Solution to $H = *\nabla\phi$
- Gerbe over S^3
- Non-Abelian: ?



\Rightarrow Choose \mathcal{G}_F , with 2-group structure: **String 2-group**
 (many other reasons for this)

- **String 2-group** \mathcal{G}_F and M-theory: long story...
- \mathcal{G}_F is analogue of $\text{Spin}(3) \cong \text{SU}(2)$ from many perspectives
- Lie differentiate (e.g. **Demessie, CS (2016)**)
- Result:

String Lie 2-algebra **string(3)** = $(\mathfrak{su}(2) \xleftarrow{\mu_1=0} \mathbb{R}[1])$ with

$$Q\xi^\alpha = -\frac{1}{2}f_{\beta\gamma}^\alpha \xi^\beta \xi^\gamma, \quad Qb = -\frac{1}{3!}f_{\alpha\beta\gamma} \xi^\alpha \xi^\beta \xi^\gamma$$

or

$$\mu_2(x_1, x_2) = [x_1, x_2], \quad \mu_3(x_1, x_2, x_3) = (x_1, [x_2, x_3])$$

where $x_{1,2,3} \in \mathfrak{su}(2)$.

Remarks:

- Can be defined for any ADE Lie algebra $\mathfrak{g} \rightarrow \mathbf{string}(\mathfrak{g})$
- Can **twist** the Weil algebra to $\tilde{W}(\mathbf{string}(\mathfrak{g}))$ by **inv. polynomial**

$(A, B, F, H) : \tilde{W}(\mathbf{string}(3)) \rightarrow W(M) = \Omega^\bullet(M)$, add ϕ

$$A \in \Omega^1(\mathbb{R}^4) \otimes \mathfrak{su}(2), \quad B \in \Omega^2(\mathbb{R}^4) \otimes \mathbb{R}, \quad \phi \in \Omega^0(\mathbb{R}^4) \otimes \mathbb{R}$$

$$H = dB + \frac{1}{2}(A, dA) + \frac{1}{3!}(A, [A, A]) = *d\phi$$

$$\Rightarrow dH = (F, F) = *\square\phi$$

$$F = dA + \frac{1}{2}[A, A] = *F$$

Passes many **consistency checks**, e.g.

- **Nice reduction** to monopoles on \mathbb{R}^3
- **BPS equations** for (1,0)-model (more later)

Elementary Solution:

$$A_\mu(x) = \frac{1}{i} \frac{\eta_{\mu\nu}^i \sigma_i (x - x_0)^\nu}{\rho^2 + (x - x_0)^2}, \quad B = 0, \quad \varphi = \frac{((x - x_0)^2 - 2\rho^2)}{((x - x_0)^2 + \rho^2)^2}$$

cf. also **Akyol, Papadopoulos 2012**

The 6d superconformal field theory

Look for candidate theory in the literature and find:

6d $(1,0)$ -model derived from tensor hierarchies
Samtleben, Sezgin, Wimmer (2011)

Open problems with this model:

- Issue 1: Choice of gauge structure unclear
- Issue 2: cubic interactions
- Issue 3: scalar fields with wrong sign kinetic term
- Issue 4: Self-duality of 3-form imposed by hand
- Issue 5: Unclear, how to fulfill “wishlist”

Previous observation:

- Gauge structure is Lie 3-algebra with “extra structure.”
Palmer, CS (2013), Samtleben et al. (2014)

New:

Schmidt, CS (2017)

- **Idea:** use $\mathfrak{string}(\mathfrak{g})$ as gauge structure in this model
- Issue: need suitable notion of **inner product** for action
- **Inner product/cyclic** L_∞ -algebras \Leftrightarrow **symplectic NQ-manifold**
- Consequence: Extend twisted $\mathfrak{string}(\mathfrak{g})$ from

$$(\mathfrak{g} \leftarrow \mathbb{R} \leftarrow \mathbb{R}) \cong \mathfrak{spin}(\mathfrak{g})$$

to symplectic graded vector space $T^*[2]\mathfrak{string}(\mathfrak{g})$:

$$\begin{array}{ccc}
 \mathbb{R}^* & \xleftarrow{\mu_1=\text{id}} & \mathbb{R}^*[1] & & \mathfrak{g}^*[2] \\
 \oplus & & \oplus & & \oplus \\
 \mathfrak{g} & & \mathbb{R}[1] & \xleftarrow{\mu_1=\text{id}} & \mathbb{R}[2]
 \end{array}$$

- This carries **natural inner product**
- Can be extended to **Lie 3-algebra**
- Has necessary **extra structure**

Field content:

- **(1,0) tensor multiplet** (ϕ, χ^i, B) , values in \mathbb{R}^2 , $\phi = \phi_s + \phi_r, \dots$
- **(1,0) vector multiplet** (A, λ^i, Y^{ij}) , values in $\mathfrak{g} \oplus \mathbb{R}$
- **C-field**, values in $\mathbb{R} \oplus \mathfrak{g}^*$

Action (schematically):

$$\begin{aligned}
 S = \int_{\mathbb{R}^{1,5}} & \left(\langle \mathcal{H}, *\mathcal{H} \rangle + \langle d\phi, *d\phi \rangle - *\langle \bar{\chi}, \not{\partial}\chi \rangle + \mathcal{H}_s \wedge *(\bar{\lambda}, \gamma_{(3)}\lambda) + *(Y, \bar{\lambda})\chi_s \right. \\
 & + \phi_s((\mathcal{F}, *\mathcal{F}) - *(Y, Y) + *(\bar{\lambda}, \nabla\lambda)) + (\bar{\lambda}, \mathcal{F}) \wedge *\gamma_{(2)}\chi_s \\
 & \left. + \mu_1(C) \wedge \mathcal{H}_s + B_s \wedge (\mathcal{F}, \mathcal{F}) + B_s \wedge ([A, A], [A, A]) \right)
 \end{aligned}$$

This solves problems 1 and 2:

- **Choice of gauge structure** for ADE-(2,0)-theories **clear**.
- **No cubic interaction term** for scalar fields

Adding **Pasti-Sorokin-Tonin-type action**:

- Recall: **PST action** has self-duality of H as equation of motion
- Bosonic part of (1,0)-theory was PST completed
Bandos, Sorokin, Samtleben (2013)
- Full PST action announced, **never appeared** (not possible?)
- With string structure, **construction possible and simplifies**

Adding **matter fields**:

- Add **hypermultiplet** to get fields of (2,0)-tensor multiplet
- General construction and couplings discussed
Samtleben, Sezgin, Wimmer (2012)
- Can make **concrete choices** with twisted string structures

⇒ A (1,0)-theory in 6d satisfying many of the “wishlist” items.

Consistency checks

- ✓ Contain an **interacting**, self-dual 2-form gauge potential
- ✓ Based on a **sound mathematical foundation**: higher bundles
- ✓ **Field content** of the $(2,0)$ -theory, $\mathcal{N} = (1,0)$ supersymmetric
- ✓ **Gauge structure** natural, match some **expectations** (ADE, ...)
- ✓ Non-trivial coupling, **interacting field theory**
- ✓ Possible restriction to **free** $\mathcal{N} = (2,0)$ tensor multiplet
- ✓ contains the **non-abelian self-dual string soliton** as BPS state
- **Reduction to 4d SYM theory with ADE gauge algebras**
- and to **3d Chern–Simons-matter models** with discrete coupling
- ? match expected **moduli space** of $\mathcal{N} = (2,0)$ -theory

Crucial consistency check: **Reduction to D-branes/SYM theory**

$$S = \int_{\mathbb{R}^{1,5}} \left(\langle \mathcal{H}, * \mathcal{H} \rangle + \langle d\phi, * d\phi \rangle - * \langle \bar{\chi}, \not{\partial} \chi \rangle + \mathcal{H}_s \wedge * (\bar{\lambda}, \gamma_{(3)} \lambda) + *(Y, \bar{\lambda}) \chi_s \right. \\ \left. + \phi_s ((\mathcal{F}, * \mathcal{F}) - *(Y, Y) + *(\bar{\lambda}, \nabla \lambda)) + (\bar{\lambda}, \mathcal{F}) \wedge * \gamma_{(2)} \chi_s \right. \\ \left. + \mu_1(C) \wedge \mathcal{H}_s + B_s \wedge (\mathcal{F}, \mathcal{F}) + B_s \wedge ([A, A], [A, A]) \right)$$

- Start from **ADE-String Lie 3-algebra**
- Anticipate 4d gauge couplings:

$$\tau = \tau_1 + i\tau_2 = \frac{\theta}{2\pi} + \frac{i}{g_{\text{YM}}^2},$$

- **VEVs** from compactification on T^2 along x^9 and x^{10}

$$\langle \phi_s \rangle = -\frac{1}{32\pi^2} \frac{\tau_2}{R_9 R_{10}} \quad \text{and} \quad \langle B_s \rangle = \frac{1}{16\pi^2} \frac{\tau_1}{R_9 R_{10}} \text{vol}(T^2)$$

- **Strong coupling expansion** around VEVs (cf. M2 \rightarrow D2)
- \Rightarrow **4d $\mathcal{N} = 4$ SYM** with ADE-gauge group and θ -term

Additional consistency check: **Reduction to M2-brane models**

- Replace $\mathbb{R}^{1,5}$ by $\mathbb{R}^{1,2} \times S^3$.
- Assumptions:
 - String Lie 3-algebra of $\mathfrak{su}(n) \times \mathfrak{su}(n)$
 - A trivial on S^3 , non-trivial on $\mathbb{R}^{1,2}$
 - B trivial on $\mathbb{R}^{1,2}$
 - B encodes **abelian gerbe** with DD class k on S^3 .
- Recall: $\mathcal{H} = dB + cs(A)$
- Then we get the **integer Chern–Simons coupling**:

$$\mathcal{H} \wedge *\mathcal{H} \rightarrow k \text{vol}_{S^3} cs(A)$$

$$\int_{\mathbb{R}^{1,5}} \mathcal{H} \wedge *\mathcal{H} \rightarrow k \int_{\mathbb{R}^{1,2}} cs(A)$$

- Altogether: **Chern–Simons matter theory** of ABJM type.
- Note: This theory has $\mathcal{N} = 4$, different potential from ABJM.

Problems and potential solutions

Our model is not the desired (2,0)-theory!

Problems:

- Free **Yang–Mills multiplet** contradicts $\mathcal{N} = (2, 0)$ SUSY
- **Moduli space of vacua** is not that of multiple M5-branes
- PST mechanism relies on $\phi_s > 0$
- Scalar field with **wrong sign kinetic term**
- Model **not compatible** with categorical equivalence

Turn **Problems** into **Hints of Solutions**:

- Scalar field with **wrong sign kinetic term**
(rigid feature of **Samtleben et al.** model)
- Model **not compatible** with **categorical equivalence**
(rigid feature of **Samtleben et al.** model)

Last point: the model of Samtleben et al. is **too rigid**:

$$(X_r)_s^t = f_{rs}^t + d_{rs}^t = f_{[rs]}^t + d_{(rs)}^t$$

Next steps (work in progress):

- Understand **twisted Weil algebras** and **categorical equivalence**
- **Rederive SUSY action** in bigger picture

Summary:

- Higher gauge theory classically underlies M-theory
- Higher analogue of $SU(2)$ is $String(3)$
- There is non-abelian self-dual string
- There is classical action with many of desired features
- However: Clear differences to $(2,0)$ -theory

Soon to come:

- ▷ Understand Categorical Equivalence, Higher Twists (WIP)
- ▷ Study $\mathcal{N} = (1,0)$ -models (next on our list)
- ▷ Better understanding of M-theory
- ▷ Link to categorified integrability, fuzzy S^3 , etc.

Towards an M5-brane model I: A 6d superconformal field theory

Christian Sämann



*School of Mathematical and Computer Sciences
Heriot-Watt University, Edinburgh*

Workshop on Higher Gauge Theory, Leeds, 1.3.2018