

# Geometry of Higher Yang-Mills Fields

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Symmetries and Geometry of Branes in String/M-Theory,  
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Based on work with:

- S Palmer, D Harland, C Papageorgakis, F Sala (**M-brane models**)
- M Wolf (**Twistor description**)
- R Szabo (**Geometric Quantization**)

There might be an effective description of M5-branes.

- **Effective description of M2-branes** proposed in 2007.
- This created lots of interest:  
**BLG-model**: >625 citations, **ABJM-model**: >917 citations

**Question**: Is there a similar description for M5-branes?

**For cautious people**:

Is there a a reasonably interesting superconformal field theory of a non-abelian tensor multiplet in six dimensions?  
(The mysterious, long-sought  $\mathcal{N} = (2, 0)$  SCFT in six dimensions)

A possible way to approach the problem: **Look at BPS subsector**

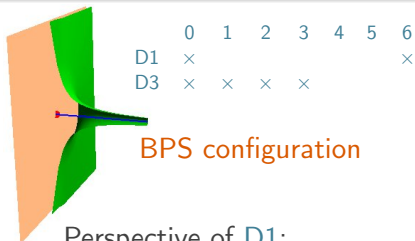
- This was how the **M2-brane models** were derived originally.
- BPS subsector is interesting itself: **Integrability**
- BPS subsector should be **more accessible** than full theory.

Things do look very promising.

- Integrability found:  
Nahm construction for self-dual strings using loop space  
CS, S Palmer & CS
- Use of loop space justified:  
M-theory suggests this, e.g. Geometric quantization of  $S^3$   
CS & R Szabo
- Integrability reasonable:  
Gauge structure of M2- and M5-brane models the same  
S Palmer & CS
- Integrability works even without loop space:  
Twistor constructions of self-dual strings and non-abelian tensor multiplets work  
CS & M Wolf
- On the way to Geometry of Higher Yang-Mills Fields:  
Explicit solutions to non-abelian tensor multiplet equations  
F Sala, S Palmer & CS

# Monopoles and Self-Dual Strings

Lifting monopoles to M-theory yields self-dual strings.



	0	1	2	3	4	5	6
D1	×						×
D3	×	×	×	×			

BPS configuration

Perspective of D1:

Nahm eqn.

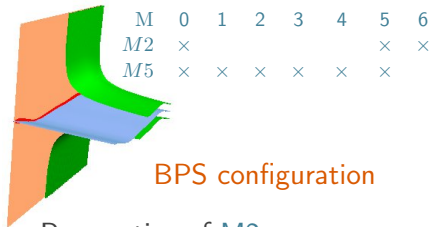
$$\frac{d}{dx^6} X^i + \varepsilon^{ijk} [X^j, X^k] = 0$$

↕ Nahm transform ↕

Perspective of D3:

Bogomolny monopole eqn.

$$F_{ij} = \varepsilon_{ijk} \nabla_k \Phi$$



	M	0	1	2	3	4	5	6
M2		×					×	×
M5		×	×	×	×	×	×	

BPS configuration

Perspective of M2:

Basu-Harvey eqn.

$$\frac{d}{dx^6} X^\mu + \varepsilon^{\mu\nu\rho\sigma} [X^\nu, X^\rho, X^\sigma] = 0$$

↕ generalized Nahm transform ↕

Perspective of M5:

Self-dual string eqn.

$$H_{\mu\nu\rho} = \varepsilon_{\mu\nu\rho\sigma} \partial_\sigma \Phi$$

In analogy with Lie algebras, we can introduce 3-Lie algebras.

$$\text{BH: } \frac{d}{ds} X^\mu + [A_s, X^\mu] + \varepsilon^{\mu\nu\rho\sigma} [X^\nu, X^\rho, X^\sigma] = 0, \quad X^\mu \in \mathcal{A}$$

## 3-Lie algebra

Obviously:  $\mathcal{A}$  is a **vector space**,  $[\cdot, \cdot, \cdot]$  **trilinear+antisymmetric**.

Satisfies a “3-Jacobi identity,” the **fundamental identity**:

$$[A, B, [C, D, E]] = [[A, B, C], D, E] + [C, [A, B, D], E] + [C, D, [A, B, E]]$$

Filippov (1985)

Gauge transformations from Lie algebra of **inner derivations**:

$$D : \mathcal{A} \wedge \mathcal{A} \rightarrow \text{Der}(\mathcal{A}) =: \mathfrak{g}_{\mathcal{A}} \quad D(A, B) \triangleright C := [A, B, C]$$

Algebra of inner derivations closes due to **fundamental identity**.

# Brief Remarks on 3-Lie Algebras

In analogy with Lie algebras, we can introduce 3-Lie algebras.

## Examples:

Lie algebra	3-Lie algebra
Heisenberg-algebra: $[\tau_a, \tau_b] = \varepsilon_{ab} \mathbb{1}, \quad [\mathbb{1}, \cdot] = 0$	Nambu-Heisenberg 3-Lie Algebra: $[\tau_i, \tau_j, \tau_k] = \varepsilon_{ijk} \mathbb{1}, \quad [\mathbb{1}, \cdot, \cdot] = 0$
$\mathfrak{su}(2) \simeq \mathbb{R}^3$ : $[\tau_i, \tau_j] = \varepsilon_{ijk} \tau_k$	$A_4 \simeq \mathbb{R}^4$ : $[\tau_\mu, \tau_\nu, \tau_\kappa] = \varepsilon_{\mu\nu\kappa\lambda} \tau_\lambda$

## Generalizations:

- **Real 3-algebras:**  $[\cdot, \cdot, \cdot]$  antisymmetric only in first two slots  
S. Cherkis & CS, 0807.0808
- **Hermitian 3-algebras:** complex vector spaces,  $\rightarrow$  ABJM  
Bagger & Lambert, 0807.0163

## Generalizing the ADHMN construction to M-branes

That is, find solutions to  $H = \star d\Phi$   
from solutions to the Basu-Harvey equation.

As M5-branes seem to require gerbes, let's start with them.

Dirac monopoles are described by principal  $U(1)$ -bundles over  $S^2$ .

Manifold  $M$  with cover  $(U_i)_i$ . **Principal  $U(1)$ -bundle** over  $M$ :

$$F \in \Omega^2(M, \mathfrak{u}(1)) \text{ with } dF = 0$$

$$A_{(i)} \in \Omega^1(U_i, \mathfrak{u}(1)) \text{ with } F = dA_{(i)}$$

$$g_{ij} \in \Omega^0(U_i \cap U_j, U(1)) \text{ with } A_{(i)} - A_{(j)} = d \log g_{ij}$$

Consider monopole in  $\mathbb{R}^3$ , **but** describe it on  $S^2$  around monopole:

$S^2$  with patches  $U_+, U_-$ ,  $U_+ \cap U_- \sim S^1$ :  $g_{+-} = e^{-ik\phi}$ ,  $k \in \mathbb{Z}$

$$c_1 = \frac{i}{2\pi} \int_{S^2} F = \frac{i}{2\pi} \int_{S^1} A^+ - A^- = \frac{1}{2\pi} \int_0^{2\pi} d\phi k = k$$

**Monopole charge:**  $k$



# Self-Dual Strings and Abelian Gerbes

Self-dual strings are described by abelian gerbes.

Manifold  $M$  with cover  $(U_i)_i$ . **Abelian (local) gerbe** over  $M$ :

$$H \in \Omega^3(M, \mathfrak{u}(1)) \text{ with } dH = 0$$

$$B_{(i)} \in \Omega^2(U_i, \mathfrak{u}(1)) \text{ with } H = dB_{(i)}$$

$$A_{(ij)} \in \Omega^1(U_i \cap U_j, \mathfrak{u}(1)) \text{ with } B_{(i)} - B_{(j)} = dA_{ij}$$

$$h_{ijk} \in \Omega^0(U_i \cap U_j \cap U_k, \mathfrak{u}(1)) \text{ with } A_{(ij)} - A_{(ik)} + A_{(jk)} = dh_{ijk}$$

**Note:** Local gerbe: principal  $U(1)$ -bundles on intersections  $U_i \cap U_j$ .

Consider  $S^3$ , patches  $U_+, U_-, U_+ \cap U_- \sim S^2$ : **bundle over  $S^2$**

Reflected in:  $H^2(S^2, \mathbb{Z}) \cong H^3(S^3, \mathbb{Z}) \cong \mathbb{Z}$

$$\frac{i}{2\pi} \int_{S^3} H = \frac{i}{2\pi} \int_{S^2} B_+ - B_- = \dots = k$$

**Charge of self-dual string:**  $k$

Describe  $p$ -gerbes + connective structure  $\rightarrow$  **Deligne cohomology**.

Gerbes are somewhat unfamiliar, difficult to work with.

Can we somehow avoid using gerbes?

By going to loop space, one can reduce differential forms by one degree.

Consider the following **double fibration**:

$$\begin{array}{ccc} & \mathcal{L}M \times S^1 & \\ ev \swarrow & & \searrow pr \\ M & & \mathcal{L}M \end{array}$$

Identify  $T\mathcal{L}M = \mathcal{L}TM$ , then:  $x \in \mathcal{L}M \Rightarrow \dot{x}(\tau) \in T\mathcal{L}M$

Transgression

$$\mathcal{T} : \Omega^{k+1}(M) \rightarrow \Omega^k(\mathcal{L}M), \quad v_i = \oint d\tau v_i^\mu(\tau) \frac{\delta}{\delta x^\mu(\tau)} \in T\mathcal{L}M$$
$$(\mathcal{T}\omega)_x(v_1(\tau), \dots, v_k(\tau)) := \oint_{S^1} d\tau \omega(x(\tau))(v_1(\tau), \dots, v_k(\tau), \dot{x}(\tau))$$

Nice properties: **reparameterization invariant**, **chain map**, ...

An abelian local gerbe over  $M$  is a principal  $U(1)$ -bundle over  $\mathcal{L}M$ .

# Transgressed Self-Dual Strings

By going to loop space, one can reduce differential forms by one degree.

Recall the **self-dual string equation** on  $\mathbb{R}^4$ :  $H_{\mu\nu\kappa} = \varepsilon_{\mu\nu\kappa\lambda} \frac{\partial}{\partial x^\lambda} \Phi$

Its **transgressed form** is an equation for a **2-form**  $F$  on  $\mathcal{L}\mathbb{R}^4$ :

$$F_{(\mu\sigma)(\nu\rho)} = \delta(\sigma - \rho) \varepsilon_{\mu\nu\kappa\lambda} \dot{x}^\kappa(\tau) \frac{\partial}{\partial y^\lambda} \Phi(y) \Big|_{y=x(\tau)}$$

Extend to full **non-abelian** loop space curvature:

$$F_{(\mu\sigma)(\nu\tau)}^\pm = (\varepsilon_{\mu\nu\kappa\lambda} \dot{x}^\kappa(\sigma) \nabla_{(\lambda\tau)} \Phi)_{(\sigma\tau)} \\ \mp (\dot{x}_\mu(\sigma) \nabla_{(\nu\tau)} \Phi + \dot{x}_\nu(\sigma) \nabla_{(\mu\tau)} \Phi - \delta_{\mu\nu} \dot{x}^\kappa(\sigma) \nabla_{(\kappa\tau)} \Phi)_{[\sigma\tau]}$$

where  $\nabla_{(\mu\sigma)} := \oint d\tau \delta x^\mu(\tau) \wedge \left( \frac{\delta}{\delta x^\mu(\tau)} + A_{(\mu\tau)} \right)$

**Goal:** Construct solutions to this equation.

# The ADHMN Construction

The ADHMN construction nicely translates to self-dual strings on loop space.

**Nahm transform:** Instantons on  $T^4 \mapsto$  instantons on  $(T^4)^*$

Roughly here:

$$T^4: \begin{cases} 3 \text{ rad. } 0 \\ 1 \text{ rad. } \infty : \text{ D1 WV} \end{cases} \quad \text{and} \quad (T^4)^*: \begin{cases} 3 \text{ rad. } \infty : \text{ D3 WV} \\ 1 \text{ rad. } 0 \end{cases}$$

**Dirac operators:**  $X^i$  solve Nahm eqn.,  $X^\mu$  solve Basu-Harvey eqn.

$$\text{IIB: } \not{D} = -\mathbb{1} \frac{d}{dx^6} + \sigma^i (iX^i + x^i \mathbb{1}_k)$$

$$\text{M: } \not{D} = -\gamma_5 \frac{d}{dx^6} + \frac{1}{2} \gamma^{\mu\nu} \left( D(X^\mu, X^\nu) - i \oint d\tau x^\mu(\tau) \dot{x}^\nu(\tau) \right)$$

**normalized zero modes:**  $\not{D}\psi = 0$  and  $\mathbb{1} = \int_{\mathcal{I}} ds \bar{\psi} \psi$

**Solution to Bogomolny/self-dual string equations:**

$$A := \int_{\mathcal{I}} ds \bar{\psi} d\psi \quad \text{and} \quad \Phi := -i \int_{\mathcal{I}} ds \bar{\psi} s \psi$$

# Remarks on The Construction

The construction is very natural and behaves as expected.

- Nahm eqn. and Basu-Harvey eqn. play analogous roles.
- Construction extends to general. Basu-Harvey eqn. (ABJM).
- One can construct many examples explicitly.
- It reduces nicely to ADHMN via the M2-Higgs mechanism.

CS, 1007.3301, S Palmer & CS, 1105.3904

## More Motivation for Loop Spaces

A recently proposed 3-Lie algebra valued tensor-multiplet implies a transgression.

3-Lie algebra valued tensor multiplet equations:

$$\nabla^2 X^I - \frac{i}{2} [\bar{\Psi}, \Gamma_\nu \Gamma^I \Psi, C^\nu] - [X^J, C^\nu, [X^J, C_\nu, X^I]] = 0$$

$$\Gamma^\mu \nabla_\mu \Psi - [X^I, C^\nu, \Gamma_\nu \Gamma^I \Psi] = 0$$

$$\nabla_{[\mu} H_{\nu\lambda\rho]} + \frac{1}{4} \varepsilon_{\mu\nu\lambda\rho\sigma\tau} [X^I, \nabla^\tau X^I, C^\sigma] + \frac{i}{8} \varepsilon_{\mu\nu\lambda\rho\sigma\tau} [\bar{\Psi}, \Gamma^\tau \Psi, C^\sigma] = 0$$

$$F_{\mu\nu} - D(C^\lambda, H_{\mu\nu\lambda}) = 0$$

$$\nabla_\mu C^\nu = D(C^\mu, C^\nu) = 0$$

$$D(C^\rho, \nabla_\rho X^I) = D(C^\rho, \nabla_\rho \Psi) = D(C^\rho, \nabla_\rho H_{\mu\nu\lambda}) = 0$$

N Lambert & C Papageorgakis, 1007.2982

Factorization of  $C^\rho = C \dot{x}^\rho$ . Here, 3-Lie algebra transgression:

$$(\mathcal{T}\omega)_x(v_1(\tau), \dots, v_k(\tau)) := \int_{S^1} d\tau D(\omega(v_1(\tau), \dots, v_k(\tau), \dot{x}(\tau)), C)$$

C Papageorgakis & CS, 1103.6192

Often: A vector short of happiness. Loop space has this vector.



# Side Remark: Quantization of $\mathbb{R}^3$

In the quantization problem, one is naturally led to loop space.

Geometric quantization prescription: (e.g. fuzzy sphere)

Special symplectic manifold  $(M, \omega)$

$\rightarrow$

line bundle  $L$  with  $(h, \nabla)$  over  $M$

$\rightarrow$

Hilbert space  $\mathcal{H}$ :  
global holomorphic sections of  $L$

Quantization map:  $[\hat{f}, \hat{g}] = i\hbar \widehat{\{f, g\}} + \mathcal{O}(\hbar^2)$

**M-theory:** 2-plectic manifold  $(M, \varpi)$ ,  $\varpi \in \Omega^3(M)$

- hol. secs. of gerbe?, quantization of one-forms? **Rogers, ...**
- **Solution:**  $\omega$  on  $\mathcal{L}M$  as  $\omega := \mathcal{T}\varpi$ , then proceed as above
- **Example:**  $\mathbb{R}^3$  with 2-plectic form  $\varpi = \varepsilon_{ijk} dx^i \wedge dx^j \wedge dx^k$ :

$$[x^i(\tau), x^j(\sigma)] = \varepsilon^{ijk} \frac{\dot{x}_k(\tau)}{|\dot{x}(\tau)|^2} \delta(\tau - \sigma) + \mathcal{O}(\theta^2)$$

CS & R Szabo, 1211.0395

- Cf. **Kawamoto & Sasakura, Bergshoeff, Berman et al. [2000]**

The duality  $D1 \leftrightarrow D3$  is a duality between Yang-Mills theories.

Question: In what sense are M2- and M5-brane models related?

Start by looking at gauge structure

Higher gauge theory describe parallel transport of extended objects.

Parallel transport of particles in representation of gauge group  $G$ :

- holonomy functor:  $\text{hol} : \text{path } p \mapsto \text{hol}(p) \in G$
- $\text{hol}(p) = P \exp(\int_p A)$ ,  $P$ : path ordering, trivial for  $U(1)$ .

Parallel transport of strings with gauge group  $U(1)$ :

- 2-holonomy functor:  $\text{hol}_2 : \text{surface } s \mapsto \text{hol}_2(s) \in U(1)$
- $\text{hol}_2(s) = \exp(\int_s B)$ ,  $B$ : connective structure on gerbe.

Nonabelian case:

- much more involved!
- no straightforward definition of surface ordering
- solution: Categorification!

see [Baez, Huerta, 1003.4485](#)

# Categorifying Gauge Groups

A Lie 2-group is a Lie groupoid with extra structure.

**Warning:** Categorification neither unique nor straightforward.

## Lie 2-group

A Lie 2-group is a

- monoidal category, morph. invertible, obj. weakly invertible.
- Lie groupoid + product  $\otimes$  obeying weakly the group axioms.

**Simplification:** use strict Lie 2-groups  $\xleftrightarrow{1:1}$  Lie crossed modules

## Lie crossed modules

Pair of Lie groups  $(G, H)$ , written as  $(H \xrightarrow{t} G)$  with:

- left automorphism action  $\triangleright: G \times H \rightarrow H$
- group homomorphism  $t: H \rightarrow G$  such that

$$t(g \triangleright h) = gt(h)g^{-1} \quad \text{and} \quad t(h_1) \triangleright h_2 = h_1 h_2 h_1^{-1}$$

**Also:** strict Lie 2-algebras  $\xleftrightarrow{1:1}$  differential crossed modules

Lie crossed modules come in a large variety.

## Lie crossed modules

Pair of Lie groups  $(G, H)$ , written as  $(H \xrightarrow{t} G)$  with:

- left automorphism action  $\triangleright: G \times H \rightarrow H$
- group homomorphism  $t: H \rightarrow G$

$$t(g \triangleright h) = gt(h)g^{-1} \quad \text{and} \quad t(h_1) \triangleright h_2 = h_1 h_2 h_1^{-1}$$

### Simplest examples:

- Lie group  $G$ , Lie crossed module:  $(1 \xrightarrow{t} G)$ .
- Abelian Lie group  $G$ , Lie crossed module:  $BG = (G \xrightarrow{t} 1)$ .

### More involved:

- Automorphism 2-group of Lie group  $G$ :  $(G \xrightarrow{t} \text{Aut}(G))$

Higher gauge theory is the dynamical theory of principal 2-bundles.

Consider a manifold  $M$  with cover  $(U_a)$

Object	Principal $G$ -bundle	Principal $(H \xrightarrow{t} G)$ -bundle
Cochains	$(g_{ab})$ valued in $G$	$(g_{ab})$ valued in $G$ , $(h_{abc})$ valued in $H$
Cocycle	$g_{ab}g_{bc} = g_{ac}$	$t(h_{abc})g_{ab}g_{bc} = g_{ac}$ $h_{acd}h_{abc} = h_{abd}(g_{ab} \triangleright h_{bcd})$
Coboundary	$g_a g'_{ab} = g_{ab} g_b$	$g_a g'_{ab} = t(h_{ab})g_{ab}g_b$ $h_{ac}h_{abc} = (g_a \triangleright h'_{abc})h_{ab}(g_{ab} \triangleright h_{bc})$
gauge pot.	$A_a \in \Omega^1(U_a) \otimes \mathfrak{g}$	$A_a \in \Omega^1(U_a) \otimes \mathfrak{g}$ , $B_a \in \Omega^2(U_a) \otimes \mathfrak{h}$
Curvature	$F_a = dA_a + A_a \wedge A_a$	$F_a = dA_a + A_a \wedge A_a$ , $F_a = t(B_a)$ $H_a = dB_a + A_a \triangleright B_a$
Gauge trafos	$\tilde{A}_a := g_a^{-1} A_a g_a + g_a^{-1} dg_a$	$\tilde{A}_a := g_a^{-1} A_a g_a + g_a^{-1} dg_a + t(\Lambda_a)$ $\tilde{B}_a := g_a^{-1} \triangleright B_a + \tilde{A}_a \triangleright \Lambda_a + d\Lambda_a - \Lambda_a \wedge \Lambda_a$

## Remarks:

- A principal  $(1 \xrightarrow{t} G)$ -bundle is a principal  $G$ -bundle.
- A principal  $(U(1) \xrightarrow{t} 1) = BU(1)$ -bundle is an abelian gerbe.
- Gauge part of  $(2,0)$ -theory:  $H = \star H$ ,  $F = t(B)$ .

Is all this machinery really useful/necessary?

3-algebras are merely special classes of differential crossed modules.

Recall the definition of a 3-algebra  $\mathcal{A}$ :

- $[\cdot, \cdot, \cdot] : \mathcal{A}^{\otimes 3} \rightarrow \mathcal{A}$
- Fundamental identity says that  $[a, b, \cdot] \in \text{Der}(\mathcal{A})$ ,  $a, b \in \mathcal{A}$ .

## Theorem

3-algebras  $\xleftrightarrow{1:1}$  metric Lie algebras  $\mathfrak{g} \cong \text{Der}(\mathcal{A})$   
faithful orthog. representations  $V \cong \mathcal{A}$   
J Figueroa-O'Farrill et al., 0809.1086

## Observations

- $\mathfrak{g} \xrightarrow{t} V$  is a simple differential crossed modules
- M2- and M5-brane models have **the same gauge structure**.
- Via Faulkner construction, **all DCMs come with  $[\cdot, \cdot, \cdot]$**
- Application of this to M2- and M5-models **looks promising**.

S Palmer & CS, 1203.5757



3-Lie algebra valued tensor multiplet equations:

$$\nabla^2 X^I - \frac{i}{2}[\bar{\Psi}, \Gamma_\nu \Gamma^I \Psi, C^\nu] - [X^J, C^\nu, [X^J, C_\nu, X^I]] = 0$$

$$\Gamma^\mu \nabla_\mu \Psi - [X^I, C^\nu, \Gamma_\nu \Gamma^I \Psi] = 0$$

$$\nabla_{[\mu} H_{\nu\lambda\rho]} + \frac{1}{4}\varepsilon_{\mu\nu\lambda\rho\sigma\tau}[X^I, \nabla^\tau X^I, C^\sigma] + \frac{i}{8}\varepsilon_{\mu\nu\lambda\rho\sigma\tau}[\bar{\Psi}, \Gamma^\tau \Psi, C^\sigma] = 0$$

$$F_{\mu\nu} - D(C^\lambda, H_{\mu\nu\lambda}) = 0$$

$$\nabla_\mu C^\nu = D(C^\mu, C^\nu) = 0$$

$$D(C^\rho, \nabla_\rho X^I) = D(C^\rho, \nabla_\rho \Psi) = D(C^\rho, \nabla_\rho H_{\mu\nu\lambda}) = 0$$

N Lambert & C Papageorgakis, 1007.2982

Factorization of  $C^\rho = C\dot{x}^\rho$ . Here, fake curvature equation:

$$\mathfrak{t} : \mathcal{A} \rightarrow \text{Der}(\mathcal{A}), \quad a \mapsto D(C, a), \quad F_{\mu\nu} = \mathfrak{t}(H_{\mu\nu\lambda} x^\lambda) =: \mathfrak{t}(B)$$

⇒ More natural interpretation as higher gauge theory.

S Palmer & CS, 1203.5757

There is a striking sequence involving division/composition algebras in physics.

Division algebras, spheres and groups:

$\mathcal{A}$	$AP^1$	$ a  = 1$	$\text{Aut}(\mathcal{A})$	Physics
$\mathbb{R}$	$\mathbb{R}P^1 \cong S^1$	$\mathbb{Z}_2 \cong S^0$	$\text{Aut}(\mathbb{R}) \cong 1$	Vortex?
$\mathbb{C}$	$\mathbb{C}P^1 \cong S^2$	$U(1) \cong S^1$	$\text{Aut}(U(1)) \cong \mathbb{Z}_2$	Monopole
$\mathbb{H}$	$\mathbb{H}P^1 \cong S^4$	$SU(2) \cong S^3$	$\text{Aut}(SU(2)) \cong SU(2)$	Instanton
$\mathbb{O}$	$\mathbb{O}P^1 \cong S^8$	$S^7$	$\text{Aut}(\mathbb{O}) \cong G_2$	?

How should we regard the unit octonions?

- By themselves, they form a Moufang loop 😞
- Better: Use Faulkner construction to get a 3-algebra  
Nambu, Yamazaki, Figueroa-O'Farrill et al.
- Therefore, we have a DCM  $(\mathfrak{g}_2 \xrightarrow{t} \mathbb{R}^8 \cong \mathbb{O})$
- This suggests sequence:  $\mathbb{Z}_2, U(1), SU(2)$ , a Lie 2-group 😊
- Not (yet) clear how useful this actually is.

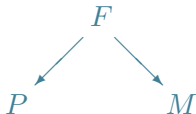
## Drop loop spaces: Principal 2-bundles over Twistor Spaces

Now that we saw the power of non-abelian gerbes, let's use them!

Using twistor spaces, one can map holomorphic data to solutions to field equations.

Recall the principle of the **Penrose-Ward transform**:

- Interested in **field equations** that are equivalent to **integrability of connections along subspaces** of spacetime  $M$
- Establish a double fibration



$P$ : **twistor space**, moduli space of subspaces in  $M$

$F$ : correspondence space

- $H^n(P, \mathfrak{G})$  (e.g. vector bundles)  $\xleftrightarrow{1:1}$  sols. to field equations.
- Explicitly appearing: **gauge transformations**, **moduli**, **symmetries of the equations**, etc.
- **BTW**: here,  $\xleftrightarrow{1:1}$  is actually a “holomorphic transgression”.

# Known Examples of Twistor Descriptions

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For Yang-Mills theories and its BPS subsectors, there is a wealth of twistor descriptions.

$$\begin{array}{ccc} \mathbb{C}^4 \times \mathbb{C}P^1 & & \\ \swarrow & & \searrow \\ \mathbb{C}P^3_{\circ} & & \mathbb{C}^4 \end{array}$$

Instantons  
hol. vector bundle

$$\begin{array}{ccc} \mathbb{C}^3 \times \mathbb{C}P^1 & & \\ \swarrow & & \searrow \\ T\mathbb{C}P^1 & & \mathbb{C}^3 \end{array}$$

Monopoles  
hol. vector bundle

$$\begin{array}{ccc} \mathbb{C}^{4|12} \times \mathbb{C}P^1 \times \mathbb{C}P^1 & & \\ \swarrow & & \searrow \\ P^{5|6} & & \mathbb{C}^{4|12} \end{array}$$

(Super) Yang-Mills  
hol. vector bundle

$$\begin{array}{ccc} \mathbb{C}^6 \times \mathbb{C}P^3 & & \\ \swarrow & & \searrow \\ P^6 & & \mathbb{C}^6 \end{array}$$

abelian  $H = \star H$   
hol. gerbe

Hughston, Murray, Eastwood, CS & M.Wolf, Mason et al.

**Note:** last twistor space reduces nicely to the above ones.

New: Penrose-Ward transform for self-dual strings.

New twistor space parameterizing hyperplanes in  $\mathbb{C}^4$ :

$$\begin{array}{ccc} \mathbb{C}^4 \times \mathbb{C}P^1 \times \mathbb{C}P^1 & & \\ \swarrow & & \searrow \\ P^3 & & \mathbb{C}^4 \end{array}$$

self-dual strings  
hol. principal 2-bundle

CS & M Wolf, 1111.2539, 1205.3108

Note:

- The **Hyperplane twistor space**  $P^3$  is the total space of the line bundle  $\mathcal{O}(1,1) \rightarrow \mathbb{C}P^1 \times \mathbb{C}P^1$ .
- The spheres  $\mathbb{C}P^1 \times \mathbb{C}P^1$  parameterize an  $\alpha$ - and a  $\beta$ -plane.
- The span of both is a **hyperplane**.
- **Nonabelian** self-dual string equations:  $H = \star d_A \Phi$ ,  $F = \mathfrak{t}(B)$ .
- **Reduces nicely** to the monopole twistor space:  $\mathcal{O}(2) \rightarrow \mathbb{C}P^1$ .

New: Penrose-Ward transform for self-dual tensor multiplet.

$$\begin{array}{ccc} & \mathbb{C}^{6|16} \times \mathbb{C}P^3 & \\ & \swarrow \quad \searrow & \\ P^{6|4} & & \mathbb{C}^{6|16} \end{array}$$

non-abelian self-dual tensor multiplet  
hol. principal 2-bundle

CS & M Wolf, 1205.3108

Note:

- $P^{6|4}$  is a straightforward SUSY generalization of  $P^6$
- EOMs, abelian:  $H = \star H$ ,  $F = \mathfrak{t}(B)$ ,  $\nabla\psi = 0$ ,  $\square\phi = 0$
- $\mathcal{N} = (2,0)$  SC non-abelian tensor multiplet EOMs!
- EOMs on superspace, remain to be boiled down (expected).
- Non-gerby Alternatives: Chu, Samtleben et al., ...

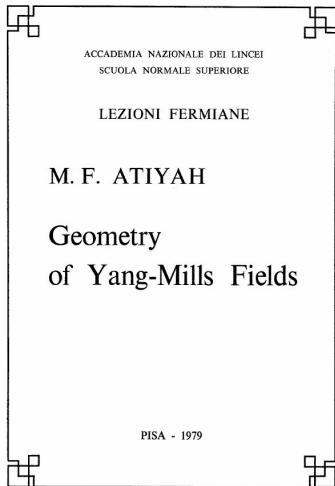
## Higher ADHM construction

Recall that the conventional ADHM and ADHMN constructions exist due to a twistor construction in the background.

Thus, there should be a direct ADHM-like construction here, too.



Translate all notions/results surrounding ADHM to higher gauge theory.



Translate this to higher gauge theory:

- Find **elementary solutions**
- Identify **moduli**
- Identify **topological charges**
- Higher **Serre-Swan theorem**
- Higher **ADHM** construction

Work in progress

F Sala & S Palmer & CS

# Elementary Solution: The Higher Instanton

The quaternionic form of Belavin et al.'s solution almost translates perfectly.

Recall the quaternionic form of the elementary instanton on  $S^4$ :

## Conformal geometry of $S^4$

Describe  $S^4$  by  $\mathbb{H} \cup \{\infty\}$ . Coordinates:  $x = x^1 + ix^2 + jx^3 + kx^4$ .  
Conformal transformations:

$$x \mapsto (ax + b)(cx + d)^{-1}, \quad a, b, c, d \in \mathbb{H}$$

SU(2)-Instanton:

$$A = \text{im} \left( \frac{\bar{x} dx}{1 + |x|^2} \right) \Rightarrow F = \text{im} \left( \frac{d\bar{x} \wedge dx}{(1 + |x|^2)^2} \right)$$

SU(2)-Anti-Instanton:

$$A = \text{im} \left( \frac{x d\bar{x}}{1 + |x|^2} \right) \Rightarrow F = \text{im} \left( \frac{dx \wedge d\bar{x}}{(1 + |x|^2)^2} \right)$$

Belavin et al. 1975, Atiyah 1979

# Elementary Solution: The Higher Instanton

The quaternionic form of Belavin et al.'s solution almost translates perfectly.

**Issue:**  $H = \pm \star H$  is sensible only on Minkowski space  $\mathbb{R}^{1,5}$ .

Recall:

- conformally compactify  $\mathbb{R}^4$ ,  $\mathbb{R}^{1,3}$  yields  $S^4$ ,  $M^c \cong S^1 \times S^3$ .
- Both  $S^4$  and  $M^c$  real slices of  $G_{2;4}$ , a quadric in  $\mathbb{C}P^5$ .

General pattern:

Conf. compact. of  $\mathbb{R}^{i,n-i} \rightarrow \mathbb{C}P^n$ : real slice of quadric in  $\mathbb{C}P^{n+1}$

This illuminates also the **conformal transformations**:

$$x = x^\mu \gamma_\mu \mapsto (ax + b)(cx + d)^{-1}$$

For certain elements  $a, d \in \mathcal{C}l_{\text{even}}(\mathbb{C}^n)$ ,  $b, c \in \mathcal{C}l_{\text{odd}}(\mathbb{C}^n)$ .

**Solution:** Quaternions have to be regarded as blocks of  $\mathcal{C}l(\mathbb{C}^4)$   
Work with blocks of the **Clifford algebra**  $\mathcal{C}l(\mathbb{C}^6)$ .

The quaternionic form of Belavin et al.'s solution almost translates perfectly.

Solution to the higher instanton equations  $H = \star H$ ,  $F = \mathfrak{t}(B)$ :

- Gauge structure:  $(\mathbb{C}^3 \otimes \mathfrak{sl}(4, \mathbb{C})) \xrightarrow{\mathfrak{t}} \mathfrak{sl}(4, \mathbb{C}) \oplus \mathfrak{sl}(4, \mathbb{C})$

$$\mathfrak{t} : h = \left( \begin{array}{c|c} h_1 & h_3 \\ \hline 0 & h_2 \end{array} \right) \mapsto \left( \begin{array}{c|c} h_1 & 0 \\ \hline 0 & h_2 \end{array} \right) \in \mathfrak{sl}(4, \mathbb{C}) \oplus \mathfrak{sl}(4, \mathbb{C}) ,$$

$h_1, h_2, h_3 \in \mathfrak{sl}(4, \mathbb{C})$ ,  $\triangleright$ : the usual commutator.

- Solution in coordinates  $x = x^M \sigma_M$ ,  $\hat{x} = x^M \bar{\sigma}_M$

$$A = \left( \begin{array}{cc} \frac{\hat{x} dx}{1+|x|^2} & 0 \\ 0 & \frac{dx \hat{x}}{1+|x|^2} \end{array} \right) \quad B = F + \left( \begin{array}{cc} 0 & \frac{\hat{x} dx \wedge d\hat{x}}{(1+|x|^2)^2} \\ 0 & 0 \end{array} \right)$$

$$F := dA + A \wedge A = \left( \begin{array}{cc} \frac{d\hat{x} \wedge dx}{(1+|x|^2)^2} + \frac{2 dx \hat{x} \wedge d\hat{x}}{(1+|x|^2)^2} & 0 \\ 0 & -\frac{dx \wedge d\hat{x}}{(1+|x|^2)^2} \end{array} \right)$$

$$H := dB + A \triangleright B = \left( \begin{array}{cc} 0 & \frac{d\hat{x} \wedge dx \wedge d\hat{x}}{(1+|x|^2)^3} \\ 0 & 0 \end{array} \right) \quad \text{but: Peiffer violated}$$

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## Summary:

- ✓ Generalized ADHMN-like construction on loop space
- ✓ Geometric quantization using loop space
- ✓ Gauge structures in M2- and M5-brane models similar
- ✓ Twistor construction of self-dual tensor fields
- ✓ 6d superconformal tensor multiplet equations
- ✓ On our way to develop Geometry of Higher Yang-Mills Fields

## Future directions:

- ▷ More general higher bundles and twistors with M Wolf
- ▷ Continue translation of ADHM with S Palmer, F Sala
- ▷ Geometric Quant. with higher Hilbert spaces with R Szabo

# Geometry of Higher Yang-Mills Fields

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