A New Representation of the Supersymmetric Fock Space by Using Supermathematics

by

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To My Parents,

who supported all my philosophical adventures.

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After clarifying the different approaches to supermathematics, we present an isomorphism between superfunctions and differential forms on supermanifolds and thus construct a new representation of the supersymmetric Fock Space. Furthermore, we examine an arbitrary one-dimensional Fermi system after canonical quantization and find that this is a Fermi oscillator. We define all the quantum mechanical tools (dual product, path integrals, extension to SUSY-oscillator, etc.) necessary to work with this system and compare our results with the approach of B.S. DeWitt.

Contents

Chapter 1

Introduction

"All pure natural sciences contain only as much proper science as mathematics can be applied in them."

Imanuel Kant

While the representation of bosonic quantum mechanical states by functions of the Hilbert Space $L_2(\mathbb{R})$ is quite clear, an analogous representation for fermionic states is not obvious. This is to be expected since there are no classical fermionic states as there are in the bosonic case, and thus we do not have a classical system for fermions to quantize.

But even if we want to construct a pseudo classical model, we face new difficulties. In a true classical system, the dynamical variables are real and thus commute, as the field of real numbers is abelian. Then canonical quantization means basically that the dynamical variables become operators, and the commutation relations between them are given (in first order of $\hbar$) by the Poisson brackets of the classical dynamical variables. Since the algebra of fermionic operators is given by anticommutation relations, we cannot start with commuting real variables, but we need objects which anticommute: $\theta^i\theta^j + \theta^j\theta^i = 0$.

This problem gave rise to several different approaches to an extension of complex

calculus called "Supermathematics". The most obvious approach was founded by F.A. Berezin, who simply introduced a set of Graßmann variables $\{\theta^i\}_i$ which anticommute and defined derivatives and integrals over these variables. Though this approach is followed by most physicists today, it has one disadvantage: while real variables represent an element of the field $\mathbb{R}$, Graßmann variables do not represent elements of a special set, but are rather elements of a ring themselves. A more logical step would be to construct an algebra Λ by interpreting the Graßmann variables as generators. This approach was followed by B.S. DeWitt, who developed a calculus over the elements of Λ, the supernumbers, rather than over Graßmann variables.
In both approaches we end up with a $\mathbb{Z}_2$-graded algebra, i.e. an algebra whose elements are even or odd, which has a lot of properties known from the algebraic formulation of supersymmetry, so that the next step is obviously to look for representations of the SUSY algebra in terms of supermathematics.
The simplest supersymmetric model is the SUSY oscillator. Since the Bose oscillator is already well known, it remains to consider the Fermi oscillator. We will repeat the representation proposed by B.S. DeWitt and then compare this to a new representation which is based on a new approach to supermanifolds by P. Cartier. The new representation of the Fermi oscillator is easily extended to the SUSY oscillator and from there to a supersymmetric Fock Space.
The second main focus of this thesis is to work out several aspects of the new approach of P. Cartier. The advantages of this approach compared to those of B.S. DeWitt and F.A. Berezin/T. Voronov are a simple construction of symmetric supermanifolds and a tensorial way of defining integration, which should be very useful for constructing path integrals over fermionic variables. We will show that the symmetric supermanifolds of P. Cartier are algebraically isomorphic to those of B.S. DeWitt, which simplifies a lot. Furthermore, we will find de Rham complexes in the set of differential forms and superfunctions on symmetric supermanifolds which

allows us to construct a new representation of a supersymmetric Fock space in terms of differential forms.

Since the work of P. Cartier is still in progress and several steps are not yet fully understood in mathematical rigour, this presentation is by no means complete and the true value of this approach cannot be estimated yet.

The structure of this thesis is as follows: In chapter one the reader is familiarized with the basics of supermathematics as it is done by T. Voronov and B.S. DeWitt. The second chapter repeats simple definitions of objects on ordinary manifolds and extends them on naive Graßmannian manifolds. Chapter three introduces the symmetric supermanifolds of P. Cartier and shows how to reduce an arbitrary supermanifold of B.S. DeWitt to these symmetric ones. In the fourth chapter we introduce terms of homology like chains, which can be regarded dual to differential forms, and de Rham complexes. Chapter five examines symmetric supermanifolds and shows the isomorphism between superfunctions and the de Rham complex of forms. This is used to construct a representation for a supersymmetric Fock space out of differential forms on supermanifolds. The sixth chapter deals with the discussion of the Fermi oscillator and its representation on symmetric supermanifolds. This representation is eventually extended to a SUSY oscillator and from there on to a supersymmetric Fock space.

Chapter 2

Introduction to Supermathematics

"The Lord created the natural numbers, everything else is the work of man."

Leopold Kronecker

Supermathematics has its historical roots in the need of anticommuting objects for describing fermionic fields in modern quantum field theory. The set of commuting (c-)numbers ($\mathbb{R}$, $\mathbb{C}$, etc.) is extended by anticommuting (a-)numbers and both together form the set of supernumbers.[1]

The known definitions of linear algebra and analysis can easily be extended to supernumbers, the definition of superforms and supermanifolds is less straightforward. The pioneer in constructing supermathematics was F.A. Berezin; B.S. DeWitt and A. Rogers proposed an alternative approach to supermanifolds, which will be followed here.

Unfortunately, both of these different approaches are needed for this thesis. The mathematically more appealing way of B.S. DeWitt is used in section 7.2 (Fermi

[1] The term "c-"number originally means "commuting" number and *not* complex number, so that there are no such objects as the "anticommuting c-numbers" mentioned by S. Weinberg.

oscillator according to B.S. DeWitt), while P. Cartier's new representation of the supersymmetric Fock space is formulated in a different way, which is followed by most physicists nowadays, worked out e.g. by Voronov.
At first, we will give an introduction to Graßmann variables and superspaces as done by Voronov, then we will present the basics of B.S. DeWitt's approach.

2.1 Graßmann Variables

Ordinary variables representing real or complex numbers commute, that means that[2] $x^i x^j = x^j x^i$ or $[x^i, x^j]_- = 0$. Though we can not write down the anticommuting equivalent of real or complex numbers, we can consider a set of anticommuting variables $\{\xi^i\}_i$:

Def. 2.1.1 *Let $\{\xi^i\}_{1\leq i\leq N}$ be a set of anticommuting variables, i.e. $\xi^i\xi^j = -\xi^j\xi^i$. These variables are called* **Graßmann variables**.

Because the Graßmann variables anticommute, obviously

$$(\xi^i)^2 = \xi^i\xi^i = -\xi^i\xi^i = 0. \tag{2.1}$$

Analytic functions of ordinary variables can be expanded in a Taylor series, e.g. for functions of one variable x: $f(x) = \sum_{i=0}^{\infty} c_i x^i$ where $c_i = f^{(i)}(x)/i!$. Because of (2.1), the corresponding Taylor series of N Graßmann variables is much simpler:

Remark 2.1.1 *A general analytic function of N Graßmann variables has the form:*

$$f(\xi^1, ..., \xi^N) = \sum_{i=0}^{N} \frac{1}{i!} c_{a_1...a_i} \xi^{a_1} ... \xi^{a_i} \tag{2.2}$$

[2]Note that in this thesis, the commutator brackets $[,]$ known from quantum mechanics will denote the supercommutator introduced below. A sign added to the brackets indicates a pure commutator $(-)$ or anticommutator $(-)$.

where the $c_{a_1 \ldots a_i}$ are complex numbers, and the sum over the a_i has to be taken according to Einstein's sum convention.[3] *The set of functions of N Graßmann variables will be denoted by $\mathcal{A}^N$.*

As on ordinary variables, we would like to introduce a derivative that works similarly to $\frac{\partial}{\partial x^\mu} x^\nu = \partial_\mu x^\nu = \delta_\mu^\nu$. We define:

Def. 2.1.2 *The left-derivative with respect to a Graßmann variable (acting from the left) is a linear operation, whose action is completely defined by:*

$$\frac{\overrightarrow{\partial}}{\partial \xi^\mu} \xi^\nu = \partial_\mu \xi^\nu := \delta_\mu^\nu. \tag{2.3}$$

A simple example of how this derivative works:

$$\partial_1 (1 + \xi^1 5 + 3\xi^2 \xi^1) = \partial_1 1 + \partial_1 \xi^1 5 + \partial_1 3 \xi^2 \xi^1 = 0 + 5 + \partial_1 3(-\xi^1 \xi^2) = 5 - 3\xi^2.$$

Note that the derivative and the corresponding Graßmann variables have to be next to each other, this is accomplished by commuting the variables. Considering $\partial_1 \partial_2 \xi^1 \xi^2 = \partial_1 \partial_2 (-\xi^2 \xi^1) = -1$ we easily see that:

Lemma 2.1.1 *The partial derivatives with respect to Graßmann variables anticommute: $\partial_\mu \partial_\nu = -\partial_\nu \partial_\mu$ and, because of this, are nilpotent: $(\partial_\mu)^2 = 0$.*

To calculate with these derivatives, we will need the modified Leibnitz-rule:

Lemma 2.1.2 *The Leibnitz-rule for derivatives with respect to Graßmann variables is:*

$$\frac{\partial}{\partial \xi^i} (\alpha \beta) = \left(\frac{\partial}{\partial \xi^i} \alpha \right) \beta + (-1)^{\tilde{\alpha}} \alpha \left(\frac{\partial}{\partial \xi^i} \beta \right). \tag{2.4}$$

Proof: If none or both α and β contain a ξ^i, then the product is zero. The first case is clear. For the second one we assume without losing generality that we can decompose $\alpha = \xi^i \alpha'$ and $\beta = \xi^i \beta'$. Our formula gives: $\partial_{\xi^i}(\alpha\beta) = \alpha'\beta + (-1)^{\tilde{\alpha}} \alpha \beta' =$

[3] *The factor $1/i!$ reflects the non-ordered indices $a_1 \ldots a_i$, and does not correspond to the $1/i!$ in the real Taylor series.*

$\alpha'\xi^i\beta' + (-1)^{\tilde{\alpha'}+1}\xi^i\alpha'\beta' = 0$. Now assume that only α contains a ξ^i. Then it is $\partial_{\xi^i}(\alpha\beta) = \left(\partial_{\xi^i}\alpha\right)\beta$, $\partial_{\xi^i}\beta = 0$, the formula is true. The remaining case is that only β contains a ξ^i: $\partial_{\xi^i}\alpha = 0$ and $\partial_{\xi^i}(\alpha\beta) = (-1)^{\tilde{\alpha}\tilde{\beta}}\partial_{\xi^i}(\beta\alpha) = (-1)^{\tilde{\alpha}\tilde{\beta}}(\partial_{\xi^i}\beta)\alpha = (-1)^{\tilde{\alpha}\tilde{\beta}}(-1)^{\tilde{\alpha}(\tilde{\beta}+1)}\alpha\partial_{\xi^i}\beta = (-1)^{\tilde{\alpha}}\alpha\partial_{\xi^i}\beta$.#

From this simple derivative, we can construct a general differential operator:

Remark 2.1.2 *An arbitrary differential operator can be written as*

$$P = \sum {c_L}^M \xi^L \partial_M. \tag{2.5}$$

Here the ${c_L}^M$ are complex numbers, L and M are multiindices, so ${c_L}^M = {c_{l_1\ldots l_i}}^{m_1\ldots m_j}$, $\xi^L = \xi^{l_1}\ldots\xi^{l_i}$ and $\partial_M = \partial_{m_1}\ldots\partial_{m_j}$. The expression has to be summed for all possible values of i and j ($0..N$ where N is the number of Graßmann variables) and the corresponding multiindices. So ${c_L}^M$ can be regarded as a complex $2^N \times 2^N$-matrix.

It can easily be shown that:

Lemma 2.1.3 *An arbitrary operator that maps a function of Graßmann variables on another such function is always a differential operator.*

Proof: (by construction): Given two functions of Graßmann variables f and g as in (2.2). Let h be the monomial of f that is of highest order in the ξ^i. There is obviously a differential operator of the form $P' = \alpha\partial_{m_1}\ldots\partial_{m_j}$ where α is a complex number and j the order of the monomial, that yields: $P'h = P'f = 1$. Now we define $P = gP'$, which is also a differential operator as it can be written in the form (2.5).#

The next step has to be the introduction of an integral of functions of Graßmann variables. As we do not have anticommuting numbers that can be plugged in for the variables, there will be nothing analoguous to the ordinary definite integral. Nevertheless, we can define an indefinite integral of a function $u(\xi^1, ..., \xi^N)$: $\mathrm{I}(u)$.

Because of lemma 2.1.3, we know that I will be a differential operator. We decide to take the simplest one that satisfies the following condition: $\forall i : \mathrm{I}(\partial_i u) = 0$.

Def. 2.1.3 *Given a function f of N Graßmann variables. The* **Berezin integral** $\mathrm{I}(f)$ *is defined by:*

$$\mathrm{I}(f) := Z\partial_1 ... \partial_N f \tag{2.6}$$

where Z is a complex constant. For algebraic calculations, Z is often set to 1, but to get similarly looking formulæ for the Fourier transform, one has to set[4] $Z = (+2\pi\mathrm{i})^{-1/2}$.

It is easily shown that this integral has the desired property: $\mathrm{I}(\partial_i u) = Z\partial_1 ... \partial_N \partial_i f = 0$, as exactly one of the nilpotent partial derivatives appears twice.

It follows immediately that $\partial_i \mathrm{I} = 0$ as well.

Given a function of Graßmann variables as in (2.2), then the Berezin integral is equal to the coefficient of the monomial of degree N:

$$u = u_0 + \sum u_i \xi^i + ... + u_{1...N} \xi^1 ... \xi^N \Rightarrow \mathrm{I}(u) = u_{1...N}. \tag{2.7}$$

The integral is often written as[5] $\int \mathrm{d}\xi^1 ... \mathrm{d}\xi^N f(\xi^1, ..., \xi^N)$. Then our definition of the Berezin integral is equivalent to

Remark 2.1.3 *The Berezin integral is a linear map, satisfying*

$$\int \mathrm{d}\xi^i 1 = 0 \text{ and } \int \mathrm{d}\xi^j \xi^i = Z\delta^{ij}. \tag{2.8}$$

The fact that the Berezin integral is defined by a differential operator gives rise to an unusual property: Consider a linear transformation of coordinates $\overline{\xi}^i = L^i{}_j \xi^j$ where L is a complex, non-singular $N \times N$-matrix. Then we obtain:

$$\frac{\partial}{\partial \overline{\xi}^i} = \frac{\partial}{\partial L^i{}_j \xi^j} = (L^{-1})_i{}^j \frac{\partial}{\partial \xi^j}. \tag{2.9}$$

[4] The "old" Z of B.S. DeWitt's will be explained later. In his new book, he has switched to this convention.

[5] We prefer to use the operator-notation for the integral.

This leads immediately to the transformation property of the integral:

$$\begin{aligned}\bar{\mathrm{I}}(u)/Z &= \frac{\partial}{\partial\bar{\xi}^1}...\frac{\partial}{\partial\bar{\xi}^N}u = (L^{-1})_1{}^{j_1}\frac{\partial}{\partial\xi^{j_1}}(L^{-1})_2{}^{j_2}\frac{\partial}{\partial\xi^{j_2}}...(L^{-1})_N{}^{j_N}\frac{\partial}{\partial\xi^{j_N}}u \\ &= \det(L^{-1})\frac{\partial}{\partial\xi^1}...\frac{\partial}{\partial\xi^N}u = \frac{1}{\det(L)}\frac{\partial}{\partial\xi^1}...\frac{\partial}{\partial\xi^N}u = \frac{1}{\det(L)}\mathrm{I}(u)/Z \quad (2.10)\end{aligned}$$

The determinant (which is an alternating multilinear form) appears because the partial derivatives anticommute and thus give rise to exactly the alternating multilinearity needed for the determinant.

The last aspect we have to consider for $\mathcal{A}^N$ is the Fourier transform. We define:

Def. 2.1.4 *The Dirac delta-function for Graßmann variables is defined by:*

$$\delta(\xi^i) := Z^{-1}\xi^i \text{ and } \delta = \delta(\xi^N...\xi^1) := Z^{-1}\xi^N...\xi^1 \quad (2.11)$$

This definition obviously preserves properties of the ordinary delta- function as: $\mathrm{I}(\delta) = 1$ and $\mathrm{I}(\delta f) = f(0)$ for $f \in \mathcal{A}^N$.

Considering the delta-function as the Fourier transform of the constant function 1, we can calculate Z:

$$\begin{aligned}\delta &= Z^{-1}\xi^i = \int \mathrm{d}\kappa \mathrm{e}^{2\pi\mathrm{i}\kappa\xi^i} = \int \mathrm{d}\kappa(1 + 2\pi\mathrm{i}\kappa\xi^i) \\ &= +2\pi\mathrm{i}\xi^i\int \mathrm{d}\kappa\kappa = +2\pi\mathrm{i}\xi^i Z. \quad (2.12)\end{aligned}$$

This equation fixes $Z = (+2\pi\mathrm{i})^{-1/2}$.

$\mathcal{A}^N$ can be extended to functions of real and Graßmann variables. Those functions are functions on the space $\mathbb{R}^{(n|\nu)}$:

Def. 2.1.5 $\mathbb{R}^{(n|\nu)}$ *is defined by the functions that can be defined on this space. These functions have the shape:*

$$F(x^1,...,x^n,\xi^1,...,\xi^\nu) = \sum_{i=0}^{\nu}\frac{1}{i!}c_{a_1...a_i}(x^1,...,x^n)\xi^{a_1}...\xi^{a_i} \quad (2.13)$$

The differences to B.S. DeWitt's space $\mathbb{R}^n_c \times \mathbb{R}^\nu_a$ will be discussed in section 4.1.

2.2 B.S. DeWitt's Formulation

In this section, we present shortly B.S. DeWitt's approach to supermathematics. Instead of using $\{\xi^i\}_i$ as anticommuting variables, he uses them as generators for an algebra, the ring of supernumbers Λ_N, which extends the field $\mathbb{C}$ to a ring with commuting and anticommuting numbers.

2.2.1 Supernumbers

The set of supernumbers is the algebra (a vector space with a vector-vector multiplication) generated by Graßmann variables:

Def. 2.2.1 *Let $\xi^\alpha, \alpha = 1, ..., N$ be a set of Graßmann variables. The algebra generated by this set is called a* **Graßmann algebra** *and will be denoted by Λ_N. For an infinite number of generators we will write Λ_∞. The Elements of Λ_N or Λ_∞ are called* **supernumbers**.

Note that here Λ_N equals $\mathcal{A}^N$ in the first approach. *So a supernumber is basically a function of N Graßmann variables.*
A basis of the vector space Λ_N is given by the set $\{1, \xi^1, ..., \xi^N, \xi^1\xi^2, ..., \xi^1...\xi^N\}$. Each generator can appear to the 0-th or to the 1-th power in a basis vector, i.e. there are two possibilities for each of the N generators, so the dimension of the vector space Λ_N is 2^N, but $\dim(\Lambda^\infty) = \aleph_0$ as there is obviously a bijection between the basis vectors and all the finite subsets of the natural numbers.
A supernumber z can be decomposed in

$$z = z_B + z_S \tag{2.14}$$

where z_B (the **body**) is an ordinary complex number and z_S (the **soul**):

$$z_S = \sum_{n=1}^{N/\infty} \frac{1}{n!} c_{a_1...a_n} \xi^{a_n} ... \xi^{a_1} \tag{2.15}$$

where the $c_{a_1...a_n}$ are complex numbers and antisymmetric under the exchange of two neighbouring indices.[6] Each of the $a_n, ..., a_1$ runs from $1..N$, and the expressions have to be summed up. Sometimes it is more convenient to have the $a_n...a_1$ ordered and get rid of the $\frac{1}{n!}$. In this case, capital letters will indicate the ordered sequence $A_n...A_1$.

Consider z_S^{N+1} for the soul of a supernumber $z \in \Lambda_N$. Each of the monomials in z_S^{N+1} has total power $N+1$ in all the ξ^i, and it is not possible to distribute $N+1$ powers on N generators without one having power two. Because of (2.1), this means $z_S^{N+1} = 0$:

Lemma 2.2.1 *For $z \in \Lambda_N$ it follows that $z_S^{N+1} = 0$. If $\xi^a z = 0$ for all a then $z = c\xi^1...\xi^N$ for $z \in \Lambda_N$ and $z = 0$ for $z \in \Lambda_\infty$.*

Theorem 2.2.1 *Let z be a supernumber with $z_B \neq 0$. Then its inverse is uniquely given by:*

$$z^{-1} = z_B^{-1} \sum_{n=0}^{N/\infty} \left(-\frac{z_S}{z_B}\right)^n. \tag{2.16}$$

Note that this sum is always defined, as it is actually not an infinite sum. Let M be the number of different Graßmann variables in z_S, then all powers with $N \geq M$ will vanish anyway.

We want to check that $zz^{-1} = 1$. Let M again be the number of different Graßmann variables in z_S.

$$\begin{aligned} zz^{-1} &= (z_B + z_S) z_B^{-1} \sum_{n=0}^{N/\infty} \left(-\frac{z_S}{z_B}\right)^n = \left(1 + \frac{z_S}{z_B}\right)\left(1 - \frac{z_S}{z_B} + \frac{z_S}{z_B}^2 - \frac{z_S}{z_B}^3 ... \frac{z_S}{z_B}^M\right) \\ &= 1 - \frac{z_S}{z_B} + \frac{z_S}{z_B} + \frac{z_S}{z_B}^2 - \frac{z_S}{z_B}^2 ... \pm \frac{z_S}{z_B}^M \mp \frac{z_S}{z_B}^M \pm \frac{z_S}{z_B}^{M+1} = 1 \end{aligned} \tag{2.17}$$

The terms $\frac{z_S}{z_B}^i$ cancel each other. The last term vanishes because of lemma (2.2.1). It is not possible to define the inverse of any nilpotent object (especially not for

[6] Note the similarity of $c_{a_1...a_n}\xi^{a_n}...\xi^{a_1}$ and an n-form.

supernumbers without body), as we would lose the associativity of multiplication:

$$z^{n-1} \neq z^n = 0 \quad \Rightarrow \quad z^{-1}z^n = 0 \neq z^{n-1} = (z^{-1}z)z^{n-1}$$

Since supernumbers do not have an inverse, if their body is vanishing, they do not form a field, but only a ring. This implies problems in generalizing objects as vector spaces to supernumbers. A supervector space will be a module over a ring rather than a module over a field.

As supernumbers introduce anticommuting objects beside the ordinary, commuting ones, we have to define terms for handling the parity of supernumbers.

Def. 2.2.2 *A* **c-number** u *commutes with every other number* z*:* $uz - zu = 0$*, an* **a-number** v_1 *anticommutes with every other a-number* v_2*:* $v_1v_2 + v_2v_1 = 0$*.* $\mathbb{C}_c$ *is the set of all c-numbers,* $\mathbb{C}_a$ *the set of all a-numbers.*

A supernumber z can obviously be uniquely decomposed in a commuting supernumber $u \in \mathbb{C}_c$ and an anticommuting one $v \in \mathbb{C}_a$:

$$z = \underbrace{z_B + \sum_{n=1}^{\infty} \frac{1}{(2n)!} c_{a_1 \ldots a_{2n}} \xi^{a_{2n}} \ldots \xi^{a_1}}_{u} + \underbrace{\sum_{n=1}^{\infty} \frac{1}{(2n+1)!} c_{a_1 \ldots a_{(2n+1)}} \xi^{a_{2n+1}} \ldots \xi^{a_1}}_{v} \tag{2.18}$$

Such a decomposition is given e.g. by:

$$z = \underbrace{2 + 4\xi^3\xi^4}_{\in \mathbb{C}_c} + \underbrace{3\xi^1 + 7\xi^1\xi^3\xi^4}_{\in \mathbb{C}_a} \tag{2.19}$$

So Λ_N is decomposed in $\mathbb{C}_c$ and $\mathbb{C}_a$ which have both dimensions 2^{N-1}: $\mathbb{C}_c \oplus \mathbb{C}_a = \Lambda_N$. We can distinguish between c-numbers, a-numbers and supernumbers of mixed type. As we will often have to include factors of (-1) in our formulæ depending on the type of numbers plugged in, we introduce a parity function which allows to write $(-1)^{\tilde{z}}$.

Def. 2.2.3 *The symbol* $\tilde{\mathbf{z}}$ *indicates the parity of a supernumber z:*

$$\tilde{z} = \begin{cases} 1 & \text{if } z \in \mathbb{C}_a \\ 0 & \text{if } z \in \mathbb{C}_c \\ \text{undefined} & \text{otherwise (mixed type)} \end{cases} \tag{2.20}$$

From now on, objects constructed from supernumbers will be characterized by their type. An **a-type object** *has parity 1, a* **c-type object** *parity 0.*

Now we can introduce the general supercommutator, valid for all kinds of superobjects, which replaces the ordinary commutator and anticommutator brackets.

Def. 2.2.4 *The supercommutator of two objects created from supernumbers is*

$$[A, B] := AB - (-1)^{\tilde{A}\tilde{B}} BA \tag{2.21}$$

For pure supernumbers A, B it is always $[A, B] = 0$. For two supervectors (see below) v_1, v_2, the commutator $[v_1, v_2^{\sim}]$ is used, where $v_2^{\sim}$ denotes the transpose of v_2. We easily obtain the following results:

$$\begin{aligned} [A, B] &= AB - (-1)^{\tilde{A}\tilde{B}} BA = -(-1)^{\tilde{A}\tilde{B}} (BA - (-1)^{\tilde{A}\tilde{B}} AB) \\ &= (-1)^{\tilde{A}\tilde{B}+1} [B, A] \qquad (2.22) \\ [A, B + C] &= AB + AC - (-1)^{\tilde{A}\widetilde{(B+C)}} (BA + CA) = [A, B] + [A, C] \qquad (2.23) \end{aligned}$$

The last formula is obvious, if B and C are of the same type, otherwise, we just define the commutator bracket to satisfy this condition.
Finally, we would like to distinguish between real and imaginary supernumbers. We define:

Def. 2.2.5 *The generators* ξ^a *are real, that is* $(\xi^a)^* = \xi^a$.

Furthermore we need rules for complex conjugation of supernumbers:

Lemma 2.2.2 *Analogue to matrix calculation we obtain the rules:*

$$\begin{aligned} \left(z+z'\right)^* &= z^*+z'^* && (2.24)\\ \left(zz'\right)^* &= z'^*z^* && (2.25)\\ \textit{and particularly: } \left(\xi^{a_1}...\xi^{a_n}\right)^* &= \xi^{a_n}...\xi^{a_1} && (2.26)\\ &= (-1)^{n(n-1)/2}\xi^{a_1}...\xi^{a_n}. && (2.27) \end{aligned}$$

The simplest definition for real and imaginary supernumbers is:

Def. 2.2.6 *A supernumber z is called real, if $z^* = z$ and imaginary if $z^* = -z$.*

A supernumber is obviously real iff body and soul are real. The soul is real iff the coefficients in (2.15) are real for $\frac{1}{2}n(n-1)$ even and imaginary otherwise. (This is a result from the inverted order of the generators after complex conjugation. The minus sign from the complex conjugation of the imaginary coefficient is compensated by the minus from the reordering of the generators.)

Def. 2.2.7 *The real supernumbers $\mathbb{R}_c$ and $\mathbb{R}_a$ are the subsets of the real elements of $\mathbb{C}_c$ and $\mathbb{C}_a$, respectively.*

Note that $\mathbb{R}_c$ is a subalgebra, while this is not the case for $\mathbb{R}_a$. It is sufficient to regard single components of a supernumber as in (2.15). Let c and c' be coefficients for two real c-numbers ($\in \mathbb{R}_c$) for monomials of nth and n'th order resp., so n and n' have to be even. Furthermore c is real if n is a multiple of 4 and imaginary otherwise, the same is true for c'. If we multiply the coefficients, we get a coefficient for a monomial of degree $n+n'$ which will be real if both or none of n and n' where a multiple of 4 and imaginary else. By analogous consideration, it is easy to show that the product of a real a-number and a real c-number is an real a-number, while the product of two real a-numbers is an *imaginary* c-number.
An alternative definition for real supernumbers is: supernumbers, where all the coefficients are real. This definition would imply that both $\mathbb{R}_c$ and $\mathbb{R}_a$ would be

subalgebras.
Since complex conjugation plays a big role in quantum mechanics, we keep the first definition.

2.2.2 Vectorspaces of Supernumbers

Usually, a vector space is defined to be a module over a field, i.e. an abelian group G together with a field F and a scalar multiplication $G \times F \to G$.
Working with supernumbers, we have no field but only a ring. B.S. DeWitt solves this problem by defining a supervector space as a module over a ring. Another approach is followed by T. Voronov, who represents a supervector space $\mathbb{R}^{n|\nu}$ by n ordinary variables $(x^i)_i$ and ν Graßmannian variables $(\xi^i)_i$ as coordinates. A function on such a supervector space for fixed ordinary coordinates is a supernumber in the formalism of B.S. DeWitt, which obviously contains T. Voronov's approach as a special case.

Supervector Spaces

Def. 2.2.8 *A* **supervector space** *is a module over the ring of supernumbers: Let Λ a Graßmann algebra and σ be an abelian group with maps $\sigma \times \sigma \to \sigma$, $\Lambda \times \sigma \to \sigma$ and $\sigma \to \sigma$ with the following properties:*

1. *$(\sigma, +)$ is an abelian group:*
   $(\forall\ X, Y \in \sigma) : (+(X, Y) := X + Y = Y + X)$ *satisfying*
   - *(a) $(+)$ is associative*
   - *(b) existence of a neutral element:* $(\exists\ \mathbf{0})(\forall\ X \in \sigma) : (X + \mathbf{0} = X)$
   - *(c) existence of an inverse element:* $(\exists\ -X \in \sigma) : (-X + X = \mathbf{0})$
2. *Multiplication of a supervector with a supernumber:*
   $(\exists\ \alpha_L, \alpha_R : \Lambda \times \sigma \to \sigma) : (\alpha_L(X) = \alpha X,\ \alpha_R(X) = X\alpha)$ *satisfying*

(a) *both maps bilinear*

(b) *normalization:* $(\forall\ X \in \sigma) : 1X = X,\ X1 = X$

(c) *associativity:* $(\forall\ X \in \sigma, \alpha, \beta \in \Lambda) : (\alpha X)\beta = \alpha(X\beta) = \alpha X\beta$

(d) *multiplication with c-number:* $(\forall\ X \in \sigma, \alpha \in \Lambda) : \alpha X = X\alpha$ *for* $\alpha \in \mathbb{C}_c$

(e) *decomposition in an* **even part** *U and an* **odd part** *V:*
$(\forall\ X \in \sigma)(\exists$ unique $U, V \in \sigma)(\forall\ \alpha \in \mathbb{C}_a) : (X = U + V,\ \alpha U = U\alpha,$
$\alpha V = -V\alpha)$.

3. *Complex conjugation of a supervector:*
$(\exists\ ^* : \sigma \to \sigma)$ *with* $(\forall\ X \in \sigma) :\ ^*(X) := X^*$ *and*

(a) $X^{**} = X$

(b) $(X + Y)^* = X^* + Y^*$

(c) $(\alpha X)^* = X^*\alpha^*$ *and* $(X\alpha)^* = \alpha^* X^*$

Then σ *is called a* **supervector space**.

We obtain immediately obvious results in supervector spaces as $0X = X0 = \mathbf{0}$ for a supervector X and $\alpha\mathbf{0} = \mathbf{0}\alpha = \mathbf{0}$ for a supernumber α, where $\mathbf{0}$ is the neutral element of the abelian group σ, the "zero supervector".

Bases of Supervector Spaces

The definition of a basis follows closely the ordinary one:

Def. 2.2.9 *A set* $\{_i e\}$ *is called* **basis** *for a supervector space* σ *iff it is linear independent:* $c^i\ {}_i e = 0 \Rightarrow c^i = 0$ *and every supervector* $X \in \sigma$ *can be constructed by linear combination:* $(\exists\ X^i \in \Lambda) : (X = X^i\ {}_i e)$.

Einstein's summation convention is modified. Instead of two different positions for indices there are four in the case of supermathematics. Sums are taken over

indices which appear in upper-lower **and** left-right position only, so i.e.: in $X^i{}_i e$ and ${}^i X e_i$ the sum over i is performed but not in $X^i e_i$. Otherwise, index dependent powers of (-1) must be introduced.

Note that "linear independence" as used in the definition for the basis has a slightly different meaning in the case of supervector spaces. A vector X can be linearly independent to a vector Y but become linearly dependent by multiplication with an odd supernumber: vX. Now the linear combination $v(vX)+0Y=0$ is zero without forcing all the coefficients to vanish.

Def. 2.2.10 *A basis is called* **pure***, if it consists only of m c-type and n a-type supervectors.*

The dimension of a vector space with an (m,n)-basis is $d=m+n$. Every finite basis $\{{}_i e\}$ can be transformed into a pure basis $({}_j f, {}_k g)$ where ${}_j f$ are c-type vectors and ${}_k g$ a-type vectors. This transformation is given by unique matrices with ${}_j f = {}_j M^i{}_i e$ and ${}_k g = {}_k N^i{}_i e$. Here, ${}_j M^i$ is obviously an $m, m+n$ and ${}_k N^i$ an $n, m+n$ matrix. Analogue to the parity of supernumbers, we define:

Def. 2.2.11 *Given a supervector $X=U+V$ where U the even part and V the odd part. Then* **parity symbol** $\tilde{X}$ *indicates the parity of the supervector X:*

$$\tilde{X} = \begin{cases} 1 & \text{if } U=0 \\ 0 & \text{if } V=0 \\ \text{undefined} & \text{otherwise} \end{cases} \tag{2.28}$$

The **parity symbol** $\tilde{i}$ *(in a supervector space with dim (m,n)) indicates the parity of the pure basisvector with index i:*

$$\tilde{i} = {}_{\tilde{i}}e = \begin{cases} 1 & \text{if } i \le m \\ 0 & \text{if } i > m \end{cases} \tag{2.29}$$

A special basis which is very useful for calculating complex conjugates is the standard basis:

Def. 2.2.12 *A pure basis where the c-type supervectors are real and the a-type supervectors are imaginary is called* **standard basis**.

A standard basis is characterized by ${}_i e^* = (-1)^{\tilde{i}}{}_i e$. We get for a real supervector: $X^i{}_i e = (-1)^{\tilde{i}}{}_i e X^{i*}$ and $X^{i*} = (-1)^{\tilde{i}\tilde{X}} X^i$.
Working with supervectors and the supermatrices introduced in the next section requires several rules for the bookkeeping of indices as e.g. for shifting indices, of which the first is:[7]

Def. 2.2.13 (Shifting indices) *Let X be a pure supervector and ${}^i X$, X^i coordinates in a pure basis. Then*

$$ {}^i X := (-1)^{\tilde{i}\tilde{X}} X^i \tag{2.30}$$

defines a possible shifting convention.

Shifting indices for impure supervectors is certainly given by separate shifts of the pure parts.

Dual Supervector Spaces

A dual space is the set of all linear functionals of the original space, i.e. linear functions mapping to the field or ring the original space is constructed over.
As in the ordinary case, there is always a dual supervector space for each finite dimensional supervector space:

Def. 2.2.14 *Let σ be a supervector space. The space* **dual** *to σ, denoted by σ^* is the set of all linear mappings $\omega : \sigma \to \Lambda$ with the notation $\omega(X) = X\omega$ (dual product).*

The dual space can be extended to a supervector space by the rules:

[7] A table which summarizes all the shifting conventions can be found at the end of this section.

Def. 2.2.15 *Let σ^* be a dual space to a supervector space σ. Let $X \in \sigma$ and $\alpha \in \Lambda$ and $\omega, \tau \in \sigma^*$. We define:*

$$X(\omega + \tau) := X\omega + X\tau \tag{2.31}$$

$$X(\alpha\omega) := (X\alpha)\omega \tag{2.32}$$

$$X(\omega\alpha) := (X\omega)\alpha := X\omega\alpha. \tag{2.33}$$

As for finite dimensional linear spaces the dual space is isomorphic to the original space, we obtain a basis by the following theorem:

Theorem 2.2.2 *Let ${}_ie$ be a basis of σ. Then a basis e^i of its dual σ^* is uniquely defined by ${}_ie e^j = {}_i\delta^j$. An element $\omega \in \sigma^*$ is uniquely fixed by its action on the basis supervectors: ${}_i\omega := {}_ie\omega$ and $\omega = e^i\ {}_i\omega$.*

The dual of a pure basis is pure, the dual of a standard basis is a standard basis for the dual with $(e^i)^* = e^i$. If σ has dimensions (m,n), σ^* has same dimensions (m,n). In the dual case, we get the following shifting conventions:

Def. 2.2.16 *A* **shifting convention for a dual supervector** *ω is given by:*

$$\omega_i := (-1)^{\tilde{i}(\tilde{\omega}+1)}{}_i\omega, \tag{2.34}$$

which yields $\omega_i = {}_i\omega^*$ The alternative dual product is defined by:

$$\omega X := (-1)^{\tilde{X}\tilde{\omega}} X\omega \tag{2.35}$$

with the further rules:

$$\omega(X + Y) := \omega X + \omega Y \tag{2.36}$$

$$(\omega + \tau)X := \omega X + \tau X \tag{2.37}$$

forcing the shifting convention:

$$e_i := (-1)^{\tilde{i}}{}_ie,\ {}^ie := e^i. \tag{2.38}$$

Linear Transformations in Supervector Spaces

Linear transformations in supervector spaces are given by left-multiplication of supervectors with supermatrices as in the ordinary case:

Def. 2.2.17 *A* **supermatrix** *is a table of supernumbers. The* **body** *of a supermatrix is the table of the supermatrix element's bodies, the* **soul** *is the remainder.*

It is easy to see that a square supermatrix has an inverse (or is nonsingular) iff its body is nonsingular.

To see how a linear transformation of a basis works, consider the following:

Let $\{{}_ie\}$ and $\{{}_i\overline{e}\}$ be two bases of a supervector space σ, related by ${}_i\overline{e} = {}_i(K^{-1})^j{}_je$. Then a supervector $X = X^i\,{}_ie$ is transformed into $X = \overline{X}^i\,{}_i\overline{e}$ with $\overline{X}^i = X^j\,{}_jK^i$. Let

$$K = \begin{pmatrix} A & C \\ D & B \end{pmatrix} \tag{2.39}$$

If $\{{}_ie\}$ and $\{{}_i\overline{e}\}$ are pure bases, then the elements of A and B are c-type, the elements of C and D a-type. This can be seen from:

$$\begin{pmatrix} \oplus & \ominus & \ominus \\ \ominus & \oplus & \oplus \\ \ominus & \oplus & \oplus \end{pmatrix} \cdot \begin{pmatrix} \ominus \\ \oplus \\ \oplus \end{pmatrix} = \begin{pmatrix} \ominus \\ \oplus \\ \oplus \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} \ominus & \oplus & \oplus \\ \oplus & \ominus & \ominus \\ \oplus & \ominus & \ominus \end{pmatrix} \cdot \begin{pmatrix} \ominus \\ \oplus \\ \oplus \end{pmatrix} = \begin{pmatrix} \oplus \\ \ominus \\ \ominus \end{pmatrix}$$

where $\oplus$ denotes a c-type element and $\ominus$ an a-type element.

If $\{{}_ie\}$ and $\{{}_i\overline{e}\}$ are standard bases, then A and B are real and C and D is imaginary.

Finally, we again have to define a parity for supermatrices, which is a little more complicated than that for supernumbers and supervectors:

Def. 2.2.18 *Let K be a Matrix as in 2.39. The* **parity symbol** *$\tilde{K}$ indicates the parity of the supermatrix K:*

$$\tilde{K} = \begin{cases} 1 & \text{if } \widetilde{A_{ij}} = \widetilde{B_{kl}} = 0 \text{ and } \widetilde{C_{ij}} = \widetilde{D_{kl}} = 1 \\ 0 & \text{if } \widetilde{A_{ij}} = \widetilde{B_{kl}} = 1 \text{ and } \widetilde{C_{ij}} = \widetilde{D_{kl}} = 0 \\ \textit{undefined} & \textit{otherwise} \end{cases} \tag{2.40}$$

As ordinary matrices, supermatrices can be transposed, but the parity of the elements has to be considered:

Def. 2.2.19 *The* **supertranspose** $K^{\sim}$ *of a supermatrix* K *is given by:*

$$ {}^{i}K^{\sim}_{j} := (-1)^{\tilde{j}(\tilde{i}+\tilde{j})}\, {}_{j}K^{i}. \tag{2.41} $$

For a matrix K in the block form (2.39), it follows:

$$ K^{\sim} = \begin{pmatrix} A^T & -D^T \\ C^T & B^T \end{pmatrix} \tag{2.42} $$

where A^T etc. denotes the ordinary matrix transposition.

The supertranspose of a product of two supermatrices is, as in the ordinary case $(KL)^{\sim} = L^{\sim}K^{\sim}$. Furthermore, supertransposition commutes with inversion: $(K^{\sim})^{-1} = (K^{-1})^{\sim}$.

We obtain further index shifting conditions by demanding e.g. that $K^{\sim\sim} = K$. From those, all others can be derived (see table at the end of this section).

Def. 2.2.20 Further index shifting rules *are defined by:*

$$ {}_{i}L^{\sim j} := (-1)^{\tilde{i}(\tilde{i}+\tilde{j})}\, {}^{j}L_{i} \tag{2.43} $$

$$ {}_{i}M^{\sim}_{j} := (-1)^{\tilde{i}+\tilde{j}+\tilde{i}\tilde{j}}\, {}_{j}M_{i} \tag{2.44} $$

$$ {}^{i}N^{\sim j} := (-1)^{\tilde{i}\tilde{j}}\, {}^{j}N^{i} \tag{2.45} $$

and

$$ K_{i}{}^{j} := (-1)^{\tilde{i}}{}_{i}K^{j} \tag{2.46} $$

$$ L^{i}{}_{j} := {}^{i}L_{j} \tag{2.47} $$

$$ M_{ij} := (-1)^{\tilde{i}}{}_{i}M_{j} \tag{2.48} $$

$$ N^{ij} := {}^{i}N^{j} \tag{2.49} $$

The Supertrace and the Superdeterminant

B.S. DeWitt introduces the superdeterminant through its relation with the supertrace. We know from ordinary linear algebra that $\delta \ln \det(A) = \mathrm{tr}(A^{-1}\delta A)$. The analogue law in the super case together with fixing the superdeterminant for the unit supermatrix defines the superdeterminant.
So at first, we need a sensible definition of a supertrace:

Def. 2.2.21 *The* **supertrace** *is (only) defined for supermatrices with one upper and one lower index:*

$$ {}_iK^j : \mathrm{str}(K) := (-1)^{\tilde{i}}\ {}_iK^i = K_i{}^i \tag{2.50}$$

$$ {}^iL_j : \mathrm{str}(L) := (-1)^{\tilde{i}}\ {}^iL_i = (-1)^{\tilde{i}} L^i{}_i. \tag{2.51}$$

From this definition we get some special cases: The supertrace of the unit matrix is obviously $\mathrm{str}(\mathbf{1}_{(m,n)}) = \mathrm{tr}(\mathbf{1}_m) - \mathrm{tr}(\mathbf{1}_n)$. The supertrace of a matrix and its supertranspose are identical: $\mathrm{str}(K^{\sim}) = \mathrm{str}(K)$. As in the ordinary case, for the supertrace of a product of two supermatrices, the order does not matter: $\mathrm{str}(MN) = \mathrm{str}(NM)$ and the supertrace is additive: $\mathrm{str}(M+N) = \mathrm{str}(M) + \mathrm{str}(N)$. Given a supermatrix K as in (2.39), its supertrace is $\mathrm{str}(K) = \mathrm{tr}(A) - \mathrm{tr}(B)$.
Now we can continue with the definition of the superdeterminant:

Def. 2.2.22 *The* **superdeterminant** *is defined by*

$$\delta \ln \mathrm{sdet}(M) := \mathrm{str}(M^{-1}\delta M) \tag{2.52}$$

together with the boundary condition

$$\mathrm{sdet}(\mathbf{1}_{(m,n)}) := 1. \tag{2.53}$$

The superdeterminant will turn out to be a c-number. The ln of this c-number is the continuation on Λ of the complex ln, see section (2.2.3)

The properties of the superdeterminant are again quite similar to the ordinary case. Given two supermatrices K and M with n the number of odd dimensions of M, then we get the rules:

$$\begin{aligned} \mathrm{sdet}(KM) &= \mathrm{sdet}(K)\mathrm{sdet}(M) && (2.54)\\ \mathrm{sdet}(M^{\sim}) &= \mathrm{sdet}(M), \text{ if } M \in \{{}_iM^j, {}^iM_j\} && (2.55)\\ \mathrm{sdet}(M^{\sim}) &= (-1)^n\mathrm{sdet}(M), \text{ if } M \in \{{}^iM^j, {}_iM_j\} && (2.56) \end{aligned}$$

In the special case, where K is a supermatrix as in (2.39), its superdeterminant is given by:

$$\mathrm{sdet}(K) = \det(A - CB^{-1}D)(\det(B))^{-1} \tag{2.57}$$

We see that the determinant is no longer a simple polynomial but a rational function, as the contribution of B in the superdeterminant appears in the denominator.

Altogether, we have the following rules for shifting, complex conjugation etc:

	definitions	implications
sbasis	$e_i := (-1)^{\tilde{i}}{}_ie$ ${}^ie := e^i$	std basis: ${}_ie^* = (-1)^{\tilde{i}}{}_ie$ $e^{i*} = e^i$
svectors	${}^iX := (-1)^{\tilde{X}\tilde{i}}X^i$ $\omega_i := (-1)^{\tilde{i}(\tilde{\omega}+1)}\,{}_i\omega$	real X: $X^{i*} = (-1)^{\tilde{i}\,\tilde{X}}X^i$ ${}_i\omega^* = (-1)^{\tilde{i}(\tilde{\omega}+1)}\,{}_i\omega$ ${}_i\omega^* = \omega_i$

	definitions	implications
smatrices	${}_iL^{\sim j} := (-1)^{\tilde{i}(\tilde{i}+\tilde{j})}\,{}^jL_i$ ${}_iM_j^{\sim} := (-1)^{\tilde{i}+\tilde{j}+\tilde{i}\tilde{j}}\,{}_jM_i$ ${}^iN^{\sim j} := (-1)^{\tilde{i}\tilde{j}}\,{}^jN^i$ $K_i{}^j := (-1)^{\tilde{i}}\,{}_iK^j$ $L^i{}_j := {}^iL_j$ $M_{ij} := (-1)^{\tilde{i}}\,{}_iM_j$ $N^{ij} := {}^iN^j$	$K^{\sim i}{}_j = (-1)^{\tilde{i}\tilde{j}}\,K_j{}^i$ $L^{\sim}{}_i{}^j = (-1)^{\tilde{i}\tilde{j}}\,L^j{}_i$ $M^{\sim}{}_{ij} = (-1)^{\tilde{i}\tilde{j}}\,M_{ji}$ $N^{\sim ij} = (-1)^{\tilde{i}\tilde{j}}\,N^{ji}$
strace	${}_iK^j : \mathrm{str}(K) := (-1)^{\tilde{i}}\,{}_iK^i$ ${}^iL_j : \mathrm{str}(L) := (-1)^{\tilde{i}}\,{}^iL_i$	$\mathrm{str}(\mathbf{1}_{(m,n)}) = tr(\mathbf{1}_m) - tr(\mathbf{1}_n)$ $\mathrm{str}(K^{\sim}) = \mathrm{str}(K)$ $\mathrm{str}(MN) = \mathrm{str}(NM)$ $\mathrm{str}(M+N) = \mathrm{str}(M) + \mathrm{str}(N)$ $\mathrm{str}(K) = tr(A) - tr(B)$
sdet	$\delta \ln \mathrm{sdet}(M) := \mathrm{str}(M^{-1}\delta M)$ $\mathrm{sdet}(\mathbf{1}_{(m,n)}) := 1$	$\mathrm{sdet}(KM) = (\mathrm{sdet}(K))(\mathrm{sdet}(M))$ $\mathrm{sdet}(M^{\sim}) = \mathrm{sdet}(M), \{{}_iM^j, {}^iM_j\}$ $\mathrm{sdet}(M^{\sim}) = (-1)^n \mathrm{sdet}(M), \{{}^iM^j, {}_iM_j\}$

2.2.3 Functions on Supernumbers

After the linear algebra is defined for the supercase, we want to proceed with a super-analysis. At first, we will define superfunctions by "analytic continuation" of complex functions. Afterwards, the derivative and the integral of such functions are introduced, which will be an extension of derivatives and integrals of Graßmann variables as already introduced in the first part of this chapter.

Remark 2.2.1 *Given an analytic function $f : \mathbb{C} \to \mathbb{C}$ and a Graßmann algebra Λ_∞. An* **analytic continuation of f on** Λ *is given by:*

$$f(z) = \sum_{n=0}^{\infty} \frac{1}{n!} f^{(n)}(z_B) z_S^n. \tag{2.58}$$

where $f^{(n)}$ is the ordinary n-th derivative of the complex function f.

We want to introduce a derivative for superanalytic functions. Therefore we have to examine the variation of a function's value due to variation of the argument:

Def. 2.2.23 *Let Λ_∞ be a Graßmann algebra, f a function $f : \Lambda_\infty \to \Lambda_\infty$. f is called* **superanalytic**, *if it satisfies*

$$\mathrm{d}f(z) = \mathrm{d}z \left[\frac{\overrightarrow{\mathrm{d}}}{\mathrm{d}z} f(z)\right] = \left[f(z)\frac{\overleftarrow{d}}{\mathrm{d}z}\right] \mathrm{d}z. \tag{2.59}$$

Certainly, all the on Λ_∞ continued complex functions as in rem.(2.2.1) are always superanalytic.

From the definition of a superanalytic function, we get two general solutions for the cases where the domain of the function is pure.

Theorem 2.2.3 *A general superanalytic function $f : \mathbb{C}_a \to \Lambda_\infty$ is a linear function:*

$$f(v) = a + bv \tag{2.60}$$

where $a, b \in \Lambda_\infty$. It is therefore superanalytic everywhere in $\mathbb{C}_a$ without any singularities.

A general superanalytic function $f : \mathbb{C}_c \to \Lambda_\infty$ has the form:

$$f(u) = \sum_{n=0}^{\infty} \frac{1}{n!} f_{a_1 \ldots a_n}(u) \xi^{a_n} \ldots \xi^{a_1}, \tag{2.61}$$

where $f_{a_1 \ldots a_n}(u)$ is a analytic continued function of the form (2.58).

For linear functions, the definition of the derivative is obvious and implies:

Let f be a superanalytic function $f : \mathbb{C}_a \to \Lambda_\infty$ with $f(v) = a + bv$ and the decomposition in commuting and anticommuting parts $b = b_c + b_a$. It follows:

$$\frac{\overrightarrow{\mathrm{d}}}{\mathrm{d}v} f(v) = b_c - b_a \text{ and } f(v)\frac{\overleftarrow{\mathrm{d}}}{\mathrm{d}v} = b_c + b_a. \tag{2.62}$$

Furthermore:

$$\frac{\overrightarrow{\mathrm{d}}}{\mathrm{d}v}f(v) = \begin{cases} f(v)\frac{\overleftarrow{\mathrm{d}}}{\mathrm{d}v} & \text{for } f:\mathbb{C}_a \to \mathbb{C}_a \\ -f(v)\frac{\overleftarrow{\mathrm{d}}}{\mathrm{d}v} & \text{for } f:\mathbb{C}_a \to \mathbb{C}_c \end{cases} \tag{2.63}$$

$$\frac{\overrightarrow{\mathrm{d}}}{\mathrm{d}u}f(u) = f(u)\frac{\overleftarrow{\mathrm{d}}}{\mathrm{d}u} \quad \text{for } f:\mathbb{C}_c \to \Lambda_\infty \tag{2.64}$$

and

$$\frac{\overrightarrow{\mathrm{d}}}{\mathrm{d}v}\frac{\overrightarrow{\mathrm{d}}}{\mathrm{d}v}f(v) = \frac{\overrightarrow{\mathrm{d}}}{\mathrm{d}v}f(v)\frac{\overleftarrow{d}}{\mathrm{d}v} = f(v)\frac{\overleftarrow{\mathrm{d}}}{\mathrm{d}v}\frac{\overleftarrow{\mathrm{d}}}{\mathrm{d}v} = 0. \tag{2.65}$$

B.S. DeWitt does not explicitly mention the derivative of the more complex functions of the form (2.61), but it is obvious, what to do: The coefficients $f_{a_1\ldots a_n}(u)$ have to be linearized, the derivative is just obtained by substituting $f_{a_1\ldots a_n}(u)$ with the first order term of its expansion.

2.2.4 Integral Calculus on Supernumbers

For the Integral Calculus, only functions on $\overline{\Lambda}$ are considered. Note that the definition of the Integral is modified as in the first part of this chapter, for getting again a dimension-independent formula for the Fourier-transformed of a function. While B.S. DeWitt uses $\int \mathrm{d}z z = Z$ with $Z = (2\pi\mathrm{i})^{1/2}$, we put $Z = (2\pi\mathrm{i})^{-1/2}$ and get the Fourier-transformed of a function by: $\hat{f}(p) = \int \mathrm{d}x f(x)\mathrm{e}^{2\pi\mathrm{i}px}$. We are told that in his forthcoming book, B.S. DeWitt switched also to this convention.

Definition of the Integral

The definition of the integral of a function with domain $\mathbb{R}_c$ is straightforward, by analytical continuation of the complex integral:

Def. 2.2.24 *Let $f(x)$ a continuated function $f:\mathbb{R}_c \to \Lambda_\infty$ as in 2.58, $F(x)$ the continuation of $\int \mathrm{d}x f(x)$ and $a, b \in \mathbb{R}_c$, not singular. Then*

$$\int_a^b \mathrm{d}x f(x) = F(b) - F(a). \tag{2.66}$$

For functions on $\mathbb{R}_a$ the definition is simpler, as there are only linear functions:

Def. 2.2.25 *Let $f(x)$ be a superanalytic function $f : \mathbb{R}_a \to \Lambda_\infty$, thus $f(x) = a + bx$. The integral $\int \mathrm{d}x f(x)$ is completely defined by:*

$$\int \mathrm{d}x := 0 \tag{2.67}$$

$$\int \mathrm{d}x x := Z \tag{2.68}$$

with $Z := (2\pi i)^{-1/2} = (2\pi)^{-1/2}\mathrm{e}^{-\pi \mathrm{i}/4}$ and the convention:

$$\int x \mathrm{d}x = -\int \mathrm{d}x x \tag{2.69}$$

As algebraists use the convention $Z = 1$, we will try to keep Z in all the formulæ instead of substituting it with its numerical value.

The integral here is given in physicist's notation as an operator ($\int \mathrm{d}x x$) rather than a bracket operator ($\int x \mathrm{d}x$). Up to a sign, both notations are equal.

As there is no difference between a definite and an indefinite integral for functions of odd supernumbers, and the integral basically equals a derivative, we get the following neat rules for shifts of the integration variable and partial integration:

$$\int \mathrm{d}x f(x+a) = \int \mathrm{d}x f(x) \tag{2.70}$$

$$\int \mathrm{d}x f(x) \frac{\overrightarrow{\mathrm{d}}}{\mathrm{d}x} g(x) = \int \mathrm{d}x f(x) \frac{\overleftarrow{d}}{\mathrm{d}x} g(x) \tag{2.71}$$

Fourier Transform

The Fourier transformation is given by a definition of the delta-distribution (which is actually a function in the case of $\mathbb{R}_a$), the Fourier transformed of the function 1.

Def. 2.2.26 *The* **delta-distribution** *for $x \in \mathbb{R}_c$ is given by*

$$\delta(x) := \int_{\mathbb{R}_c} \mathrm{d}p \mathrm{e}^{2\pi \mathrm{i} p x} := \lim_{\epsilon \to +0} \int_{\mathbb{R}_c} \mathrm{d}p \mathrm{e}^{2\pi(\mathrm{i}px - \epsilon p^2)} \tag{2.72}$$

The **delta-distribution** *for $x \in \mathbb{R}_a$ is given by*

$$\delta(x) := Z^{-1}x(= \int \mathrm{d}p \mathrm{e}^{2\pi \mathrm{i} p x}) \tag{2.73}$$

with $Z := (2\pi\mathrm{i})^{-1/2} = (2\pi)^{-1/2}\mathrm{e}^{-\pi\mathrm{i}/4}$ as in def. (2.2.25).

It is this definition, which fixes Z so as to conserve as many properties as possible of the ordinary delta-distribution.

For $x \in \mathbb{R}_c$ we get the relation

$$\delta(x) = \sum_{n=0}^{\infty} \frac{1}{n!} \delta_{\mathbb{R}}^{(n)}(x_B) x_S^n = \delta(-x) \tag{2.74}$$

which is similar to the delta-distribution in the complex case. It is easy to see from the definition that this is different for $x \in \mathbb{R}_a$:

$$\delta(-x) = Z^{-1}(-x) = -\delta(x) \tag{2.75}$$

In both cases, we get the known formula

$$\int_{\mathbb{R}_c} f(x)\delta(x)\mathrm{d}x = f(0) \text{ and } \int_{\mathbb{R}_a} f(x)\delta(x)\mathrm{d}x = f(0) \tag{2.76}$$

Integration over $\mathbb{R}_c^n \times \mathbb{R}_a^\nu$

Before we start with defining the integral over[8] $\mathbb{R}_c^n \times \mathbb{R}_a^\nu$, we introduce a simplifying notation by T.Voronov: c-type dimensions/indices and numbers will be denoted by latin letters, a-type dimensions/indices and numbers by greek letters. If an expression is valid for both types, capital latin letters will be used:

$$x^A = (x^a, \chi^\alpha) \tag{2.77}$$

Furthermore, we will use the abbreviation

$$_{IJ,}f_{,KL} := \frac{\overrightarrow{\partial}}{\partial x^I} \frac{\overrightarrow{\partial}}{\partial x^J} f \frac{\overleftarrow{\partial}}{\partial x^K} \frac{\overleftarrow{\partial}}{\partial x^L}. \tag{2.78}$$

[8]Note that $\mathbb{R}_c^n \times \mathbb{R}_a^\nu$ is *not* a supervector space, as it is not closed under multiplication with supernumbers.

B.S. DeWitt defines the integral by giving "volume elements". This approach is often criticized by other authors, as strictly speaking, there is no measure for a-type dimensions. Once more we come along situations in which volume elements can be defined without the existence of a measure.

Def. 2.2.27 *Without introducing a measure, we formally define the* **volume elements***:*

$$\mathrm{d}^n x := \mathrm{d}x^1 ... \mathrm{d}x^n \tag{2.79}$$

$$\mathrm{d}^\nu \chi := \mathrm{i}^{\nu(\nu-1)/2} \mathrm{d}\chi^1 ... \mathrm{d}\chi^\nu \tag{2.80}$$

$$\mathrm{d}^{n,\nu} x := \mathrm{i}^{\nu(\nu-1)/2} \mathrm{d}x^1 ... \mathrm{d}x^n \mathrm{d}\chi^1 ... \mathrm{d}\chi^\nu \tag{2.81}$$

When integrating out the a-type dimensions, we get the formula

$$\int \mathrm{d}^{n,\nu} x f(x) = Z^\nu (-\mathrm{i})^{\nu(\nu-1)/2} \int \mathrm{d}^n x g(x^1, ..., x^n) \tag{2.82}$$

where $g(x^1, ..., x^n)\chi^1 ... \chi^\nu$ is the term of n-th order in the power expansion of f in $\mathbb{R}_a$.

It is obvious that integration over a-dimensions is equivalent to differentiation:

$$\int \mathrm{d}\chi^1 ... \mathrm{d}\chi^\nu f(x) = (-1)^{\nu(\nu-1)/2} Z^\nu f(x) \frac{\overleftarrow{\partial}}{\partial \chi^1} ... \frac{\overleftarrow{\partial}}{\partial \chi^\nu} \tag{2.83}$$

(For the integral to be well defined, f must satisfy sufficiently rapidly:

$$\lim_{x_B \to \infty} f(x) = 0. \tag{2.84}$$

We will not examine this constraint any closer and suppose that all functions appearing in B.S. DeWitt's approach to the Fermi oscillator satisfy this condition.)

Under a transformation of coordinates, the superintegral behaves quite similarly to the ordinary case:

Theorem 2.2.4 *For a homogenous linear transformation* $\overline{x}^I = B^I{}_J x^J$ *the volume element transforms as:*

$$\mathrm{d}^{n,\nu} \overline{x} = (det(B))^{-1} \mathrm{d}^{n,\nu} x. \tag{2.85}$$

For an inhomogenous transformation $\overline{x}^i = \overline{x}^i(x)$ the volume element transforms as:

$$\mathrm{d}^{n,\nu}\overline{x} = J d^{n,\nu}x \text{ with } J = \mathrm{sdet}(\overline{x}^i{}_{,j}) \tag{2.86}$$

Finally, we want to have a look a the result for a Gaussian integral:
Let M be a Matrix

$$M = \begin{pmatrix} A & C \\ -C^T & B \end{pmatrix}, \tag{2.87}$$

where $A = A^T$, $B = -B^T$ and $A_{ij} \in \mathbb{R}_c$, $B_{ij} \in i\mathbb{R}_c$ and $C_{ij} \in i\mathbb{R}_a$. Then the **Gaussian Integral I** is:

$$I := \int_{\mathbb{R}_c^n \times \mathbb{R}_a^\nu} \mathrm{d}^{n,\nu}x \exp(\frac{\mathrm{i}}{2} x^I{}_I M_J x^J) = Z^{(n+\nu)}(\mathrm{sdet}(M))^{-\frac{1}{2}}. \tag{2.88}$$

Chapter 3

Analysis on Manifolds

"The knowledge of the divine
cannot be obtained by the mathematically uneducated one."
Nikolaus von Kues/Cusanus

For the representation of the supersymmetric Fock space proposed in this thesis, we need a calculus on ordinary manifolds, from which (n, n)-dimensional supermanifolds are derived by parity change of the fiber coordinates.
After briefly defining manifolds and the basic objects related to them, we will repeat the basic definitions and results for ordinary calculus on manifolds, before we construct the supermanifolds.

3.1 Ordinary Manifolds

Manifolds are a generalization of points, lines, and surfaces to arbitrary dimensions. With their help, calculus on curved spaces often needed in physics can be defined. A rough definition of a manifold is: "something, that locally looks like $\mathbb{R}^m$". More exactly:

Def. 3.1.1 *Let M be a topological space. Let $\{(U_i, \phi_i)\}_i$ be a family of pairs with U_i*

open sets that cover M: $\bigcup U_i = M$ and ϕ_i smooth homeomorphisms $U_i \to U_i' \in \mathbb{R}^m$ with $\psi_{ij} = \phi_i \circ \phi_j^{-1}$ infinitely differentiable for arbitrary i, j where ψ_{ij} is defined. Then M is a **manifold** *of dimension m. The pairs (U_i, ϕ_i) are called* **charts**, *the whole family $\{(U_i, \phi_i)\}_i$ is called an* **atlas**. *$\phi_i(x)$ is called the* **local coordinates** *of x denoted by $(x^1, ..., x^m)$.*

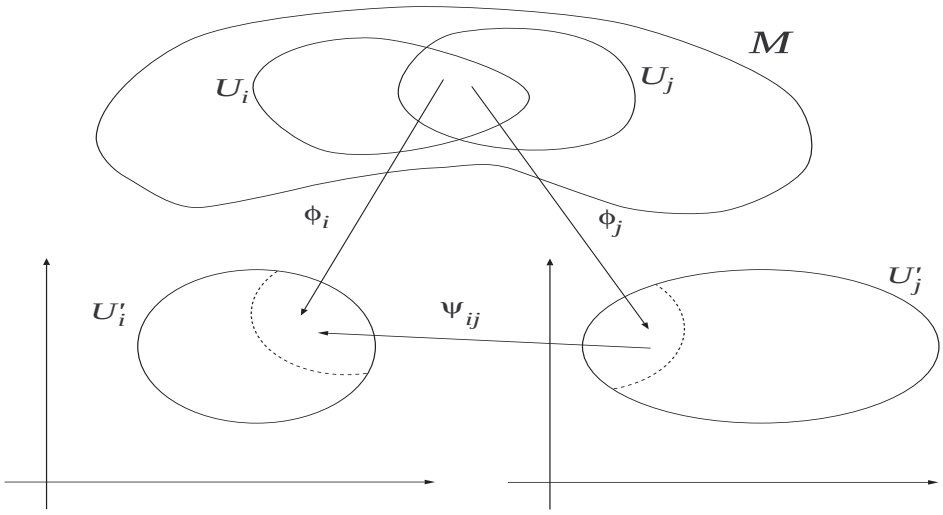

Figure 3.1: Two Charts ϕ_i and ϕ_j of a manifold M mapping open subsets of M on open subsets of $\mathbb{R}^n$, $\psi_{ij} = \phi_i \circ \phi_j^{-1}$ is smooth where defined.

A simple example for a manifold is $\mathbb{R}^m$, a little bit more complicated one is the circle, where

$$
\begin{aligned}
&M = \{(x,y) | x, y \in \mathbb{R}, x^2 + y^2 = 1\} \\
&U_1 = M \backslash \{(x,y) | x < 0\} \quad \phi_1(x,y) = \arcsin(y) \qquad \phi_1^{-1}(\theta) = (\cos(\theta), \sin(\theta)) \\
&U_2 = M \backslash \{(x,y) | x > 0\} \quad \phi_2(x,y) = \arcsin(-y) \quad \phi_2^{-1}(\theta) = (\cos(\theta), -\sin(\theta))
\end{aligned}
$$

In general, given a set $N = \{(x^1, ..., x^n) | f_i(x^1, ..., x^n) = 0\}$ with arbitrary smoothness of the f_i's and the matrix $(\partial_k f_i(x))$ having maximal rank for every point $x \in N$, then N is a manifold. (In the case of the circle, we had $f_1 = x^2 + y^2 - 1$.)
The next object that is naturally defined, is a vector. Since there is no "origin" of a

manifold, and the meaning of the term "straight line" is rather useless, vectors can only be tangent vectors at certain points:

Def. 3.1.2 *Given a manifold M and a curve $c : (a, b) \to M$ where (a, b) is an open interval containing 0. Let $x = c(0)$. Then the vector $\frac{\mathrm{d}c(t)}{\mathrm{d}t}\Big|_{t=0}$ is a* **tangent vector** *of M at x. The set of tangent vectors is called the* **tangent space** *T_xM at x. The union of all tangent spaces is called the* **tangent bundle** *$TM := \bigcup T_xM$.*

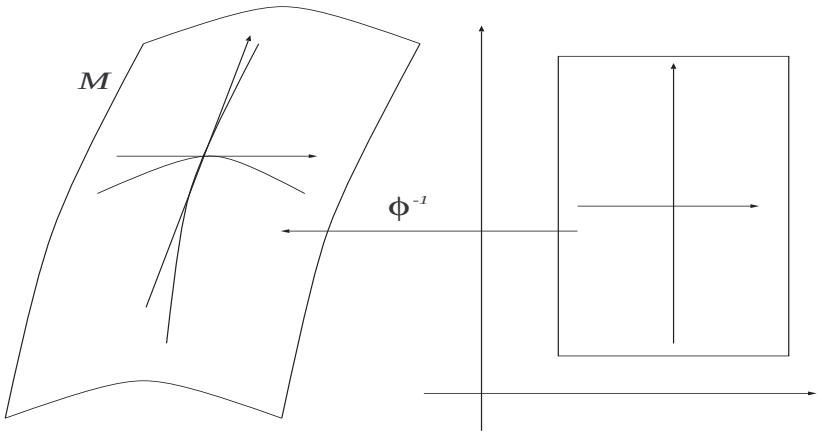

Figure 3.2: Infinitesimally varying the chart coordinates of a point x on M leads to vectors in the tangent space T_xM. In this case, the parameters of the curves on M are the chart coordinates.

To find a basis for T_xM, we consider the change of x under the change of its local coordinates.

Remark 3.1.1 *A basis for T_xM is given by the tuple*

$$(e_k)_k = (\partial_k x)_k = \left(\frac{\partial \phi^{-1}(x^1, ..., x^n)}{\partial x^k}\Big|_{\phi(x)} \right)_k . \tag{3.1}$$

An arbitrary vector of T_xM is given by $X(x) = X^\mu \partial_\mu(x)$.

Note that the directional derivative $X^\mu \partial_\mu$ acts on functions $f : M \to \mathbb{R}$ independently of the chart. This allows us to define

Def. 3.1.3 *The action of a vector on a function is given by:*

$$Xf = X[f] = X^\mu \partial_\mu f \tag{3.2}$$

Def. 3.1.4 *If a vector X is assigned smoothly to each point of a manifold M, $X(x)$ is called a* **vector field** *on M at x. If $X[f]$ is a smooth function for all smooth functions f, then X is a vector field.*

Graßmannian manifolds, i.e. manifolds with local coordinates $(\xi^i)_i$ which obey the Graßmann algebra $\xi^i\xi^j = -\xi^j\xi^i$, can naively be defined in the same way as ordinary manifolds, including tangent vectors and the action of vectors on function (directional derivative). This is due to the fact that only linear combinations of basis vectors but no multiplication of coordinates are needed in our definitions. Nevertheless, we will introduce Manifolds with odd coordinates in another way, allowing us to define an invariant volume element for Berezin integration.

3.2 Analysis on Ordinary Manifolds

The basis of modern analysis is the language of forms, which are a special case of tensors (i.e. antisymmetric tensors of type $(0, r)$). Tensors are a generalization of the concepts of scalars, vectors and matrices, so of objects which behave linearly under suitable multiplication.

Def. 3.2.1 *Let $\mathbb{X}$ be a linear space over a field $\mathbb{K}$ with the dual $\mathbb{X}^\top$. The set $\mathcal{T}_q^p(\mathbb{X})$ consists of all multilinear forms M mapping p elements of $\mathbb{X}^\top$ and q elements of $\mathbb{X}$ to $\mathbb{K}$:*

$$M : \mathbb{X}^\top \times ... \mathbb{X}^\top \times \mathbb{X} \times ... \times \mathbb{X} \to \mathbb{K}. \tag{3.3}$$

The elements of $\mathcal{T}_q^p(\mathbb{X})$ are called **tensors of type (p,q)** *(p-fold contravariant and q-fold covariant tensor).*

With this definition, a scalar is a tensor of rank $(0,0)$, a vector has rank $(1,0)$ and a matrix $(1,1)$.

The terms "covariant" and "contravariant" refer to the transformation properties of a tensor under change of coordinates. Under such a change from coordinates $(x^i)_i$ to coordinates $(\bar{x}^i)_i$, covariant tensor components transform like $\bar{A}_i = \frac{\partial x^j}{\partial \bar{x}^i} A_j$, contravariant components like $\bar{A}^i = \frac{\partial \bar{x}^i}{\partial x^j} A^j$.

Two tensors can be multiplied to get tensors of higher types by the following rule:

Def. 3.2.2 *The tensor product* $\otimes : \mathcal{T}_q^p \times \mathcal{T}_{q'}^{p'} \to \mathcal{T}_{q+q'}^{p+p'}$ *of a tensor* μ *of type* (p,q) *and a tensor* ν *of type* (p',q') *is given by:*

$$(\mu \otimes \nu)(x^1, ..., x^{p+p'}; y_1, ...y_{q+q'}) = \\ \mu(x^1, ..., x^p; y_1, ..., y_q)\nu(x^{p+1}, ..., x^{p+p'}; y_{q+1}, ..., y_{q+q'}) \quad (3.4)$$

where $(x^i)_i$ *are covectors (from the dual space* $\mathbb{X}^\top$*) and* $(y_i)_i$ *are vectors (from the space* $\mathbb{X}$*).*

An example for the tensor product is the combination of different subspaces of the configuration space in quantum mechanics. While in classical mechanics, these subspaces are combined to the configuration space by the direct sum $\mathbb{X} = \bigoplus \mathbb{X}_i$, quantum mechanical subspaces are combined by the tensor product, e.g. $|nlm\rangle = |n\rangle \otimes |lm\rangle$ for the hydrogen wave functions. The subspaces of a Fock space, the multiparticle Hilbert spaces are also created by tensor products of one particle wave functions ($\mathbb{H}^3 = \mathbb{H}^1 \otimes \mathbb{H}^1 \otimes \mathbb{H}^1$) which are the summed up to the Fock space: $\mathbb{F} = \bigoplus \mathbb{H}^n$.

The opposite operation that reduces the rank of a tensor is called a contraction:

Def. 3.2.3 *The* **contraction** *maps a tensor of type* (p,q) *to a tensor of type* $(p-1, q-1)$ *by the rule*

$$Contraction(\tau) = \tau(...e^i...; ...e_i...) \quad (3.5)$$

where $(e_i)_i$ and $(e^i)_i$ are bases for the space and its dual resp.

An example for a contraction is the trace of a matrix $A^i{}_k$, which is simply $A^i{}_i$. The dual product of two vectors $u^\top$ and v can also be interpreted as the contraction of the tensor product $\langle u^\top | v \rangle = (u \otimes v)^i{}_i$.
From now on, we will consider tensors on the tangent space of a manifold at a certain point: T_pM. The basis vectors of this space will be the vectors $(e_i = \partial_i)_i$ as shown above. As T_pM is a vector space, there exists its dual, the cotangent space:

Def. 3.2.4 *Given a manifold M and x a point on M. Then the dual space of the tangent space T_xM is called the* **cotangent space** *T_x^*M. The union of all cotangent spaces is called the cotangent bundle $T^*M := \bigcup T_x^*M$.*

The basis of T_p^*M will be denoted by $(e^i = \mathrm{d}x^i)_i$. As the elements of T_pM are tensors of type $(1,0)$, the elements of its dual have type $(0,1)$ and by multiplying these tensors and contracting we can define a dual product:

$$\langle \mathrm{d}x^\nu, \partial/\partial x^\mu \rangle = \frac{\partial x^\nu}{\partial x^\mu} = \delta^\nu_\mu \tag{3.6}$$

(Note that due to linearity, the dual product is completely defined by the product of the basis vectors.) The elements of T_p^*M are also called "one-forms" the generalization of which are differential forms:

Def. 3.2.5 *A* **differential form** *of order r, or an r-***form** *is a totally antisymmetric tensor of type (0,r).*

Note that the elements of T_p^*M were obviously of type $(0,1)$. Since these forms map just one vector to a scalar, the antisymmetry in the exchange of the vectors does not occur in this case. If one multiplied two r-forms by the tensor product to get forms of higher order, the antisymmetry will in general be broken. So a new product is needed:

Def. 3.2.6 *The* **wedge product** *of r one-forms is given by:*

$$\mathrm{d}x^{\mu_1} \wedge \mathrm{d}x^{\mu_2} \wedge ... \wedge \mathrm{d}x^{\mu_r} = \sum_P \mathrm{sgn}(P(1..r))\mathrm{d}x^{\mu_{P(1)}} \otimes \mathrm{d}x^{\mu_{P(2)}} \otimes ... \otimes \mathrm{d}x^{\mu_{P(r)}} \quad (3.7)$$

where the sum is taken over all permutations P.

Immediate results are $\mathrm{d}x^\mu \wedge \mathrm{d}x^\nu = \mathrm{d}x^\mu \otimes \mathrm{d}x^\nu - \mathrm{d}x^\nu \otimes \mathrm{d}x^\mu = -\mathrm{d}x^\nu \wedge \mathrm{d}x^\mu$ and $\mathrm{d}x^\mu \wedge \mathrm{d}x^\mu = 0$. Furthermore, it is:

$$\mathrm{d}x^{\mu_1} \wedge \mathrm{d}x^{\mu_2} \wedge ... \wedge \mathrm{d}x^{\mu_r} = \mathrm{sgn}(P)\mathrm{d}x^{\mu_{P(1)}} \wedge \mathrm{d}x^{\mu_{P(2)}} \wedge ... \wedge \mathrm{d}x^{\mu_{P(r)}} \quad (3.8)$$

The wedge product for two arbitrary forms is given by a trivial extension:

$$(\omega \wedge \pi)(X_1, X_2, ..., X_{r+s}) = \frac{1}{r!s!} \sum_{P(1..r+s)} \mathrm{sgn}(P)\omega(X_{P(1)}, ...X_{P(r)})\pi(X_{P(r+1)}, ...X_{P(r+s)}) \quad (3.9)$$

where the X_i are vectors.

With the wedge product, the general form of an r-form in explicit coordinates is:

$$\omega = \frac{1}{r!}\omega_{\mu_1\mu_2...\mu_r}\mathrm{d}x^{\mu_1} \wedge \mathrm{d}x^{\mu_2} \wedge ... \wedge \mathrm{d}x^{\mu_r}, \quad (3.10)$$

where the $\omega_{\mu_1\mu_2...\mu_r}$ is totally antisymmetric:

$$\omega_{\mu_1\mu_2...\mu_r} = \mathrm{sgn}(P)\omega_{P(\mu_1)P(\mu_2)...P(\mu_r)} \quad (3.11)$$

so the 2-form $\omega = 3\mathrm{d}x^1 \wedge \mathrm{d}x^3 + 6\mathrm{d}x^1 \wedge \mathrm{d}x^2$ has nonvanishing components $\omega_{13} = -\omega_{31} = 3$ and $\omega_{12} = -\omega_{21} = 6$.

We will label the set of r-forms on a manifold M at a point p with $\Omega^r_p(M)$. Forms of maximal order $n = \dim(M)$ are called **top forms**.

The set of all smooth fields of r-forms on M will be denoted by $\Omega^r(M)$ as particularly $C^\infty(M) = \Omega^0(M)$.

There are two other operations which allow to increase and decrease the order of a form by 1. The first one is called the exterior derivative, the second operation is a contraction with a vector.

Def. 3.2.7 *The* **exterior derivative** d_r *is a map* $\Omega^r(M) \to \Omega^{r+1}(M)$ *whose action on a form as in (3.10) is defined by:*

$$\mathrm{d}_r\omega = \frac{1}{r!}\left(\frac{\partial}{\partial x^\nu}\omega_{\mu_1,\ldots\mu_r}\right)\mathrm{d}x^\nu \wedge \mathrm{d}x^{\mu_1} \wedge \ldots \wedge \mathrm{d}x^{\mu_r} \tag{3.12}$$

($\mathrm{d}_r\omega$ *is often called the* **differential of** ω*.) An element of* $\ker(\mathrm{d}_r)$ *is called a* **closed** r**-form**, *an element of* $\mathrm{im}(\mathrm{d}_{r-1})$ *is called an* **exact** r**-form**.

The index r is often dropped. This definition implies $\mathrm{d}f = \frac{\partial f}{\partial x^i}\mathrm{d}x^i$, so d on a function (a 0-form) is the gradient of the function. If one considers a 1- and a 2-form on a three dimensional manifold as vectors in the following sense:

$$\omega_1 = \begin{pmatrix} \omega_x \\ \omega_y \\ \omega_z \end{pmatrix} \text{ and } \omega_2 = \begin{pmatrix} \omega_{xy} \\ \omega_{yz} \\ \omega_{zx} \end{pmatrix} \tag{3.13}$$

with bases $(\mathrm{d}x, \mathrm{d}y, \mathrm{d}z)$ and $(\mathrm{d}x\wedge\mathrm{d}y, \mathrm{d}y\wedge\mathrm{d}z, \mathrm{d}z\wedge\mathrm{d}x)$ resp., then the exterior derivative of a 1-form is the rotation (rot) of this vector, in the case of a 2-form, it is the divergence (div) of usual, 3 dimensional vector calculus.
It follows also that $\mathrm{d}^2\omega = 0$, as

$$\mathrm{d}^2\omega = \frac{1}{(r+1)!}\frac{1}{r!}\frac{\partial^2\omega_{\mu_1\ldots\mu_r}}{\partial x^\lambda \partial x^\nu}\mathrm{d}x^\lambda \wedge \mathrm{d}x^\nu \wedge \mathrm{d}x^{\mu_1} \ldots \wedge \mathrm{d}x^{\mu_r} \tag{3.14}$$

This expression has to be invariant under exchange of the dummy variables λ and ν, i.e. symmetric. Since the partial derivatives are symmetric and the wedge product is antisymmetric, the overall expression has to be antisymmetric. Now the only form of order $r \geq 2$ being symmetric and antisymmetric at the same time is the form 0.[1]
An example of the exterior derivative in a physical context: Given the electromagnetic potential as the vector $A = (\phi, \mathbf{A})$, the electromagnetic tensor is given by $F = \mathrm{d}A$. It follows immediately that $\mathrm{d}F = \mathrm{d}^2A = 0$, which summarizes the two

[1] Calling "0" a r-form with $r \geq 2$ seems a little strange, meant is a general r-form as in (3.10) with coefficients $\omega_{\mu_1\ldots\mu_r} = 0$.

Maxwell equations containing the magnetic field B. The opposite operation to the exterior derivative which lowers the order of a form is the interior product[2]:

Def. 3.2.8 *Let $X, X_1, ..., X_{r-1}$ be vector fields over a manifold M ($X, X_i \in \mathcal{X}(M)$). The* **interior product** *is a map $\mathrm{i}_X : \Omega^r(M) \to \Omega^{r-1}(M)$ acting on a form $\omega \in \Omega^r$ by the following rule:*

$$\mathrm{i}_X\omega(X_1, ..., X_{r-1}) := \omega(X, X_1, ..., X_{r-1}) \tag{3.15}$$

If $\omega \in \Omega^0$, we define explicitely: $\mathrm{i}_X\omega = 0$.[3]

If a vector field is given by $X = X^\mu \partial/\partial x^\mu$, then the action of the interior product on a form in coordinates as in (3.10) is given by:

$$\mathrm{i}_X\omega \frac{1}{(r-1)!} X^\nu \omega_{\nu\mu_2...\mu_r} \mathrm{d}x^{\mu_2} \wedge ... \wedge \mathrm{d}x^{\mu_r} \tag{3.16}$$

Since the exterior derivative, the interior product is nilpotent: $(\mathrm{i}_X)^2 = 0$. Given an arbitrary r-form ω of the form (3.10), then[4]

$$\begin{aligned} \mathrm{i}_X(\mathrm{i}_X\omega) &= \mathrm{i}_X \frac{1}{(r-1)!} X^\nu \omega_{\nu\mu_2...\mu_r} \mathrm{d}x^{\mu_2} \wedge ... \wedge \mathrm{d}x^{\mu_r} \\ &= \frac{1}{(r-2)!} \frac{1}{(r-1)!} X^\sigma X^\nu \omega_{\nu\sigma\mu_3...\mu_r} \mathrm{d}x^{\mu_3} \wedge ... \wedge \mathrm{d}x^{\mu_r}. \end{aligned} \tag{3.17}$$

Again, the last expression has to be symmetric under exchange of the dummy variables σ and ν. The exchange of the vector components is symmetric, but the exchange in $\omega_{\nu\sigma\mu_3...\mu_r}$ is antisymmetric. This causes $\mathrm{i}_X(\mathrm{i}_X\omega)$ to vanish. The last object we want to consider is the Lie-derivative $\mathcal{L}_X$. The original reason for introducing $\mathcal{L}_X$ is, that one wants to calculate the change of tangent vectors along a flow (a special curve on a manifold, generated by a vector field). As tangent vectors at different points of a manifold belong to different tangent spaces, their difference is

[2] which must not be confused with the inner product

[3] *This definition is analog to $\mathrm{d}\omega = 0$ for $\omega \in \Omega^n$ (top forms) and yields the desired results for the de Rham complexes in chapter 5.2.*

[4] Strictly speaking, ω has to be of rank ≥ 2, for smaller rank, we have immediately $(\mathrm{i}_X)^2\omega = 0$.

ill defined. The Lie-derivative transports one of the vectors in the tangent space of the other one and thus allows a comparison. Here we are only interested in the action of the Lie-derivative on forms:

Def. 3.2.9 *The* **Lie derivative** *of a form ω is given by*

$$\mathcal{L}_X\omega = (\mathrm{i}_X\mathrm{d} + \mathrm{d}\mathrm{i}_X)\omega. \tag{3.18}$$

3.3 Fiber Bundles

The tangent and cotangent bundles of a manifold, TM and T^*M, introduced in the last section are spaces generated by assigning certain spaces to each point of a manifold. These constructs play an important role for analysis on manifolds. They are called fiber bundles and examples for them are vector bundles and principle bundles.

Def. 3.3.1 *Given two sets $\mathcal{B}$, $\mathcal{M}$ and a surjective map π:*

$$\pi : \mathcal{B} \to \mathcal{M} \tag{3.19}$$

Then π is called a **fiber**, *$\mathcal{B}$ is the* **bundle space** *and $\mathcal{M}$ the* **base space**.
The inverse image of π is called the **fiber** *F_x* **over** *x:*

$$F_x := \pi^{-1}(x). \tag{3.20}$$

The fibers of different points of M are obviously disjunct and the complete bundle space is obtained by the union of the fibers: $\mathcal{B} = \bigcup_x F_x$. Because of the decomposition in fibers, a bundle is often called a **fiber bundle**. (Furthermore, one does usually not distinguish strictly between a bundle and its bundle space.)
A function, which assigns an object of F_x to each point x of M is called a section (or a cross section):

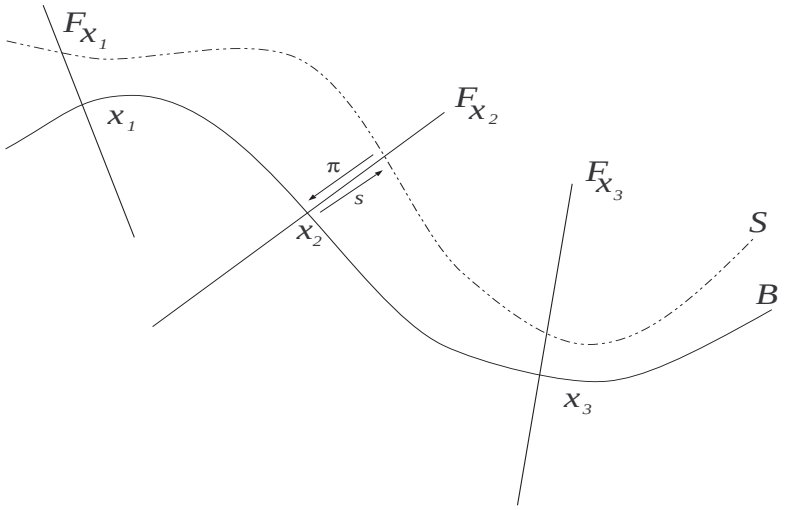

Figure 3.3: An example for a fiber bundle: To each point of a one dimensional base space B there is a one dimensional fiber F_x assigned (three points are shown as an example). The dotted line S represents a section of the fiber bundle.

Def. 3.3.2 *A map $s : \mathcal{M} \to \mathcal{B}$ satisfying $\pi(s(x)) = x$ for all $x \in M$ is called a* **section** *of the bundle.*

Let us look at an explicit example for the introduced objects:
Consider the space of complex numbers $\mathbb{C}$ together with the map $\pi(z) := \mathrm{Re}(z)$. The bundle is here obviously the projection of a complex number on its real part, the bundle space is $\mathbb{C}$, the base space is $\mathbb{R}$. So for each $x \in \mathbb{R}$, the set $\mathrm{i}\mathbb{R}$ has been assigned. A section can easily be defined by $(x, f(x))$ where $f(x)$ is a function mapping a real number to an imaginary number.
If the spaces assigned to each element of the base space are vector spaces then the fiber bundle is called a **vector bundle**. Examples for vector bundles are the tangential space of a manifold TM (tangent bundle) and the cotangential space T^*M (cotangent bundle):

Remark 3.3.1 *Given a manifold M of dimension n (base space) with local coordinates $q^1, ..., q^D$ the tangential bundle TM (the bundle space, coordinates*

$q^1, ..., q^D, \dot{q}^1, ..., \dot{q}^n$) can be mapped on M by dropping the tangential coordinates $\dot{q}^1, ..., \dot{q}^n$. This operation will be denoted by π_M. For the cotangent bundle (coordinates $p_1, ..., p_n$), the cotangential coordinates are similarly dropped by π_M.

Considering the manifold of this example as the configuration space of a physical system, we see easyly that the Lagrangian formulation of mechanics operates on the tangent bundle, while the Hamiltonian formalism works with the cotangent bundle (the momenta p_i transform covariant under change of coordinates).

3.4 Naive Analysis on Graßmannian Manifolds

Graßmannian[5] manifolds are manifolds with anticommuting chart coordinates. For analysis on Graßmannian manifolds, analogue objects as in the ordinary case can be defined. Due to the noncommutative algebra, an adjustments of the wedge product has to be made, most of the other formulæ remain unchanged, but can occasionally be simplified. This analysis is called "naive", as it is regarded to be not very successful in covering all the properties of supermanifolds.

The definition of a tensor (Def. 3.2.1) is not changed. The field $\mathbb{K}$ is the set of the complex/real numbers but coordinates in the linear space will be generators for the Graßmann algebra $(\xi^i)_i$. The additional fact, that the generators obey the algebra $\xi^i\xi^j = -\xi^j\xi^i$ is simply ignored, a tensor remains a multilinear map of elements of the linear space and its dual to the field. The same is true for tensor operations. The tensor product and the contraction are unaffected by the use of Graßmannian variables. The tangent space of a Graßmannian manifold is spanned by the vectors $(\partial/\partial\xi^\mu)_\mu$, its dual, the cotangent space by $(\mathrm{d}\xi^\mu)_\mu$, so the dual product is changed to $\langle \mathrm{d}\xi^\mu, \partial/\partial\xi^\nu\rangle = \partial\xi^\mu/\partial\xi^\nu = \delta^\nu_\mu$. The first important difference between ordinary and Graßmannian analysis appears in forms. Since differential forms are totally antisymmetric for a symmetric algebra, we want Graßmannian forms for the

[5] not to be confused with Graßmann manifolds defined for example in [8, p. 109]

antisymmetric Graßmann algebra to be totally symmetric. (As we will see later on, this definition will lead to a preservation of the property $\mathrm{d}^2 = 0$ for the exterior derivative.)

Def. 3.4.1 *A* **Graßmannian form of order** r *is a totally symmetric tensor of type* $(0, r)$.

Similarly to this, the wedge product has to be adjusted by dropping the sign of the permutation:

Def. 3.4.2 *The* **wedge product** *of r Graßmannian 1-forms is given by:*

$$\mathrm{d}\xi^{\mu_1} \wedge \mathrm{d}\xi^{\mu_2} \wedge ... \wedge \mathrm{d}\xi^{\mu_r} = \sum_P \mathrm{d}\xi^{\mu_{P(1)}} \otimes \mathrm{d}\xi^{\mu_{P(2)}} \otimes ... \otimes \mathrm{d}\xi^{\mu_{P(r)}} \tag{3.21}$$

where the sum is again taken over all permutations P.

The missing sign of the permutation reflects the fact that the antisymmetry of the wedge product together with the antisymmetry of the differentials of the generators yields a total symmetry: $\mathrm{d}\xi^i \wedge \mathrm{d}\xi^j = -(-\mathrm{d}\xi^j \wedge \mathrm{d}\xi^i)$.
A difference, which will be important later on, is that for Graßmann variables ξ_i we get $\mathrm{d}\xi^\mu \wedge \mathrm{d}\xi^\mu \neq 0$, while $\mathrm{d}x^\mu \wedge \mathrm{d}x^\mu = 0$ in the ordinary case.
The $\mathrm{sgn}(P)$ has to be dropped in all the formulæ, especially in (3.8) which will read

$$\mathrm{d}\xi^{\mu_1} \wedge \mathrm{d}\xi^{\mu_2} \wedge ... \wedge \mathrm{d}\xi^{\mu_r} = \mathrm{d}\xi^{\mu_{P(1)}} \wedge \mathrm{d}\xi^{\mu_{P(2)}} \wedge ... \wedge \mathrm{d}\xi^{\mu_{P(r)}} \tag{3.22}$$

and in (3.9).
The general form of a Graßmannian r-form in coordinates equals again the ordinary case.

$$o = \frac{1}{r!} o_{\mu_1\mu_2...\mu_r} \mathrm{d}\xi^{\mu_1} \wedge \mathrm{d}\xi^{\mu_2} \wedge ... \wedge \mathrm{d}\xi^{\mu_r}, \tag{3.23}$$

but the $o_{\mu_1\mu_2...\mu_r}$ is again totally symmetric:

$$o_{\mu_1\mu_2...\mu_r} = o_{P(\mu_1)P(\mu_2)...P(\mu_r)} \tag{3.24}$$

so the 2-form $o = 3\mathrm{d}\xi^1 \wedge \mathrm{d}\xi^2 + 4\mathrm{d}\xi^1 \wedge \mathrm{d}\xi^1$ has nonvanishing components $o_{12} = o_{21} = 3$ and $o_{11} = 8$. (Note that the components with i equal indices are the bare components multiplied by $i!$. In our example, we have two similar indices in o_{11}, so we have to take the bare coefficient 4 and multiply by 2!. This explains $o_{11} = 8$.) The two operations acting on forms, the exterior derivative (def. 3.2.7) and the interior product (def. 3.2.8) remain unchanged.

The set of Graßmannian r-forms on a manifold M_G at a point p will certainly be denoted by $O^r_p(M)$, the set of all smooth fields of Graßmannian r-forms on M by $O^r(M)$.

The "top forms" are not as easily found as in the ordinary case, as there is no form with maximal order due to the lack of total antisymmetry.

Though the differentials of Graßmannian variables are symmetric, this time the partial derivatives are antisymmetric as discussed above (e.g. $\partial_\mu \partial_\nu \xi^\nu \xi^\mu \alpha = \alpha = \partial_\nu \partial_\mu \xi^\mu \xi^\nu \alpha$). This preserves the property $\mathrm{d}^2 = 0$ for the same reasons as in the ordinary case, see (3.14). Again, the interior product is nilpotent: $(\mathrm{i}_\Xi)^2 = 0$. Looking at equation (3.17) we note, that in the Graßmannian case the components of the form are symmetric under exchange of the dummy variables, but this time the coordinates of the vector field X are antisymmetric, making $\mathrm{i}_\Xi(\mathrm{i}_\Xi o)$ vanish again. The Lie derivative also has to be adjusted by:

Def. 3.4.3 *The* **Lie derivative** *of a Graßmannian form o is given by*

$$\mathcal{L}_\Xi o = (\mathrm{i}_\Xi \mathrm{d} - \mathrm{d}\mathrm{i}_\Xi) o. \tag{3.25}$$

This definition differs from the ordinary case, but it makes the equations in theorem (3.5.2) and in remark (3.5.2) look similar.

Summarizing, we get the interesting properties:

	Coordinates	Partial Derivatives	Differentials
ordinary	x^k symmetric	$\partial/\partial x^k$ symmetric	$\mathrm{d}x^k$ antisymmetric
Graßmannian	ξ^k antisymmetric	$\partial/\partial \xi^k$ antisymmetric	$\mathrm{d}\xi^k$ symmetric

3.5 Further Results Needed for the Operator Algebra

In this section, a generalized exterior derivative (the e_f operator) is introduced and its commuatation relation with the contraction are examined.

Def. 3.5.1 *Let f be a function on a manifold M, $f \in C^\infty(M)$ and ω a form on M. The operator e_f is a map $\Omega^r(M) \to \Omega^{r+1}(M)$ acting on a form by the rule:*

$$\mathrm{e}_f \omega := \mathrm{d}f \wedge \omega. \tag{3.26}$$

For functions and forms of Graßmann variables, we define:

$$\mathrm{e}_\phi o := \mathrm{d}\phi \wedge o. \tag{3.27}$$

The following theorem is only a preparation for theorem 3.5.2:

Theorem 3.5.1 *Let M be a manifold, X a vector field on M and $\omega \in \Omega^r$ and $\pi \in \Omega^s$ forms on M. Then*

$$\mathrm{i}_X(\omega \wedge \pi) = (\mathrm{i}_X \omega) \wedge \pi + (-1)^r \omega \wedge (\mathrm{i}_X \pi) \tag{3.28}$$

Proof: Let $X_2, ..., X_{r+s}$ be vector fields on M and $X_1 = X$. Then

$$\begin{aligned} \mathrm{i}_X(\omega \wedge \pi)(X_2, ..., X_{r+s}) &= (\omega \wedge \pi)(X_1, X_2, ..., X_{r+s}) \\ &= \frac{1}{r!s!} \sum_{P(1..r+s)} \mathrm{sgn}(P) \omega(X_{P(1)}, ...X_{P(r)}) \pi(X_{P(r+1)}, ...X_{P(r+s)}) \end{aligned}$$

Furthermore

$$
\begin{aligned}
&(\mathrm{i}_X\omega) \wedge \pi(X_2, ..., X_{r+s}) \\
&= \frac{1}{(r-1)!s!} \sum_{P(2..r+s)} \mathrm{sgn}(P)\omega(X_1, X_{P(2)}, ..., X_{P(r)})\pi(X_{P(r+1)}, ..., X_{P(r+s)}) \\
&\omega \wedge (\mathrm{i}_X\pi)(X_2, ..., X_{r+s}) \\
&= \frac{1}{r!(s-1)!} \sum_{P(2..r+s)} \mathrm{sgn}(P)\omega(X_{P(1)}, ..., X_{P(r+1)})\pi(X_1, X_{P(r+2)}, ..., X_{P(r+s)})
\end{aligned}
$$

After adding both lines, it remains to prove that

$$
\begin{aligned}
&\sum_{P(1..r+s)} \mathrm{sgn}(P)\omega(X_{P(1)}, ...X_{P(r)})\pi(X_{P(r+1)}, ..., X_{P(r+s)}) \\
= &\sum_{P(2..r+s)} \mathrm{sgn}(P)(r\omega(X_1, X_{P(2)}, ..., X_{P(r)})\pi(X_{P(r+1)}, ..., X_{P(r+s)}) \\
&+(-1)^{rs}\omega(X_{P(2)}, ..., X_{P(r+1)})\pi(X_1, X_{P(r+2)}, ..., X_{P(r+s)})
\end{aligned}
$$

This is obviously true, as there are r places to put X_1 in the first term and s in the second one. The $(-1)^r$ reflects the fact, that we get a minus sign from permuting $X_1, X_2, ..., X_{r+s}$ to $X_2, ..., X_{r+1}, X_1, X_{r+2}, ..., X_{r+s}$ on the $r+1$th position for odd r.#

Remark 3.5.1 *Since forms on a Graßmannian manifold are symmetric, the* $(-1)^r$ *has to be dropped in this case. So the last theorem would read:*
Let M_G *be a Graßmannian manifold,* Ξ *a vector field on* M_G *and* $o \in O^r$ *and* $p \in O^s$*forms on* M_G*. Then*

$$\mathrm{i}_\Xi(o \wedge p) = (\mathrm{i}_\Xi o) \wedge p + o \wedge (\mathrm{i}_\Xi p) \tag{3.29}$$

Theorem 3.5.2 *Given a manifold* M*, a vector field* X *on* M *and a smooth function* f *on* M*. Then*

$$\mathrm{i}_X\mathrm{e}_f + \mathrm{e}_f\mathrm{i}_X = Xf(= \mathcal{L}_X f) \tag{3.30}$$

Proof: Let ω be an arbitrary form on M. From theorem (3.5.1) we get

$$\begin{aligned} \mathrm{i}_X(\mathrm{e}_f\omega) + \mathrm{e}_f\mathrm{i}_X\omega &= (\mathrm{i}_X\mathrm{d}f)\wedge\omega + (-1)^1\mathrm{d}f\wedge(\mathrm{i}_X\omega) + \mathrm{d}f\wedge(\mathrm{i}_X\omega) \\ &= (\mathrm{i}_X\mathrm{d}f)\wedge\omega = Xf\omega. \ \# \end{aligned}$$

Remark 3.5.2 *Again, for Graßmannian manifolds we get a similar result with an other sign: Given a Graßmannian manifold M_G, a vector field Ξ on M_G and a smooth function ϕ on M_G. Then*

$$\mathrm{i}_\Xi\mathrm{e}_\phi - \mathrm{e}_\phi\mathrm{i}_\Xi = \Xi\phi(= \mathcal{L}_\Xi\phi) \tag{3.31}$$

After calculating the commutation relations between e_f and i_X, let us proceed with the remaining relations:

Theorem 3.5.3 *The operators e_f and i_X obey the following relations:*

$$\left.\begin{aligned} \mathrm{e}_f\mathrm{e}_g + \mathrm{e}_g\mathrm{e}_f &= 0 \\ \mathrm{i}_X\mathrm{i}_Y + \mathrm{i}_Y\mathrm{i}_X &= 0 \end{aligned}\right\} \textit{ for ordinary manifolds and} \tag{3.32}$$

$$\left.\begin{aligned} \mathrm{e}_\phi\mathrm{e}_\gamma - \mathrm{e}_\gamma\mathrm{e}_\phi &= 0 \\ \mathrm{i}_\Xi\mathrm{i}_\Psi - \mathrm{i}_\Psi\mathrm{i}_\Xi &= 0 \end{aligned}\right\} \textit{for Graßmannian manifolds.} \tag{3.33}$$

Proof: 1. Given an arbitrary form $\omega = \frac{1}{r!}\omega_{\mu_1\ldots\mu_r}\mathrm{d}x^{\mu_1}\ldots\mathrm{d}x^{\mu_r}$ on an ordinary manifold. Then it is:

$$\begin{aligned} \mathrm{e}_f\mathrm{e}_g\omega &= \mathrm{d}f\wedge\mathrm{d}g\wedge\omega = \frac{1}{r!}\omega_{\mu_1\ldots\mu_r}\mathrm{d}f\wedge\mathrm{d}g\wedge\mathrm{d}x^{\mu_1}\ldots\mathrm{d}x^{\mu_r} \\ &\overset{(3.8)}{=} -\frac{1}{r!}\omega_{\mu_1\ldots\mu_r}dg\wedge\mathrm{d}f\wedge\mathrm{d}x^{\mu_1}\ldots\mathrm{d}x^{\mu_r} = -\mathrm{e}_g\mathrm{e}_f\omega \end{aligned}$$

$$\begin{aligned} \mathrm{i}_X\mathrm{i}_Y\omega &= \mathrm{i}_X Y^\nu\omega_{\nu\mu_1\ldots\mu_{r-1}}\mathrm{d}x^{\mu_1}\wedge\ldots\wedge\mathrm{d}x^{\mu_{r-1}} \\ &= X^\lambda Y^\nu\omega_{\nu\lambda\mu_1\ldots\mu_{r-2}}\mathrm{d}x^{\mu_1}\wedge\ldots\wedge\mathrm{d}x^{\mu_{r-2}} \\ &\overset{(3.11)}{=} -Y^\nu X^\mu\omega_{\lambda\nu\lambda_1\ldots\mu_{r-2}}\mathrm{d}x^{\mu_1}\wedge\ldots\wedge\mathrm{d}x^{\mu_{r-2}} = -\mathrm{i}_Y\mathrm{i}_X\omega. \end{aligned}$$

2. Note that we used antisymmetric properties of the ordinary forms in (3.8) and (3.11). The corresponding symmetric properties for Graßmannian forms are (3.22) and (3.24), therefore the minus sign in the upper calculation has to be dropped:

$$\begin{aligned} \mathrm{e}_\phi \mathrm{e}_\gamma \omega &= +\mathrm{e}_\gamma \mathrm{e}_\phi \omega \text{ and} \\ \mathrm{i}_\Xi \mathrm{i}_\Psi \omega &= +\mathrm{i}_\Psi \mathrm{i}_\Xi \omega. \ \# \end{aligned} \tag{3.34}$$

Chapter 4

Supermanifolds

"Structures are the weapons of mathematicians."

Bourbaki

For supersymmetric models of field theory, it is sufficient to consider supermanifolds with equally many bosonic as fermionic dimensions, as each particle has a supersymmetric partner with opposite parity (the electron has the selectron, the photon the photino, etc.).
Furthermore, we will show that each general supermanifold can be reduced invariantly to a symmetric supermanifold.

4.1 The spaces $\mathbb{R}^{(n|\nu)}$ and $\mathbb{R}^n_c \times \mathbb{R}^\nu_a$

Essentially, there are two approaches to the definition of a superspace, i.e. a space described by even and odd variables. T. Voronov's superspace $S_V := \mathbb{R}^{(n|\nu)}$ is defined by the functions having S_V as their domain which are functions of n real and ν Graßmann variables. An element of B.S. DeWitt's superspace $S_D := \mathbb{R}^n_c \times \mathbb{R}^\nu_a$ is represented by a tuple of n real c-type and ν real a-type supernumbers.
The most important difference is that in the first case (S_V), the Graßmann variables

are used as coordinates while the elements of the tuples representing S_D are elements of Λ_∞, which is the algebra generated by infinitely many Graßmann variables.
While S_V can be considered as a vector space with $n+\nu$ ordinary (c-type) vectors as a basis, S_D is in general not closed under multiplication with a-type supernumbers and thus not a vector space:
Assume e.g. $n > \nu$ and multiply an element of S_D by an a-type supernumber. This yields a tuple of n a-type and ν c-type supernumbers which is certainly not an element of S_D.
Nevertheless, S_D is a subset of a supervector space of dimension (n, ν). Since we encoded the parity of the vectors in their coordinate-tuples, the basis again has to consist of $n + \nu$ c-type supervectors.
The dimensions, counted in terms of real numbers are $n+\nu$ for S_V and $n \cdot \dim(\mathbb{R}_c) + \nu \cdot \dim(\mathbb{R}_a) = \infty$ for S_D.
Though the two spaces are algebraically equivalent (they have the same multiplication rules for their coordinates), they will show different behaviour when we represent the phase space of the Fermi oscillator by them: While on S_D there are infinitely many coherent state (this set is nevertheless undercomplete), on S_B the only eigenstate of the annihilation operator is the vacuum state $|0\rangle$.

4.2 Symmetric Supermanifolds

Def. 4.2.1 *A* **symmetric supermanifold** *is a topological space which can be locally described by n real coordinates and n Graßmann coordinates.*

(Symmetric means here that there is an equal number of c-type and a-type coordinates.) A neat way of obtaining a supermanifold of dimension (n, n) is by changing the parity of the fiber coordinates of a tangent bundle or a cotangent bundle.

Def. 4.2.2 *The* **parity operator** Π *acts on a fiber bundle by changing the parity of the fiber coordinates.*

Given the tangent bundle TM of an n-dimensional manifold as in remark 3.3.1, then Π replaces the fiber coordinates $(\dot{q}^i)_i$ by Graßmannian variables $(\xi^i)_i$. While an element of T_xM was given by $\dot{q}^i\partial_i$, an element of ΠT_xM has the form $\xi^i\partial_i$.
Similarly, if we would start with a naive Graßmannian manifold M_G which has Graßmann variables as local coordinates, the fiber coordinates of ΠTM_G become ordinary variables.
So the objects ΠTM, ΠT^*M, ΠTM_G and ΠT^*M_G are symmetric supermanifolds.
The cartesian product of ΠT_xM and ΠT_x^*M at each point of M will be denoted by Σ_M:

$$\Sigma_M := \Pi TM \underset{M}{\times} \Pi T^*M \tag{4.1}$$

Here the symbol $\times_M$ indicates that the tangent and cotangent spaces are combined over equal points of M.
Σ_M has obviously coordinates $(q^1, ..., q^n, \xi^1, ..., \xi^n, \pi_1, ..., \pi_n)$, where the q^i are c-type, the ξ^i and the π^i are a-type. Such an object will be called a dualized symmetric supermanifold:

Def. 4.2.3 *Given an ordinary manifold of dimension M. Then Σ_M is called a* **dualized symmetric supermanifold**.

We will basically work on dualized symmetric supermanifolds, as the tangent and cotangent spaces remain dual to each other, and allow us to define invariant objects. It is obvious, that by dropping one group of a-type variables, we obtain a symmetric supermanifold.
A superfunction is a function which depends on c-type and a-type variables:

Def. 4.2.4 *Given a set of commuting variables $x^1, ..., x^m$ and a set of anticommut-*

ing variables $\eta^1, ..., \eta^n$. *A map*

$$F(x, \eta) = \sum_{i=0}^{n} \frac{1}{i!} c_{a_1 ... a_i}(x) \eta^{a_1} ... \eta^{a_i} \tag{4.2}$$

where the $c_{a_1 ... a_i}(x)$ *are functions mapping* m *commuting variables to* $\mathbb{R}$*, is called a* **superfunction**.

Given an ordinary manifold M of dimension n with parity-changed tangent and cotangent bundle ΠTM and ΠT^*M. Functions on ΠTM and ΠT^*M are obviously superfunctions.

4.3 General Supermanifolds and their Reduction to Symmetric Supermanifolds

We want to define a general supermanifold as introduced by B.S. DeWitt and show, that it can be reduced to a symmetric supermanifold without loss of generality.
As manifolds are topological spaces, we have to introduce a topology[1] for a supervector space which induces the topology on the supermanifold.
First, we define two projections: π, which maps $\mathbb{R}^n_c \times \mathbb{R}^\nu_a$ on $\mathbb{R}^n$ by stripping the soul off the coordinates:

$$\pi(x^1, ..., x^n, \chi^1, ..., \chi^\nu) = (x^1_B, ..., x^n_B) \tag{4.3}$$

and b, which maps $\mathbb{R}^n_c \times \mathbb{R}^\nu_a$ on $\mathbb{R}^n_c \times \mathbb{R}^\nu_a$ essentially in the same way as π but without changing the type or the number of coordinates:

$$b(x^1, ..., x^n, \chi^1, ..., \chi^\nu) = (x^1_B, ..., x^n_B, 0, ..., 0) \tag{4.4}$$

Obviously, $\pi(x) = \pi(b(x))$ for each supervector x. Now we are ready to define a topology on $\mathbb{R}^n_c \times \mathbb{R}^\nu_a$.

[1] A topology is the set of subsets of a space, which we want to be open. At least the empty set ($\emptyset$) and the space itself have to be open. Furthermore, the union and the intersection of open sets have to be open again.

Def. 4.3.1 *A subset X of $\mathbb{R}^n_c \times \mathbb{R}^\nu_a$ will be called open, if it has the form $X = \pi^{-1}(Y)$ and Y is an open set in $\mathbb{R}^n$.*

With this definition, we loose all information on the soul components of the supervectors. There have been also attempts to introduce a topology for the soul components, but they were rather unsuccessful. This is most likely related to the fact, that soul components can be considered being infinitesimally close to the body component: Just take the example of supernumbers. For each supernumber $z = z_B + z_S$ there is a N which is larger than the number of different generators appearing in the soul z_S so that $z_S^N = 0$. Similarly to $\epsilon^2 = 0$ for infinitesimal ϵ, this can be interpreted as an infinitesimal distance between two supernumbers with the same body. As differences between supervectors are expressed in terms of supernumbers, it is clear that this concept is also valid in the case of supervector spaces.
Furthermore, our space is not Hausdorff[2] any more but only projectively Hausdorff. Now we are ready to generalize the definition of a manifold by:

Def. 4.3.2 *A* **supermanifold of pseudo-dimension (n,ν)** *is a topological space which is locally diffeomorphic to the space*[3] $\mathbb{R}^n_c \times \mathbb{R}^\nu_a$.

Locally, such a supermanifold has coordinates $x^1, ..., x^n$ (c-type) and $\eta^1, ..., \eta^\nu$ (a-type). An arbitrary coordinate transformation should not change the number of a-type and the number of c-type coordinates as otherwise stating that a supermanifold has pseudo-dimension (n, ν) instead of $(n + \nu)$ was senseless. B.S. DeWitt allows the following coordinate transformations (CT1):

$$\bar{x}^m = \sum_{r=0}^{\nu} \sum_{s=0}^{\infty} c_1 \frac{\partial^s X^m_{\mu_1 ... \mu_r}(x_B)}{\partial x_B^{n_1} ... \partial x_B^{n_s}} x_S^{n_1} ... x_S^{n_s} \eta^{\mu_r} ... \eta^{\mu_1} \tag{4.5}$$

$$\bar{\eta}^\mu = \sum_{r=0}^{\nu} \sum_{s=0}^{\infty} c_2 \frac{\partial^s X^\mu_{\mu_1 ... \mu_r}(x_B)}{\partial x_B^{n_1} ... \partial x_B^{n_s}} x_S^{n_1} ... x_S^{n_s} \eta^{\mu_r} ... \eta^{\mu_1} \tag{4.6}$$

[2] A space is called Hausdorff, if for two different points there is always an open set containing only one of them.

[3] *Since the space $\mathbb{R}^n_c \times \mathbb{R}^\nu_a$ is not a supervector space, we call (n, ν) "pseudo-dimension" to clarify the difference to $n + \nu$, the dimension of the supervector space containing $\mathbb{R}^n_c \times \mathbb{R}^\nu_a$.*

where $c_1 = \frac{\mathrm{i}^{r(r+1)/2}}{r!s!}$ and $c_2 = \frac{\mathrm{i}^{r(r-1)/2}}{r!s!}$ are just complex constants and the $X^m(x_B)$ or $X^\mu(x_B)$ are C^∞- functions with values in $\mathbb{R}_c$ or $\mathbb{R}_a$, so that the parity of the equation is matched.

These expressions basically correspond to a Taylor expansion in several variables.

One could think that it is possible to assign a body to each point in the following way:

"Given a supermanifold M of pseudo-dimension (n, ν) with a chart ϕ mapping M on $\mathbb{R}^n_c \times \mathbb{R}^\nu_a$. Then for a point p in M its body is given by $\phi^{-1} \circ b \circ \phi(p)$."

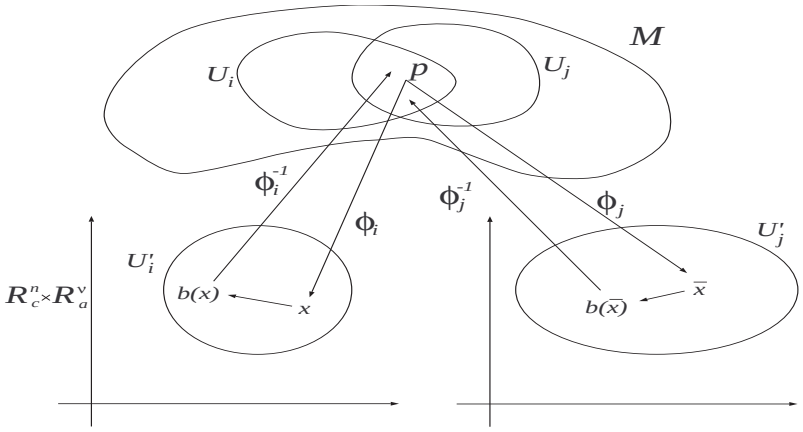

Figure 4.1: The naive definition of the body $p_B = \phi^{-1} \circ b \circ \phi(p)$ of a point is not invariant under (CT1).

Unfortunately, this definition is not invariant under (CT1), what is easy to prove. Invariance would require: $b(\bar{x}(x)) = \bar{x}(b(x))$. Considering the explicit formulæ for (CT1), we see that for $b(\bar{x}(x))$ only contributions for $r = s = 0$ can remain, as all other terms contain Graßmann generators. We are left with $b(\bar{x}^m) = c_1 b(X^m_o(x_B)$ and $b(\bar{\eta}^\mu) = 0$. In the case of $\bar{x}(b(x))$ we get again only terms for $r = s = 0$, as all other terms of x originally containing Graßmann variables vanish from $b(x)$. Thus we are left with $\bar{x}^m(b(x)) = c_1 X^m_0(x_B)$ and $\bar{\eta}^\mu(b(x)) = c_2 X^\mu_0(x_B)$. One could try to adjust (CT1), e.g. by starting the sum for the odd coordinates with $r = 1$, but even

then, usually $c_1 b(X_o^m(x_B) \neq c_1 X_0^m(x_B)$.

Another approach that could be tried is to call the "even part" of the supermanifold, i.e. the inverse image with fixed odd coordinates, the body. It is easy to see, that this breaks invariance under (CT1) even more seriously. Invariance would require:

$$\bar{x}(x,0) = \bar{x}(x,\eta) \tag{4.7}$$

$$\bar{\eta}(x,0) = 0, \tag{4.8}$$

and those equations are obviously not satisfied when (CT1) is applied to arbitrary coordinates.

As we want to create a structure on the supermanifold, to reduce its arbitrary complexity and eventually to obtain isomorphic symmetric supermanifolds, we would like to define at least the body of a supermanifold. As parts of such a manifold can obviously not be defined to be the body without loosing invariance under (CT1), we have to take the whole manifold and combine subsets by an equivalence relation to points. This is the approach also followed by B.S. DeWitt (see "Supermanifolds", p.55).

We define the subsets by:

Def. 4.3.3 *Given a supermanifold M with chart ϕ mapping M on $\mathbb{R}_c^n \times \mathbb{R}_a^\nu$, then the set $A(x) := \phi^{-1} \circ \pi^{-1} \circ \pi \circ \phi(x)$ is called an* **aura**.

Instead of the term aura, B.S. DeWitt uses "soul subspace", the term "halo" was discarded as it is already used in mathematics in a different context.[4] Obviously, $A(x)$ contains x and is invariant under coordinate transformation (CT1) which is

[4] According to esoteric teaching, the aura surrounds the body of entities and consists of their different "souls", so the analogy fits.

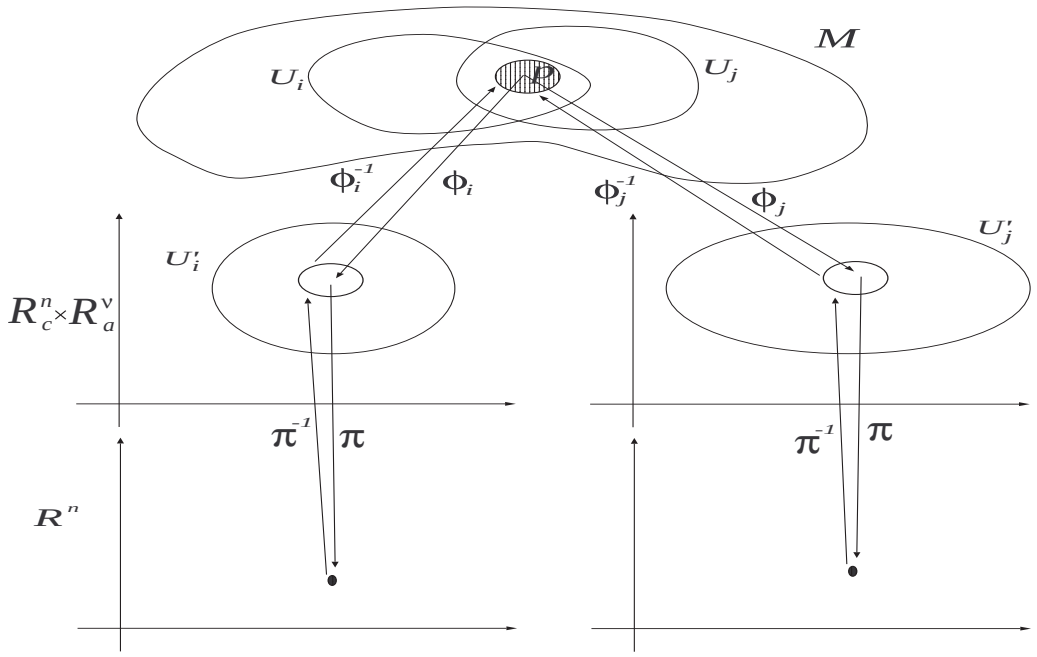

Figure 4.2: Auræ, as the one shown here (shaded area), are invariant under coordinate transformation: $\phi_i^{-1} \circ \pi^{-1} \circ \pi \circ \phi_i = \phi_j^{-1} \circ \pi^{-1} \circ \pi \circ \phi_j$ This enables us to define the body of a supermanifold.

easily seen:

$$\begin{aligned} c_1 b(X_0^m(x_B)) &= c_1 b(X_0^m(x_B)) \\ \pi(\bar{x}^m(x + x_S)) &= \pi(\bar{x}^m) \\ \pi(\bar{x}(x + x_S)) &= \pi(\bar{x}) \\ \pi(\bar{x}(x + x_S)) &= \pi(\bar{x}(x) + \bar{x}_S) \end{aligned}$$

The manifold $M' = \{A(x)|x \in M\}$ which has auræ of M as points and the chart $\pi \circ \phi$ mapping M on $\mathbb{R}^n$ takes over the topology of M which ignores completely the aura. Altogether M' is an ordinary real manifold, which we can call the body of M.

Def. 4.3.4 *The real manifold $M' = \{A(x)|x \in M\}$ with chart $\pi \circ \phi$ mapping M on $\mathbb{R}^n$ is called the* **body** *of M.*

We now introduce an equivalence class where elements are said to be equivalent, iff they are elements of the same aura

$$x_1 \sim x_2 \Leftrightarrow A(x_1) = A(x_2). \tag{4.9}$$

Choosing a representative of each aura we obtained a slicing of our supermanifold: the set of representatives forms an ordinary manifold M_R and fibers at each point x of M_R are given by the sets of points in M equivalent to x: $F_x = \{p \in M | p \sim x\}$. We will call this picture a **sliced supermanifold**.

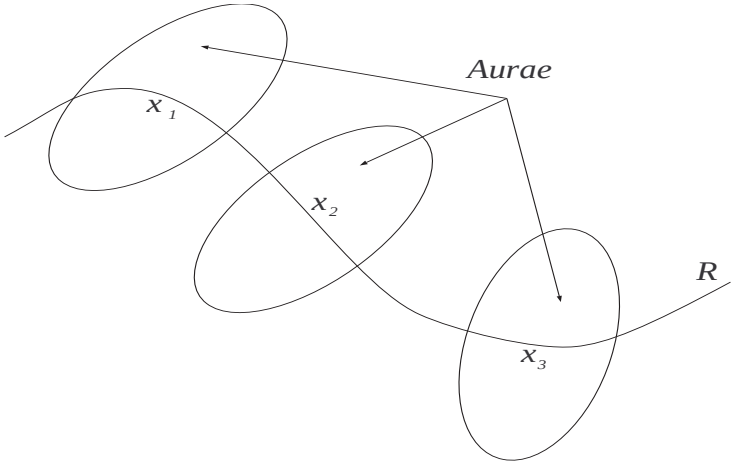

Figure 4.3: A sliced supermanifold consists of a real manifold R, which is the set of representatives for the auræ. The fiber at a point x is the set of equivalent points to the representative x: $F_x = \{p | p \sim x\}$ and can thus be regarded as the auræ attached to each representative.

This "slicing" can be considered invariant: The auræ are certainly invariant, but much more important is the fact that a different choice of representatives for each auræ is just a section of the slices. Since there is no metric on the manifold and no topology on the auræ, replacing the manifold with a section on its fiber does not destroy isomorphy, especially since the fibers are independent of each other. Particularly, there is no need to impose any kind of smoothness condition on the choice

of representatives, since, again, the topology is not influenced by soul components. A sliced supermanifold is locally described by n real coordinates referring to the body of the supermanifold and n c-type and ν a-type coordinates, containing Graßmann generators: $(x^1, ..., x^n, y^1, ..., y^n, \eta^1, ..., \eta^\nu)$. The intrinsic coordinate transformation for such a sliced supermanifold is (CT2):

$$\bar{x}^m = X^m(x) \tag{4.10}$$

$$\bar{y}^m = \sum_{r=0}^{\nu} \sum_{s=0}^{\infty} c_1 \frac{\partial^s Y^m_{\mu_1 ... \mu_r}(x)}{\partial x^{n_1} ... \partial x^{n_s}} y^{n_1} ... y_S^{n_s} \eta^{\mu_r} ... \eta^{\mu_1} \tag{4.11}$$

$$\bar{\eta}^\mu = \sum_{r=0}^{\nu} \sum_{s=0}^{\infty} c_2 \frac{\partial^s Y^\mu_{\mu_1 ... \mu_r}(x)}{\partial x^{n_1} ... \partial x^{n_s}} y^{n_1} ... y^{n_s} \eta^{\mu_r} ... \eta^{\mu_1}. \tag{4.12}$$

The constants are the same as in (CT1), $X^m(x)$ is an arbitrary bijective function, mapping real numbers to real numbers and Y^m, Y^μ are functions[5] $\mathbb{R}^n \to \mathbb{R}_c \backslash \mathbb{R}$ or $\mathbb{R} \to \mathbb{R}_a$ so that the parity of the equations is matched.

The next step is to really linearize the slices. It is clear, that varying $(y^i)_i$ and $(\eta^\iota)_\iota$ for one x by bodyless values, we obtain the whole slice at x. Since the coordinates are bodyless themselves, multiplication with a supernumber does not change this and we can consider this space as a vector space over the ring of supernumbers. The intrinsic coordinate transformations here are linear maps:

$$\bar{x}^m = X^m(x) \tag{4.13}$$

$$\bar{y}^m = Y(x)^m_n y^n + Y(x)^m_\nu \eta^\nu \tag{4.14}$$

$$\bar{\eta}^\mu = \Upsilon(x)^\mu_n y^n + \Upsilon(x)^\mu_\nu \eta^\nu, \tag{4.15}$$

where $Y(x)^m_n$ and $\Upsilon(x)^\mu_\nu$ are maps $\mathbb{R}^n \to \mathbb{R}_c$ and $Y(x)^m_\nu$ and $\Upsilon(x)^\mu_n$ are maps $\mathbb{R}^n \to \mathbb{R}_a$.

Up to here, we reduced an arbitrary supermanifold of pseudo-dimension (n, ν) to a manifold of dimension n with an even vector bundle of dimension n and an odd

[5] Excluding $\mathbb{R}$ makes sure, that the range consists only of supernumbers without body.

vector bundle with dimension ν. Using an ordinary Manifold M_R with dimension large enough, this means:

$$M \simeq TM_R \underset{M_R}{\times} \Pi T_R^M \simeq TM_R \underset{M_R}{\times} \Pi T^* M_R \simeq ... \tag{4.16}$$

The isomorphism $\simeq$ is not exact, M is usually (i.e. for $n \neq \nu$) isomorphic to a subset of M_R. This reduction certainly works only if we only care for the algebra of the coordinates and if we are not interested in the way it is represented (i.e. by Graßmann and ordinary variables or variables allowing bodyless supernumbers.)
It is possible to add the bodyless even coordinates and the coordinates on the manifold to new coordinates: $z^i = x^i + y^i$. As the splitting in body and soul is unique, this mapping can always be inverted. Furthermore, considering the even fibers together with the real manifold as a new (product) manifold, we finally obtained the set of coordinates z^i for the manifold and fiber coordinates η^i. This construction is isomorphic (again, using the right amount of dimensions and dropping the superfluous ones) to a symmetric supermanifold, if we consider only the algebra of the coordinates and not their representation:

$$M \simeq \Pi T M_R \tag{4.17}$$

Summarizing, all the work we do on symmetric supermanifolds can be directly translated on arbitrary supermanifolds (e.g. those proposed by B.S. DeWitt) which will simplify our work considerably.

Chapter 5

Homology Groups and de Rham Complexes

"Math is like love; a simple idea, but it can get complicated."

R. Drabek

After defining forms on manifolds, we want to examine subsets of our manifolds over which we can integrate the forms. In the theory of homology groups, topological spaces are usually constructed from simplices, which are the generators of chains. Those chains provide possible areas of integration and thus can be considered dual to the space of n-forms. Both the chains and the forms build de Rham complexes, which are dual to each other.

5.1 Homology Groups

An n-simplex is the closed envelope of $n+1$ points of the $\mathbb{R}^n$, which are not all in a proper linear subspace. A homeomorph image of an n-simplex is a topological n-simplex. In classical topology, one tries to construct topological spaces from topological simplices, which does not work for general topological spaces. Therefore

today the more convenient singular simplices are used:

Def. 5.1.1 *The* **q-standard simplex** Δ^q *is given by the set:*

$$\Delta^q := \{x \in \mathbb{R}^{q+1} | \forall i \in [0, 1, ..., q] : x_i \geq 0, x_0 + ... + x_q = 1\}. \tag{5.1}$$

From the standard simplices, the singular simplices are constructed by:

Def. 5.1.2 *Given a topological space* T*. If* f *is a continuous map* $f : \Delta^q \to T$ *then* f *is called a* **singular** q**-Simplex**.

The simplest way of getting an $(q-1)$-simplex from a q-simplex is obviously to take the boundary of the second one. We define:

Def. 5.1.3 *The* **boundary operator** ∂ *acts on a* q*-simplex by the rule:*

$$\partial f := \sum_{j=0}^{q} (-1)^j f^{(j)} \text{ where } f^{(j)}(x_0, ..., x_{q-1}) := f(x_0, ..., x_{j-1}, 0, x_j, ..., x_{q-1}) \tag{5.2}$$

Note that the sum is just a formal addition, it is *not* a vector addition. Furthermore, each $f^{(j)}$ is a singular, $(q-1)$-boundary simplex provided with an orientation.
The singular simplexes can be used as basis of a vector space of chains:

Def. 5.1.4 *Given a non-empty subset* $N \subset M$ *with a relation "*$\geq$*" for pairs* (u, v) *with* $u, v \in N$*. Iff for all* $u, v \in N$ *it is always* $u \geq v$ *or* $v \geq u$ *then* N *is called a* **chain**.

Def. 5.1.5 *A* **singular** q**-chain** *is an arbitrary linear combination of singular* q*-simplices:*

$$C := a_1 f_1 + ... + a_m f_m \text{ where } a_1, ..., a_m \in \mathbb{R} \text{ or } \mathbb{C} \tag{5.3}$$

If $T \supset \text{im}(f_i)$ *is a manifold and all the* f_i *are smooth maps, then the chain is also called smooth. The set of all smooth* q*-chains will be denoted by* $\mathfrak{C}_q(M)$

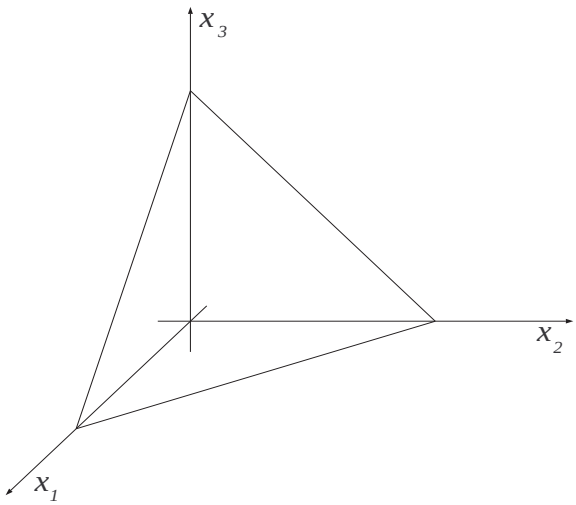

Figure 5.1: The standard 2-simplex is a triangle with the corners on the coordinate axes.

Note that the sum is again just formal but represents again the orientation of the singular simplex in the chain. The orientation of the chain takes the role of the "$\geq$" relation.

We can define sums of singular q-chains by adding the corresponding linear combinations, so the sum of two q-chains is again a q-chain.

The boundary operator for chains is obtained by linear continuation of the boundary operator for singular simplices:

Def. 5.1.6 *Let C be a q-chain as in (5.3). The action of the boundary operator ∂ on C is given by:*

$$\partial C := a_1 \partial f_1 + ... + a_m \partial f_m. \tag{5.4}$$

Finally it remains to define the integral of forms on chains, which is also done by linear continuation:

Def. 5.1.7 *Given a q-form ω and a q-chain C as in (5.3). Then the integral of ω*

on C is given by:

$$\int_C \omega := a_1 \int_{f_1} \omega + ... + a_m \int_{f_m} \omega. \tag{5.5}$$

5.2 De Rham Complexes

A de Rham complex is a special Hilbert complex, for which the linear operators are the exterior derivatives of n-forms. Since the exterior derivative is an elliptic differential operator, the de Rham complex is also an elliptic complex.

Def. 5.2.1 *Let $(H_i)_{0\leq i\leq n}$ be a sequence of Hilbert spaces and $H_{n+1} := \{0\}$. Let $(\mathrm{D}_i)_{0\leq i\leq n}$ be a sequence of linear operators D_k whose domain $\mathcal{D}_k = \mathrm{dom}(\mathrm{D}_k)$ is dense in H_k and $\mathrm{im}(\mathrm{D}_k) \subset H_{k+1}$. We demand furthermore that $\mathrm{im}(\mathrm{D}_k) \subset \mathrm{D}_{k+1}$ and $\mathrm{D}_{k+1} \circ \mathrm{D}_k = 0$.*
Then the sequence

$$0 \longrightarrow \mathcal{D}_0 \xrightarrow{\mathrm{D}_0} \mathcal{D}_1 \xrightarrow{\mathrm{D}_1} ... \xrightarrow{\mathrm{D}_n} \mathcal{D}_n \xrightarrow{\mathrm{D}_{n+1}} 0 \tag{5.6}$$

is called a **Hilbert complex**, *denoted by $(\mathrm{D}, \mathcal{D})$ where $\mathrm{D} = \bigoplus \mathrm{D}_k$ and $\mathcal{D} = \bigoplus \mathcal{D}_k$.*

The condition $\mathrm{D}_{k+1} \circ \mathrm{D}_k = 0$ can also be written as $\mathrm{im}(D_k) \subset \ker(D_{k+1})$.
If the direction of the sequence is reversed, the dual complex is obtained:

Def. 5.2.2 *Given a Hilbert complex $(\mathrm{D}, \mathcal{D})$. Then the sequence*

$$0 \longrightarrow \mathcal{D}_n \xrightarrow{\mathrm{D}^*_{n-1}} \mathcal{D}_{n-1} \xrightarrow{\mathrm{D}^*_{n-2}} ... \xrightarrow{\mathrm{D}^*_0} \mathcal{D}_0 \longrightarrow 0 \tag{5.7}$$

is again a Hilbert complex. It is called the **dual complex** *of $(\mathrm{D}, \mathcal{D})$ and denoted by $(\mathrm{D}^*, \mathcal{D}^*)$ where again $\mathrm{D}^* = \bigoplus \mathrm{D}^*_k$ and $\mathcal{D}^* = \bigoplus \mathcal{D}^*_k = \bigoplus \mathcal{D}_{n-k}$.*

Now consider the following complex (the definitions of $\Omega^\bullet$ etc. are the same as in chapter 3):

Def. 5.2.3 *Let $\Omega^k(M)$ be the set of all smooth fields of k-forms on a manifold M: $\Omega^k(M) = C^\infty(\wedge^k T^*M)$ and $\Omega^\bullet = \bigoplus \Omega^k$. Let d_k be the exterior derivative: $\mathrm{d}_k : \Omega^k \longrightarrow \Omega^{k+1}$. Then the sequence*

$$0 \longrightarrow \Omega^0 \xrightarrow{\mathrm{d}_0} \Omega^1 \xrightarrow{\mathrm{d}_1} ... \xrightarrow{\mathrm{d}_n} \Omega^n \longrightarrow 0 \tag{5.8}$$

is called a **de Rham complex.**

We can show that:

Lemma 5.2.1 *The de Rham complex is the Hilbert complex (d_k, Ω).*

Proof: Consider the scalar product $(\omega, \eta) := \int_M \omega \wedge *\eta$.[1] Completion of Ω^k with respect to this scalar product yields obviously a Hilbert space. Since the action of the operator d_k is well defined on Ω^k, $\mathrm{im}(\mathrm{d}_k) \subset \Omega^{k+1} = \mathrm{dom}(\mathrm{d}_{k+1})$ and $\mathrm{d}_{k+1}\mathrm{d}_k = 0$, the de Rham complex has all the properties of a Hilbert complex.#

The **dual of the de Rham complex** is obviously given by (i_X, Ω):

$$0 \longrightarrow \Omega^n \xrightarrow{\mathrm{i}_X} \Omega^{n-1} \xrightarrow{\mathrm{i}_X} ... \xrightarrow{\mathrm{i}_X} \Omega^0 \longrightarrow 0 \tag{5.9}$$

Note that i_X can be applied to any Ω^k. If we introduce the (superfluous) index k to denote the action of i_X on Ω^{n-k}: $\mathrm{i}_{X,k}$, we can write: $\mathrm{im}(\mathrm{i}_{X,k}) \subset \Omega^{n-(k+1)} = \mathrm{dom}(\mathrm{i}_{X,k+1})$ and $\mathrm{i}_{X,k+1}\mathrm{i}_{X,k} = \mathrm{i}_X\mathrm{i}_X = 0$, as previously shown.
The dual of a de Rham complex will also be called a de Rham complex.

5.3 The Complex of Chains

Given an n-dimensional Manifold M, then we can construct two complexes on M: the de Rham complex

$$0 \longrightarrow \Omega^0 \xrightarrow{\mathrm{d}_0} \Omega^1 \xrightarrow{\mathrm{d}_1} ... \xrightarrow{\mathrm{d}_n} \Omega^n \longrightarrow 0 \tag{5.10}$$

[1] The Hodge operator '*' is a map $* : \Omega^k \longrightarrow \Omega^{n-k}$, which shall not be introduced explicitely.

and a complex of singular chains (or currents):

$$0 \longrightarrow \mathfrak{C}_n \xrightarrow{\partial} \mathfrak{C}_{n-1} \xrightarrow{\partial} ... \xrightarrow{\partial} \mathfrak{C}_0 \longrightarrow 0. \tag{5.11}$$

Here, $\mathfrak{C}_0$ is the set of points on M, $\mathfrak{C}_1$ the set of lines, etc. The manifold M itself is an element of $\mathfrak{C}_n$.

As we noticed above, we can always integrate a q-form ω over a q-chain Γ. In the special case where $q = 0$ (ω is a function on M and Γ is the formal sum of points on M), we define $\int_\Gamma \omega = \sum \omega(\Gamma_i)$.

As the integral is a linear map, we can regard $\mathfrak{C}_q$ and Ω^q as dual to each other with the dual product:

$$\langle \Gamma, \omega \rangle := \int_\Gamma \omega, \text{ where } \Gamma \in \mathfrak{C}_q, \, \omega \in \Omega^q. \tag{5.12}$$

We see an obvious connection between forms and currents, if we write a current by a form and a delta-function:

$$\Gamma' = \mathrm{d}^n x \delta^{n-q}(f(x)) \text{ for } \Gamma \in \mathfrak{C}_q, \tag{5.13}$$

where the original singular chain is a solution of $f(x) = 0$.

With this substitution we can write:

$$\int_\Gamma \omega = \int_M \Gamma' \omega. \tag{5.14}$$

So we can assign an $n - q$-form Γ' to each q-chain Γ, or equally identify Ω^q with $\mathfrak{C}_{n-q}$.

Now we change the index of the chains by raising the index, this, in the language of Homology, is equivalent to inverting its sign: $\mathfrak{C}_{n-q} \to \mathfrak{C}^{q-n}$.

Altogether, we obtained the Poincaré duality:

$$\Omega^q \xrightarrow{id} \mathfrak{C}^{q-n} = \mathfrak{C}_{n-q} \xrightarrow{dual} \Omega^{n-q} \tag{5.15}$$

The two complexes now are:

$$0 \longrightarrow \quad \Omega^0 \quad \xrightarrow{\mathrm{d}_0} \Omega^1 \xrightarrow{\mathrm{d}_1} ... \xrightarrow{\mathrm{d}_n} \Omega^n \longrightarrow 0 \tag{5.16}$$

$$0 \longrightarrow \mathfrak{C}^{-n} \xrightarrow{\partial} ... \xrightarrow{\partial} \mathfrak{C}^{-1} \xrightarrow{\partial} \quad \mathfrak{C}^0 \quad \longrightarrow 0 \tag{5.17}$$

where this time $\mathfrak{C}^{-p}$ is dual to Ω^p.

Consider the following example of the dual product: $\langle \Gamma, 1 \rangle = 0$ with $\Gamma \in \mathfrak{C}^0$. This is equivalent to $\Gamma = \partial \Gamma'$ with $\Gamma' \in \mathfrak{C}^{-1}$: Given the formal linear combination of lines Γ', we have to evaluate the 0-form 1 for each line at its endpoint and its starting point and subtract them, which yields obviously 0. On the other hand it is possible to construct a Γ' with $\Gamma' = \partial\Gamma$. Note that $\Gamma = \sum a_m p_m$ and $\langle \Gamma, 1 \rangle = \sum a_m 1(p_m) = \sum a_m = 0$. Each positive a_m in the sum represents an endpointp_m of $|a_m|$ lines, each negative a_m the starting point of $|a_m|$ lines, so altogether we have as many endpoints as starting points. By connecting them arbitrarily and taking the formal sum of those lines, we constructed Γ' with $\partial\Gamma' = \Gamma$.

The next step is to define the action of several operators on chains. (A general differential operator is an object of the form $P = \sum c_\alpha \partial_\alpha$, where α is a multiindex, the set of all differential operators of order q is denoted by $\mathcal{D}^q$, the set of all differential operators by $\mathcal{D}^\bullet = \bigoplus \mathcal{D}^q$.)

Def. 5.3.1 *We define:*

$$\langle \Gamma^{-q} \mathrm{P}, \omega^q \rangle \quad := \quad \langle \Gamma^{-q}, \mathrm{P}\omega^q \rangle \qquad \mathrm{P} \text{ : } \textit{differential operator} \tag{5.18}$$

$$\langle \Gamma^{-q} \mathrm{e}_f, \omega^{q-1} \rangle \quad := \quad \langle \Gamma^{-q}, \mathrm{e}_f \omega^{q-1} \rangle = \langle \Gamma^{-q}, \mathrm{d}f \wedge \omega^{q-1} \rangle \tag{5.19}$$

$$\langle \Gamma^{-q} \mathrm{i}_X, \omega^{q+1} \rangle \quad := \quad \langle \Gamma^{-q}, \mathrm{i}_X \omega^{q+1} \rangle \tag{5.20}$$

$$\langle \Gamma^{-(p+q)} \omega^p, \omega^q \rangle \quad := \quad \langle \Gamma^{-(p+q)}, \omega^p \wedge \omega^q \rangle \tag{5.21}$$

where Γ^{-q} denotes a q-chain (element of $\mathfrak{C}^{-q}$) and ω^q a q-form.

So, $\mathfrak{C}^{-p}$ is a $\mathcal{D}^\bullet$-right module, and we can multiply a chain by a form and the operators $\mathrm{e}(\mathrm{d}f)$ and i_X.

Together with $PQ\omega = P(Q(\omega))$ it follows that $\Gamma PQ = (\Gamma P)Q$. The special case $\langle \Gamma^{-p}, \omega^p \rangle = \langle \Gamma^{-p}\omega^p, 1, \rangle$ shows that the degree of a product of a form and a chain should certainly be defined as $\deg(\Gamma\omega) = \deg(\Gamma) + \deg(\omega)$.

Note that the products with chains are not explicitly defined, but only in the context of its dual product with a form.

Chapter 6

A New Representation of the Fock Space

"An equation only makes sense to me, if it expresses a thought of God."

Srinivasa Ramanujan

6.1 Invariant Objects on Manifolds and their Counterparts on Symmetric Supermanifolds

Our goal in this section is to introduce operators and invariant objects on an ordinary manifold and its tangent and cotangent bundles and extend them to symmetric supermanifolds.

Let us briefly consider the action of a linear change of coordinates on some of the objects defined on $S_M = TM \times_M T^*M$ and $\Sigma_M = \Pi TM \times_M \Pi T^*M$. Given an invertible matrix C which transforms the basis vectors $\bar{e}_i = C_i{}^j\, e_j$ and thus the coordinates of a vector with the inverse: $\bar{x}^i = (C^{-1})^i{}_j\, x^j$. (Note that e_i is a vector

but x^i a real number.) We immediately get the transformation rules:

$$\dot{\bar{x}}^i = \frac{\mathrm{d}\bar{x}^i}{\mathrm{d}t} = \frac{\partial \bar{x}^i}{\partial x^j}\frac{\mathrm{d}x^j}{\mathrm{d}t} = (C^{-1})^i{}_j\, \dot{x}^i \tag{6.1}$$

$$\bar{\partial}_i = \frac{\partial}{\partial \bar{x}^i} = \frac{\partial x^j}{\partial \bar{x}^i}\frac{\partial}{\partial x^j} = C_i{}^j \partial_j \tag{6.2}$$

It follows, that $\dot{x}^i\partial_i$ is invariant, as expected. For the cotangent space we have to consider the invariant dual product:

$$\langle p_i \mathrm{d}x^i, \dot{x}^j\partial_j\rangle = \langle \bar{p}_i \mathrm{d}\bar{x}^i, \dot{\bar{x}}^j\bar{\partial}_j\rangle \tag{6.3}$$

$$p_i \cdot \dot{x}^j\langle \mathrm{d}x^i, \partial_j\rangle = \bar{p}_i \cdot \dot{\bar{x}}^j\langle \mathrm{d}\bar{x}^i, \bar{\partial}_j\rangle \tag{6.4}$$

$$p_i\dot{x}^i = \bar{p}_i\dot{\bar{x}}^i \tag{6.5}$$

The result is not only that $p_i\dot{x}^i$ is invariant, but that p_i transforms inversely to x^i: $\bar{p}_i = C_i{}^j\, p_j$ and, as $p_i\mathrm{d}x^i$ is invariant, $\mathrm{d}\bar{x}^i = (C^{-1})^i{}_j\, \mathrm{d}x^j$. The fact that $\mathrm{d}x^i$ and x^i transform similarly is consistent with the picture that $\mathrm{d}x^i$ is an infinitesimal part of x^i.

Altogether, objects with an upper index transform with C^{-1} (covariantly) and objects with a lower index with C (contravariantly):

covariantly transforming	A_i (gen. covector field)		∂_i	p_i	π_i
contravariantly transforming	X^i (gen. vector field)	q^i	$\mathrm{d}x^i$	$\dot{q}^i$	ξ^i

A generic vector field is a smooth section in TM: $X : M \to TM$ and given by the coordinates $(q^1, ..., q^N, x^1(q), ..., x^N(q))$, a generic covector field is a smooth section in T^*M: $A : M \to T^*M$ and given by coordinates $(q^1, ..., q^N, A_1(q), ..., A_N(q))$.

Invariant objects are now products of one co- and one contravariant transforming object on M. We introduce the simplifying notation:

Def. 6.1.1 *For the product of two opposingly transforming objects, we introduce the abbreviation "$\cdot$" which will mean $a \cdot b = a_i b^i$ if a transforms covariantly and b contravariantly and $a \cdot b = a^i b_i$ otherwise.*

Given a manifold M of dimension n with local coordinates $q^1, ..., q^n$. The vector bundle TM/T^*M is the set of all tangential/cotangential spaces of all points on M. Elements of T_xM are described by coordinates $\dot{q}^1, ..., \dot{q}^n$, elements of T_x^*M by coordinates $p_1, ..., p_n$. So altogether we have the topological space

$$S_M := TM \underset{M}{\times} T^*M \tag{6.6}$$

with coordinates $(q^1, ..., q^n, \dot{q}^1, ..., \dot{q}^n, p_1, ..., p_n)$ and similarly Σ_M:

$$\Sigma_M := \Pi TM \underset{M}{\times} \Pi T^*M \tag{6.7}$$

with coordinates $(q^1, ..., q^n, \xi^1, ..., \xi^n, \pi_1, ..., \pi_n)$.
On T^*M, we can define the following objects:

- the 1-form $\lambda := p \cdot \mathrm{d}q$
- the 2-form $\omega := \mathrm{d}\lambda = \mathrm{d}p_i \wedge \mathrm{d}q^i$
- the volume element (Liouville form) $\frac{1}{N!}\omega \wedge ... \wedge \omega = \mathrm{d}p_1 \wedge \mathrm{d}q^1 \wedge ... \wedge \mathrm{d}p_N \wedge \mathrm{d}q^N$

and the corresponding objects on ΠT^*M:

- the 1-form $\lambda := \pi \cdot \mathrm{d}q$
- the 2-form $\omega := \mathrm{d}\lambda = \mathrm{d}\pi_i \wedge \mathrm{d}q^i$
- the volume element $\frac{1}{N!}\omega \wedge ... \wedge \omega = \mathrm{d}\pi_1 \wedge \mathrm{d}q^1 \wedge ... \wedge \mathrm{d}\pi_N \wedge \mathrm{d}q^N$

The Poisson bracket is generalized to the Schouten bracket, which respects the parity of the functions:

$$\{u, v\}_{p,q,Schouten} = \{u, v\}_{p,q} = \left(\partial_{q^i} u\right)\left(\partial_{p_i} v\right) - (-1)^{\tilde{u}\tilde{v}}\left(\partial_{p_i} u\right)\left(\partial_{q^i} v\right). \tag{6.8}$$

6.2 $\Omega^\bullet$ as Ascending de Rham Complex

Imagine an ordinary manifold M with dimension n with the fields of forms given by $\Omega^\bullet = \bigoplus_{i=0}^{n} \Omega^i$.

$\Omega^\bullet$ can be consideres as the de Rham complex

$$0 \to \Omega^0(M) \xrightarrow{\mathrm{d}} \Omega^1(M) \xrightarrow{\mathrm{d}} ... \xrightarrow{\mathrm{d}} \Omega^n(M) \xrightarrow{\mathrm{d}} 0 \tag{6.9}$$

as seen in the previous chapter.

The complex ends on the right side, as d is a map $\Omega^n(M) \to \Omega^{n+1}(M)$ and all forms Ω^k with $k \geq n$ vanish. (A form of order $n+1$ contains at least one of the n nilpotent one-forms twice.)

Further operators acting on this de Rham complex are M_f which multiplies a form by a function and thus is a map $\Omega^n \times \Omega^0 \to \Omega^n$, the Lie derivative $\mathcal{L}_X$ which is a map $\Omega^n \times \mathcal{X}(M) \to \Omega^n$ and the wedge product with an exact 1-form: $\mathrm{e}_f = [\mathrm{d}, M_f] = \mathrm{d}f\wedge$ mapping $\Omega^n \to \Omega^{n+1}$. Together with the last operator, all elements of $\Omega^\bullet$ can be constructed starting with the constant function $f(x) = 1$ on a manifold $\Omega^0 = C^\infty(M)$: Given an element of $\Omega^n(M)$: $\omega(x) = \omega_{i_1,...,i_n}(x)\mathrm{d}x^{i_1}...\mathrm{d}x^{i_n}$, we can construct $\omega(x)$ by $\omega(x) = \mathrm{e}_{x^{i_1}}...\mathrm{e}_{x^{i_n}} M_{\omega_{i_1,...,i_n}(x)} \cdot 1$ and each element of $\Omega^\bullet$ is just a sum of elements of $\Omega^n(M)$.

We want to show that the set of superfunctions on ΠTM is isomorphic to $\Omega^\bullet(M)$:

$$C^\infty(\Pi TM) \simeq \Omega^\bullet(M) \tag{6.10}$$

The structures of a superfunction and an arbitrary (dual) form are obviously equal:

$$\begin{aligned} F(x,\xi) &= \sum_k \frac{1}{k!} f(x)_{i_1,...,i_k} \xi^{i_1}...\xi^{i_k} \\ \omega(x) &= \sum_k \frac{1}{k!} \omega(x)_{i_1,...,i_k} \mathrm{d}x^{i_1} \wedge ... \wedge \mathrm{d}x^{i_k} \end{aligned}$$

$f(x)_{i_1,...,i_k}$ and $\omega(x)_{i_1,...,i_k}$ are both functions $\mathbb{R}^n \to \mathbb{R}$ and totally antisymmetric under the exchange of an index, so the isomorphicity is clear.

It remains to find the operators for superfunctions which correspond to the operators for $\Omega^\bullet$. The multiplication with a function $f \in C^\infty(M)$ remains obviously the same. The exterior derivative becomes $\mathrm{d} \to \delta = \xi^i \partial_{x^i}$, and from those two definitions we obtain $\mathrm{e}_f = [\delta, M_f] = (\partial_{x^i} f)\xi^i$. The obvious choice for the interior product is $\mathrm{i}_X = X^i \partial_{\xi^i}$.

With those operators, we can map the de Rham complex of forms on a de Rham complex of superfunctions and vice versa.

6.3 $\mathcal{V}_\bullet$ as Descending de Rham Complex

As we saw in the previous chapter, the dual complex to the complex of forms is the complex of chains which can be identified with the complex of forms, where $\mathfrak{C}_p$ corresponds to Ω^{n-p}. We denote this complex by $\mathcal{V}_\bullet$.

Here, we want to show that this complex is isomorphic to a complex on $C^\infty(\Pi T^* M)$, where the only difference to the previous section is, that the odd coordinates $(\pi_i)_i$ now transform covariantly, while the $(\xi^i)_i$ transformed contravariantly.

The structures of a superfunction and an arbitrary form are obviously equal:

$$\begin{aligned} F(x,\pi) &= \sum_k \frac{1}{k!} f(x)^{i_1,\dots,i_k} \pi_{i_1} \dots \pi_{i_k} \\ \omega(x) &= \sum_k \frac{1}{k!} \omega(x)^{i_1,\dots,i_k} \mathrm{d}x_{i_1} \wedge \dots \wedge \mathrm{d}x_{i_k} \end{aligned}$$

where $f(x)^{i_1,\dots,i_k}$ and $\omega(x)^{i_1,\dots,i_k}$ are contravariantly transforming real functions which are totally antisymmetric under exchange of indices and

$$\mathrm{d}x_{i_1} \wedge \dots \wedge \mathrm{d}x_{i_k} = \bigwedge_{j \notin \{i_1,\dots,i_k\}} \mathrm{d}x^j. \tag{6.11}$$

As $\mathrm{d}x_{i_1} \wedge \dots \wedge \mathrm{d}x_{i_k}$ is generated by contracting an invariant[1] top-form with a covariantly transformig tensor, it is clear, that this expression has to transform covariantly,

[1] under unitary transformations

too.

After clarifying that the structure is identical, let us again proceed with defining the operators.

The multiplication remains the same. The exterior derivative is $\mathrm{d} \to \delta = \partial_{\pi_i}\partial_{x^i}$. This leads to $\mathrm{e}_f = [\delta, M_f] = (\partial_{x^i} f)\partial_{\pi_i}$. The obvious choice for the interior product is $\mathrm{i}_X = X^i \pi_i$.

Again we have shown the isomorphism

$$C^\infty(\Pi T^* M) \simeq \mathcal{V}_\bullet(M) \tag{6.12}$$

and found the corresponding de Rham complex on $C^\infty(\Pi T^* M)$.

Altogether, we have the relations:

on $\Omega^\bullet(\mathbb{M})$	on $C^\infty(\Pi T\mathbb{M})$	on $\Omega_\bullet(\mathbb{M})$	on $C^\infty(\Pi T^*\mathbb{M})$
M_f	M_f	M_f	M_f
$\mathrm{d} = \mathrm{d}x^i \wedge \partial_{x^i}$	$\xi^i \partial_{x^i}$	$e^i \partial_{x^i}$	$\partial_{\pi_i}\partial_{x^i}$
$e_f = \mathrm{d}f\wedge$	$(\partial_{x^i} f)\xi^i$	$(\partial_{x^i} f)e^i$	$(\partial_{x^i} f)\partial_{\pi_i}$
i_X by contr.	$X^i \partial_{\xi^i}$	$X^i \mathrm{d}x^i\wedge$	$X^i \pi_i$

6.4 Complexes on Graßmannian Manifolds

Obviously we could have also started with a Graßmannian manifold, where $\Pi T M_G$ has coordinates $(\xi^1, ..., \xi^n, x^1, ..., x^n)$ and we would have found an isomorphism between naive differential forms on M_G and superfunctions:

$$\begin{aligned} C^\infty(\Pi T M_G) &\simeq O^\bullet(M_G) \\ C^\infty(\Pi T^* M_G) &\simeq \mathcal{V}_\bullet(M_G) \end{aligned}$$

This is due to the fact that Graßmann forms are totally symmetric and thus not nilpotent.

Since the development of this formalism works in exactly the same way as in the previous sections, we refrain from repeating this.

Altogether we see that a superfunction can be mapped on a wedge product of forms:

$$\begin{aligned} F(x,\xi) &= \sum_{i=0}^{\infty} f_{k_1,\ldots,k_i}(x)\xi^{k_1}\ldots\xi^{k_i} \\ &= \sum_{i=0}^{\infty}(\sum_{j=0}^{\infty} a_{m_1,\ldots,m_j;k_1,\ldots,k_i}x^{m_1}\ldots x^{m_j})\xi^{k_1}\ldots\xi^{k_i} \\ &\simeq \sum_{i=0}^{\infty}(\sum_{j=0}^{\infty} a_{m_1,\ldots,m_j;k_1,\ldots,k_i}\mathrm{d}\xi^{m_1}\wedge\ldots\wedge\mathrm{d}\xi^{m_j})\wedge\mathrm{d}x^{k_1}\wedge\ldots\wedge\mathrm{d}x^{k_i} \end{aligned}$$

So we can map a function on ΠTM on a form of M.

6.5 The Fock Spaces

A Fock space is basically given by an harmonic quantum mechanical oscillator for each point in momentum/position space. In analogy to the bosonic case in quantum mechanics, where states are represented by functions $L_2(\mathbb{R}) \subset C^\infty(\mathbb{R})$, we expect fermionic states to be represented by functions of odd variables, which is shown for the Fermi oscillator in the following chapter.

As we know, that the sets of superfunctions are isomorphic to sets of forms or chains, we expect to find the structure of a Fock space in $\Omega^\bullet$.

The structure we are looking for is:

$$\begin{aligned} \hat{a}_i|0\rangle &:= 0 \\ \left[\hat{a}_i,\hat{a}_j^\dagger\right]_\mp &= \hat{a}_i\hat{a}_j^\dagger \mp \hat{a}_j^\dagger\hat{a}_i := \delta_{ij} \end{aligned}$$

where i runs over the degrees of freedom and the upper sign refers to a bosonic Fock space, the lower sign to a fermionic one.

6.5.1 Fermionic Hilbert spaces

If we want forms to represent a fermionic Hilbert space, we have the following operators available to act on them:

$$\mathrm{d}, \mathrm{e}_f, \mathrm{i}_X, \mathcal{L}_X \tag{6.13}$$

As proven in chapter 3, the operators e_f and i_X follow (anti)commutation relations similar to creation and annihilation operators. With a countable set of possible quantum states $(i)_i$ for each particle, we define the fermionic creation and annihilation operators to be:

$$\hat{f}^\dagger(i) = \mathrm{e}_{x^i} \text{ and } \hat{f}(i) = \mathrm{i}_{e_i} \tag{6.14}$$

where $e_i = \partial_i$ (see 3.1). Certainly one could exchange "creation" and "annihilation" in the previous identification and get Dirac's hole theory; but this would lead to two inconveniences, which will become clearer after discussing the bosonic Hilbert space (first, we loose the common vacuum of fermions and bosons, second, we have to work with codimensions in the bosonic case).
Thus we get from theorem (3.5.3) the relations:

$$\hat{f}^\dagger(i)\hat{f}^\dagger(j) + \hat{f}^\dagger(j)\hat{f}^\dagger(i) = \mathrm{e}_{x^i}\mathrm{e}_{x^j} + \mathrm{e}_{x^j}\mathrm{e}_{x^i} = 0 \tag{6.15}$$

$$\hat{f}(i)\hat{f}(j) + \hat{f}(j)\hat{f}(i) = \mathrm{i}_{e_i}\mathrm{i}_{e_j} + \mathrm{i}_{e_j}\mathrm{i}_{e_i} = 0 \tag{6.16}$$

Eventually, theorem (3.5.2) yields the last relation:

$$\hat{f}(i)\hat{f}^\dagger(j) + \hat{f}^\dagger(j)\hat{f}(i) = \mathrm{i}_{e_i}\mathrm{e}_{x^j} + \mathrm{e}_{x^j}\mathrm{i}_{e_i} = \partial_i x^j = \delta_i^j. \tag{6.17}$$

The vacuum state will be given by $|0\rangle \equiv 1$, all other states are generated by acting with the fermionic operators $\hat{f}^\dagger$ on the vacuum state.

Remark 6.5.1 *The* **n-particle fermionic Hilbert space** *is represented by all forms of order n:* $\mathbb{H}^n \simeq \Omega^n$. *The* **fermionic Fock space** *is the direct sum of all Hilbert spaces:* $\mathbb{F} = \bigoplus \mathbb{H}^n \simeq \Omega^\bullet$.

The fact, that the Fock space is an infinite direct sum over an infinite dimensional space is mathematically not trivial. But since we only allow sums where nearly all (all except for finite many) are zero, this representation of the Fock space is well-defined.

To see how the Pauli principle works in this representation is quite easy: two particles in the same state would be represented by a form $...\mathrm{d}x^i \wedge ... \wedge \mathrm{d}x^i...$ which is immediately zero.

Let us finally have a closer look at the action of the fermionic operators on different states. It is

$$\begin{aligned} \hat{f}^\dagger(i)|\omega\rangle &= e_{x^i}\omega = \frac{1}{r!}\omega_{\mu_1\ldots\mu_r}\mathrm{d}x^i \wedge \mathrm{d}x^{\mu_1} \wedge ... \wedge \mathrm{d}x^{\mu_r} && (6.18) \\ &= \begin{cases} 0 & \text{if } \exists(\mu_2...\mu_r) : \omega_{i\mu_2\ldots\mu_r} \neq 0 \\ \mathrm{d}x^i & \text{if } \omega = |0\rangle = 1 \\ ... & \end{cases} && (6.19) \\ \hat{f}(i)|\omega\rangle &= i_{e_i}\omega = \frac{1}{(r-1)!}\omega_{i\mu_2\ldots\mu_r}\mathrm{d}x^{\mu_2} \wedge ... \wedge \mathrm{d}x^{\mu_r} && (6.20) \\ &= \begin{cases} 1 = |0\rangle & \text{if } \omega = \mathrm{d}x^i \\ 0 & \text{if } \omega = |0\rangle = 1 \\ ... & \end{cases} && (6.21) \end{aligned}$$

6.5.2 Bosonic Hilbert spaces

For the bosonic Fock space, we proceed analogously to the fermionic case. We define our bosonic creation and annihilation operators by:

$$\hat{b}^\dagger(i) = \mathrm{e}_{\xi^i} \text{ and } \hat{b}(i) = \mathrm{i}_{\epsilon_i}. \tag{6.22}$$

From theorem (3.5.3) we get the commutation relations:

$$\begin{aligned} \hat{b}^\dagger(i)\hat{b}^\dagger(j) - \hat{b}^\dagger(j)\hat{b}^\dagger(i) &= \mathrm{e}_{\xi^i}\mathrm{e}_{\xi^j} - \mathrm{e}_{\xi^j}\mathrm{e}_{\xi^i} = 0 && (6.23) \\ \hat{b}(i)\hat{b}(j) - \hat{b}(j)\hat{b}(i) &= \mathrm{i}_{\epsilon_i}\mathrm{i}_{\epsilon_j} - \mathrm{i}_{\epsilon_j}\mathrm{i}_{\epsilon_i} = 0 && (6.24) \end{aligned}$$

and from remark (3.5.2) the last one:

$$\hat{b}(i)\hat{b}^\dagger(j) - \hat{b}^\dagger(j)\hat{b}(i) = \mathrm{i}_{\epsilon_i}\mathrm{e}_{\xi^j} - \mathrm{e}_{\xi^j}\mathrm{i}_{\epsilon_i} = \partial_i\xi^j = \delta_i^j. \tag{6.25}$$

The vacuum state is again given by $|0\rangle \equiv 1$.

Remark 6.5.2 *The* **n-particle bosonic Hilbert space** *is represented by all Graßmannian forms of order n:* $\mathbb{H}^n \simeq O^n$. *The* **bosonic Fock space** *is the direct sum of all Hilbert spaces:* $\mathbb{F} = \bigoplus \mathbb{H}^n \simeq O^\bullet$.

In this case, the Pauli principle does not yield any constraint as the forms are totally symmetric and $\mathrm{d}\xi^i \wedge \mathrm{d}\xi^i \neq 0$. The action of the bosonic creation and annihilation operators on states is given by:

$$\hat{b}^\dagger(i)|o\rangle = e_{\xi^i}o = \frac{1}{r!}o_{\mu_1\ldots\mu_r}\mathrm{d}\xi^i \wedge \mathrm{d}\xi^{\mu_1} \wedge \ldots \wedge \mathrm{d}\xi^{\mu_r} \tag{6.26}$$

$$= \begin{cases} 0 & \text{if } o \text{ top form} \\ \mathrm{d}\xi^i & \text{if } o = |0\rangle = 1 \\ (\mathrm{d}\xi^i\wedge)^{r+1}1 & \text{if } o = (\mathrm{d}\xi^i\wedge)^r 1 \\ \ldots & \end{cases} \tag{6.27}$$

$$\hat{b}(i)|o\rangle = i_{\epsilon_i}o = \frac{1}{(r-1)!}o_{i\mu_2\ldots\mu_r}\mathrm{d}\xi^{\mu_2} \wedge \ldots \wedge \mathrm{d}\xi^{\mu_r} \tag{6.28}$$

$$= \begin{cases} 1 = |0\rangle & \text{if } o = \mathrm{d}\xi^i \\ 0 & \text{if } o = |0\rangle = 1 \\ r(\mathrm{d}\xi^i\wedge)^{r-1}1 & \text{if } o = (\mathrm{d}\xi^i\wedge)^r 1 \\ \ldots & \end{cases} \tag{6.29}$$

A difference to the ordinary bosonic creation and annihilation operators is the fact that only in the case of i_{ϵ_i} the occupation number of i is relevant.[2]
Nevertheless, the number operator $\hat{b}^\dagger(i)\hat{b}(i)$, that counts the particles in a quantum state i, is still working. First two preliminary Lemmata:

[2]Usually it is $\hat{b}|i\rangle = \sqrt{i}|i-1\rangle$ and $\hat{b}^\dagger|i\rangle = \sqrt{i+1}|i+1\rangle$.

Lemma 6.5.1 *Given a form* $o = (\mathrm{d}\xi^i\wedge)^n 1$. *Its representation in coordinates is* $o = (1/r!)o_{\mu_1\ldots\mu_n}\mathrm{d}\xi^{\mu_1}\ldots\mathrm{d}\xi^{\mu_n}$, *where* $o_{i\ldots i} = r!$ *and all other* $o_{\mu_1\ldots\mu_n} = 0$. *It follows:*

$$\begin{aligned} i_{\epsilon_i}(\mathrm{d}\xi^i\wedge)^n 1 &= \frac{1}{(r-1)!}o_{i\mu_2\ldots\mu_n}\mathrm{d}\xi^{\mu_2}\ldots\mathrm{d}\xi^{\mu_n} \\ &= \frac{r!}{(r-1)!}\mathrm{d}\xi^i\ldots\mathrm{d}\xi^i = r(\mathrm{d}\xi^i\wedge)^{r-1}1 \end{aligned} \tag{6.30}$$

Lemma 6.5.2 *Given a state* i *and a form* o *not containing* $\mathrm{d}\xi^i$. *Then for its representation in coordinates* $o_{\mu_1\ldots\mu_n} = 0$ *if one of the* $\mu_j = i$, *which yields:*

$$i_{\epsilon_i}o = \frac{1}{(r-1)!}o_{i\mu_2\ldots\mu_n}\mathrm{d}\xi^{\mu_2}\ldots\mathrm{d}\xi^{\mu_n} = 0 \tag{6.31}$$

Remark 6.5.3 *Given a state* $o = (\mathrm{d}\xi^i\wedge)^n \wedge o_r$ *where* o_r *is an arbitrary form not containing* $\mathrm{d}\xi^i$. *Then:*

$$\begin{aligned} \hat{b}^\dagger(i)\hat{b}(i)o &= \hat{b}^\dagger(i)(i_{\epsilon_i}(\mathrm{d}\xi^i\wedge)^n o_r) \\ &\overset{3.29}{=} \hat{b}^\dagger(i)(i_{\epsilon_i}(\mathrm{d}\xi^i\wedge)^n)o_r + (\mathrm{d}\xi^i\wedge)^n(i_{\epsilon_i}o_r)) \\ &\overset{(6.30),(6.31)}{=} \hat{b}^\dagger(i)(n(\mathrm{d}\xi^i\wedge)^{n-1}o_r + 0) = n(\mathrm{d}\xi^i\wedge)^n o_r = no \end{aligned} \tag{6.32}$$

Chapter 7

The Harmonic Oscillators in Quantum Mechanics

"Physics is actually too difficult for physicists."

David Hilbert

In this chapter we review first the harmonic oscillators from the algebraic point of view. Then we repeat the approach of B.S. DeWitt and eventually develop our own representation in the framework presented in the chapters before.
As the bosonic oscillator is well-known and its representation by functions of even variables is clear, we will explicitly discuss only the Fermi oscillator.

7.1 Algebraic Considerations

7.1.1 Bose and Fermi Oscillators

A complete definition of the quantum mechanical, harmonic oscillators is given by the equations:

$$\begin{aligned} \hat{a}|0\rangle &:= 0 \\ \left[\hat{a}, \hat{a}^\dagger\right]_\mp &= \hat{a}\hat{a}^\dagger \mp \hat{a}^\dagger\hat{a} := 1 \end{aligned}$$

and

$$\hat{H} := \frac{\hbar\omega}{2}\left[\hat{a}^\dagger, \hat{a}\right]_\pm = \hbar\omega\left(\hat{a}^\dagger\hat{a} \pm \frac{1}{2}\right).$$

The additional demand that $\hat{H}$ has at least one eigenvector. Here, the upper sign defines the bosonic oscillator, the lower sign the fermionic one. $\hat{a}^\dagger$ and $\hat{a}$ are called creation and annihilation operators, $|0\rangle$ is called the vacuum state. All matrix elements and energy eigenvalues can be evaluated from this set of equations, particularly no more commutation relations are needed.

To clarify the picture, let us comment the equations and introduce new identifiers for the operators. Beside the vacuum, we define the state:

$$\frac{\overbrace{\hat{a}^\dagger ... \hat{a}^\dagger}^{n \text{ operators}}}{\sqrt{n!}}|0\rangle = |n\rangle \tag{7.1}$$

The action of an operator $\hat{A}^\dagger$ is given by:

$$\langle\alpha|\hat{A}^\dagger = \hat{A}|\alpha\rangle \text{ and } \left(\hat{A}^\dagger\right)^\dagger = \hat{A}. \tag{7.2}$$

Thus we easily see that $\langle i|j\rangle = \delta_{ij}$, so that the states $|i\rangle$ form an orthonormal basis of a vector space. Furthermore, $\hat{a}$ and $\hat{a}^\dagger$ act on $|n\rangle$ according to:

$$\begin{aligned} \hat{a}|n\rangle &= \frac{\hat{a}\hat{a}^\dagger ... \hat{a}^\dagger}{\sqrt{n!}}|0\rangle = \frac{(1 + \hat{a}^\dagger\hat{a})... \hat{a}^\dagger}{\sqrt{n!}}|0\rangle = ... = \frac{n}{\sqrt{n!}}\sqrt{n-1!}|n-1\rangle = \sqrt{n}|n-1\rangle \\ \hat{a}^\dagger|n\rangle &= \frac{\hat{a}^\dagger\hat{a}^\dagger\hat{a}^\dagger ... \hat{a}^\dagger}{\sqrt{n!}}|0\rangle = \frac{\sqrt{n+1!}}{\sqrt{n!}}|n\rangle = \sqrt{n+1}|n+1\rangle \end{aligned}$$

The number operator $\hat{N} = \hat{a}^\dagger \hat{a}$ has eigenvectors $|n\rangle$ with eigenvalues n:

$$\hat{N}|n\rangle = \hat{a}^\dagger \hat{a}|n\rangle = \hat{a}^\dagger \sqrt{n}|n-1\rangle = n|n\rangle \tag{7.3}$$

Thus the Hamilton operator $\hat{H} = \hbar\omega\left(\hat{a}^\dagger \hat{a} \pm \frac{1}{2}\right)$ has the same eigenvectors, but with eigenvalues $\hbar\omega\left(n \pm \frac{1}{2}\right)$.

From now on, we denote the bosonic creation and annihilation operators by $\hat{b}^\dagger$ and $\hat{b}$, the fermionic ones by $\hat{f}^\dagger$ and $\hat{f}$. We get the explicit equations:

$$\left[\hat{b}, \hat{b}^\dagger\right]_- = 1, \ \left[\hat{b}, \hat{b}\right]_- = 0, \ \left[\hat{b}^\dagger, \hat{b}^\dagger\right]_- = 0 \tag{7.4}$$

$$\hat{H} = \hbar\omega\left(\hat{b}^\dagger \hat{b} + \tfrac{1}{2}\right) \tag{7.5}$$

and for the fermionic case:

$$\left[\hat{f}, \hat{f}^\dagger\right]_+ = 1, \ \left[\hat{f}, \hat{f}\right]_+ = 0, \ \left[\hat{f}^\dagger, \hat{f}^\dagger\right]_+ = 0 \tag{7.6}$$

$$\hat{H} = \hbar\omega\left(\hat{f}^\dagger \hat{f} - \tfrac{1}{2}\right) \tag{7.7}$$

While the Bose oscillator has an infinite set of energy eigenstates: $|n\rangle_B$ where $n \in \mathbb{N}$, the nilpotence of the fermionic operators $\hat{f}^2 = (\hat{f}^\dagger)^2 = 0$ allows only the eigenstates $|0\rangle_F$ and $|1\rangle_F$.

There are many definitions for the pseudo-classical states of a harmonic oscillator, called "coherent states". They are states with minimal joint uncertainty of position and momentum or eigenstates of the annihilation operator. We will use the last property as a definition, so that a coherent state $|a\rangle$ satisfies:

$$\hat{a}|a\rangle = a|a\rangle \tag{7.8}$$

The coherent states for the bose oscillator are easily verified to be:

$$|b\rangle = \mathrm{e}^{-\frac{|b|^2}{2}} \sum_{n=1}^{\infty} \frac{b^n}{\sqrt{n!}}|n\rangle = \mathrm{e}^{-\frac{|b|^2}{2}} \mathrm{e}^{b\hat{b}^\dagger}|0\rangle. \tag{7.9}$$

They are overcomplete and linearly dependent.

In the fermionic case, eigenstates of the annihilation operator are more complicated

to obtain. We can decompose an arbitrary state in $|f\rangle = f_1|0\rangle + f_2|1\rangle$ and thus we get the equation

$$\hat{f}|f\rangle = (-1)^{\tilde{f}_2} f_2|0\rangle = f(f_1|0\rangle + f_2|1\rangle) \tag{7.10}$$

where the $(-1)^{\tilde{f}_2}$ has to be included as $\hat{f}$ is a-type. The conditions for obtaining eigenvalues are $f \cdot f_1 = (-1)^{\tilde{f}_2} f_2$ and $f \cdot f_2 = 0$. A first solution is given by $f = f_2 = 0$ (and $f_1 = 1$ so that the state is normalized), a second one by $f = -f_2$ both a-type objects with $f^2 = f_2^2 = f \cdot f_2 = 0$ and $f_1 = 1$ again. So the number of coherent states in the fermionic case depends on if and how many a-type objects we want to allow. In any case, this set is undercomplete, as pure a-type objects do not have an inverse[1], so that it is impossible to obtain $|1\rangle$ from $f_2|1\rangle$. B.S. DeWitt's ring of supernumbers allows an infinite number of a-type objects, while our new representation has only the two Graßmann variables ξ and π available. The implications of that will be discussed below.

7.1.2 The Supersymmetric Oscillator

Roughly speaking, in supersymmetry for each bosonic degree of freedom there is a fermionic one. As we can imagine a Fock space as the tensor product of oscillators at each point in position or momentum space, we need a bosonic and a fermionic oscillator for each point in the simplest case of a supersymmetric Fock space.
Constraining ourselves to only one point in space, we obtain the supersymmetric oscillator, whose states are tensor products of the states of a bosonic and fermionic oscillator with the same frequency and a common vacuum:

$$|\alpha\rangle_{SUSY} = |\alpha\rangle_{Bose} \otimes |\alpha\rangle_{Fermi} \tag{7.11}$$

$$|0\rangle_{SUSY} = |0\rangle_{Bose} \otimes |0\rangle_{Fermi} = |0\rangle_{Bose} \otimes |0\rangle_{Bose} = |0\rangle_{Fermi} \otimes |0\rangle_{Fermi} \tag{7.12}$$

[1] If they had, we would loose the associativity of multiplication: $\xi^{-1}(\xi\xi) = 0 \neq \xi = (\xi^{-1}\xi)\xi$

As the bosonic and fermionic creation and annihilation operators act on different parts of the tensor product, we do not want them to interfere:

$$\begin{aligned}\left[\hat{b},\hat{f}\right] &= \left[\hat{b},\hat{f}\right]_- := 0\\ \left[\hat{b},\hat{f}^\dagger\right] &= \left[\hat{b},\hat{f}^\dagger\right]_- := 0\end{aligned}$$

The Hamilton operator is just the sum of the bosonic and fermionic Hamilton operators, which cancels the constant terms so that the vacuum energy vanishes:

$$\hat{H}_{SUSY} = \hat{H}_{Bose} + \hat{H}_{Fermi} = \hbar\omega\left(\hat{b}^\dagger\hat{b} + \frac{1}{2}\right) + \hbar\omega\left(\hat{f}^\dagger\hat{f} - \frac{1}{2}\right) = \hbar\omega\left(\hat{b}^\dagger\hat{b} + \hat{f}^\dagger\hat{f}\right). \tag{7.13}$$

The energy eigenvalues are $\hbar\omega(n+m)$ with $n \in \mathbb{N}$ and $m \in \{0,1\}$:

$$\hat{H}|n\rangle_{Bose} \otimes |m\rangle_{Fermi} = \hbar\omega(n+m)|n\rangle_{Bose} \otimes |m\rangle_{Fermi} \tag{7.14}$$

where each eigenstate (except for the vacuum) is twice degenerated: both $|n\rangle_{Bose} \otimes |0\rangle_{Fermi}$ and $|n-1\rangle_{Bose} \otimes |1\rangle_{Fermi}$ have energy eigenvalue $\hbar\omega n$. States containing the fermionic or the bosonic vacuum are called pure bosonic or fermionic resp.
We define the two simplest SUSY operators

$$\hat{Q}_+ = \sqrt{2}\hat{b}\hat{f}^\dagger \text{ and } \hat{Q}_- = \sqrt{2}\hat{b}^\dagger\hat{f} \tag{7.15}$$

The operator $\hat{Q}_+$ annihilates a boson and creates a fermion, the operator $\hat{Q}_-$ creates a boson and annihilates a fermion. With those operators, the Hamilton operator can be rewritten with a commutator expression as we could for the pure oscillators:

$$\begin{aligned}\hat{H}_{SUSY} &= \frac{\hbar\omega}{2}[\hat{Q}_+,\hat{Q}_-]_+ = \hbar\omega(\hat{b}^-\hat{b}^+\hat{f}^+\hat{f}^- + \hat{b}^+\hat{b}^-\hat{f}^-\hat{f}^+)\\ &= \hbar\omega((1+\hat{b}^+\hat{b}^-)\hat{f}^+\hat{f}^- + \hat{b}^+\hat{b}^-(1-\hat{f}^+\hat{f}^-))\\ &= \hbar\omega(\hat{f}^+\hat{f}^- + \hat{b}^+\hat{b}^-)\end{aligned} \tag{7.16}$$

Obviously, the two SUSY operators commute with the Hamiltonian, as the energy eigenstate which is obtained by exchanging a fermionic with a bosonic particle has

the same eigenvalue as the original state:

$$[\hat{H}_S, \hat{Q}_+]_- = [\hat{H}_S, \hat{Q}_-]_- = 0 \tag{7.17}$$

As the operators $\hat{Q}_+$ and $\hat{Q}_-$ are not hermitian: $(\hat{Q}_\pm)^\dagger = \hat{Q}_\mp$, it may be more convenient to work with the hermitian operators

$$\hat{Q}_1 = \hat{Q}_+ + \hat{Q}_- \text{ and } \hat{Q}_2 = -\mathrm{i}(\hat{Q}_+ - \hat{Q}_-) \tag{7.18}$$

They obey the anticommutation relations $[\hat{Q}_1, \hat{Q}_2]_+ = 0$ and the Hamilton operator in terms of them is:

$$\hat{H} = \frac{\hbar\omega}{2}\hat{Q}_1^2 = \frac{\hbar\omega}{2}\hat{Q}_2^2. \tag{7.19}$$

7.2 The Fermi Oscillator According to B.S. DeWitt

In this section, we present the results by B.S. DeWitt, who derived the Fermi oscillator from a pseudo classical model using supermathematics. The configuration space in this model is $\mathbb{R}_c^0 \times \mathbb{R}_a^2$, so we have two a-type dynamical variables. The most important results are an undercomplete set of coherent states which still allows a partition of unity (due to the infinite set obtained by allowing the states to be multiplied with a supernumber) and a pseudo classical trajectory, which should be regarded rather critically.

Several calculations in this chapter make use of special properties of supernumbers. Since the reader is not expected to be familiar with these properties, most relations are derived explicitly. Contrary to B.S. DeWitt, we use $\int \mathrm{d}\chi\chi = (2\pi\mathrm{i})^{-1/2}$ to be able to define $\delta(x) = \int \mathrm{e}^{2\pi\mathrm{i}px}\mathrm{d}p$.

7.2.1 Basics and Equation of Motion

In the bosonic case, the action for the harmonic oscillator is given by

$$S = \int \frac{1}{2}(\dot{x}^2 - \omega^2 x^2)\mathrm{d}t. \tag{7.20}$$

The dynamical equation can easily derived from Hamilton principle[2]:

$$0 \equiv \ddot{x} + \omega^2 x \tag{7.21}$$

In this case, the configuration space is usually chosen to be $\mathbb{R}$, but with the extension of real numbers to real supernumbers, it would be more natural to choose $\mathbb{R}_c^1 \times \mathbb{R}_a^0$ ($\mathbb{R}_c^1 \times \mathbb{R}_a^0 \supsetneq \mathbb{R}$), thus the variables x and ω are c-type. The transition to the quantum system is done by introducing linear hermitian operators which correspond to the

[2]The Hamilton principle requires the variation of the action to vanish ($\delta S = 0$), which implies $\frac{\delta S[x(t),\dot{x}(t)]}{\delta x(t')} \equiv 0$. Furthermore $\frac{\delta S[x(t),\dot{x}(t)]}{\delta x(t')} = \frac{\partial L[x,\dot{x}]}{\partial x} - \frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial L[x,\dot{x}]}{\partial \dot{x}}$, hence the Lagrangian equation of motion is the differential equation corresponding to the functional equation $\delta S = 0$.

classical dynamical variable and its derivative. The commutator of these operators is obtained from the second functional derivative of the action:

$$\frac{\delta^2 S}{\delta x(t)\delta x(t')} = -\left(\frac{\partial^2}{\partial t^2}+\omega^2\right)\delta(t,t') \tag{7.22}$$

$$\left[\hat{x}(t),\dot{\hat{x}}(t')\right] := \mathrm{i}\bar{G}(t,t') \tag{7.23}$$

with $\bar{G}(t,t')$ the difference of advanced and retarded Green's function of the operator $-\left(\frac{\partial^2}{\partial t^2}+\omega^2\right)$. The fermionic oscillator is treated in the same way as the bosonic oscillator. Starting with a superclassical model, this time with a-type dynamical variables, one obtains the quantum system by canonical quantization.

It is obvious, that naively copying of the bosonic action does not work, as the square of an a-type variable is always zero. To achieve a coupling that appears linearly in the dynamical equation as in (7.21), a two-dimensional configuration space is needed. The dynamical variable χ and a matrix M are defined by:

$$\chi := \begin{pmatrix}\chi_1\\ \chi_2\end{pmatrix} \tag{7.24}$$

$$M := \begin{pmatrix}0 & 1\\ -1 & 0\end{pmatrix} \tag{7.25}$$

where χ_1 and χ_2 are real a-type variables, so that the configuration space is $\mathbb{R}^0_c\times\mathbb{R}^2_a$ in the fermionic case. We define the action of our system by

$$S=\int\frac{\mathrm{i}}{2}(\chi^{\sim}\dot{\chi}+\omega\chi^{\sim}M\chi)\mathrm{d}t, \tag{7.26}$$

where $\chi^{\sim}$ denotes the transpose of χ and ω is a positive, real number. The Lagrangian is

$$L=\frac{1}{2}\left(\chi_1\dot{\chi}_1+\chi_2\dot{\chi}_2+\omega(\chi_1\chi_2-\chi_2\chi_1)\right) \tag{7.27}$$

The equation of motion is directly obtained from Hamilton principle:

$$\begin{aligned} 0 &\equiv \frac{\overrightarrow{\delta}}{\delta\chi(t)}S \equiv \begin{pmatrix} \frac{\partial L}{\partial \chi_1} - \frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial L}{\partial \dot{\chi}_1} \\ \\ \frac{\partial L}{\partial \chi_2} - \frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial L}{\partial \dot{\chi}_2} \end{pmatrix} = \frac{\mathrm{i}}{2}\begin{pmatrix} \dot{\chi}_1 + \dot{\chi}_1 + \omega\chi_2 + \omega\chi_2 \\ \\ \dot{\chi}_2 + \dot{\chi}_2 - \omega\chi_1 - \omega\chi_1 \end{pmatrix} \\ &= \mathrm{i}(\dot{\chi} + \omega M\chi). \end{aligned} \tag{7.28}$$

Because of $M^2 = -\mathbf{1}_2$, deriving this equation with respect to t and substituting $\dot{\chi}$ using the original equation yields an equation of motion similar to the bosonic case:[3]

$$0 \equiv \ddot{\chi} + \omega^2\chi \tag{7.29}$$

For the transition to the quantum system, we need again the second functional derivative, this time with respect to an a-type supervector χ:

$$\frac{\overrightarrow{\delta}}{\delta\chi(t)}S\frac{\overleftarrow{\delta}}{\delta\chi(t')} = \mathrm{i}\left(\mathbf{1}_2\frac{\partial}{\partial t} + \omega M\right)\delta(t,t') \tag{7.30}$$

The advanced and retarded Green's functions of the operator $\mathrm{i}\left(\mathbf{1}_2\frac{\partial}{\partial t} + \omega M\right)$, satisfying ${}_{i,}S_{,k}G^{\pm kj} = -{}_i\delta^j$, are given by

$$G^-(t,t') = \mathrm{i}\theta(t,t')\begin{pmatrix} \cos\omega(t-t') & -\sin\omega(t-t') \\ \sin\omega(t-t') & \cos\omega(t-t') \end{pmatrix} \tag{7.31}$$

$$G^+(t,t') = \mathrm{i}\theta(t',t)\begin{pmatrix} \cos\omega(t'-t) & -\sin\omega(t'-t) \\ \sin\omega(t'-t) & \cos\omega(t'-t) \end{pmatrix} \tag{7.32}$$

As we work in this case with pure a-type variables, the supercommutator which was a pure commutator in the bosonic case becomes a pure anticommutator here. With the extended definition of the commutator for supernumbers

$$[x,y] = xy - (-1)^{\tilde{x}\tilde{y}}yx \tag{7.33}$$

where $\tilde{x}$ is the parity of the supernumber x (see def. 2.2.4), we can write

$$[\hat{\chi}(t), \hat{\chi}^{\sim}(t')] = \mathrm{i}\bar{G}(t,t') := \begin{pmatrix} \cos\omega(t-t') & -\sin\omega(t-t') \\ \\ \sin\omega(t-t') & \cos\omega(t-t') \end{pmatrix} \tag{7.34}$$

[3]The choice of the fermionic action which lead to this equation is unique up to a factor and a constant if only analytic functions are allowed.

with $\bar{G}(t,t') = G^+(t,t') - G^-(t,t')$. This implies $[\hat{\chi}_1(t), \hat{\chi}_1(t)] = 1$, $[\hat{\chi}_2(t), \hat{\chi}_2(t)] = 1$ and $[\hat{\chi}_1(t), \hat{\chi}_2(t)] = 0$. Regarding the supercommutator at equal times, we find

$$(\hat{\chi}_1)^2 = (\hat{\chi}_2)^2 = \frac{1}{2}, \tag{7.35}$$

so that the eigenvalues of $\hat{\chi}_1$ and $\hat{\chi}_2$ are $\pm 1/\sqrt{2}$. Since the supercommutator of the operators does not vanish and because of the uncertainty principle, the eigenvalues cannot be specified simultaneously.

7.2.2 Mode Functions and the Hamiltonian

The general solution of the dynamical equation (7.28) is given by:

$$\chi(t) = a u(t) + a^* u^*(t) \tag{7.36}$$

with

$$u(t) := \begin{pmatrix} 1/\sqrt{2} \\ \mathrm{i}/\sqrt{2} \end{pmatrix} \mathrm{e}^{\mathrm{i}\omega t} \qquad u(t)^\dagger = \begin{pmatrix} 1/\sqrt{2} \\ -\mathrm{i}/\sqrt{2} \end{pmatrix} \mathrm{e}^{-\mathrm{i}\omega t} \tag{7.37}$$

The functions $u(t)$ and $u^*(t)$ are called **mode functions**.

The supercommutator function can be rewritten in terms of $u(t)$. Choosing the decomposition $\bar{G}(t,t') = G^{(+)}(t,t') - G^{(-)}(t,t')$[4], the **positive** and **negative frequency functions** may be defined as:

$$G^{(+)}(t,t') := -u(t)u^\dagger(t') \tag{7.38}$$

$$G^{(-)}(t,t') := -\mathrm{i}u^*(t)u^\sim(t') = -G^{(+)}(t,t')^*. \tag{7.39}$$

The coefficient a in equation (7.36), an arbitrary complex a-number in the superclassical case, is a non-self-adjoint a-type operator in the quantum system. From $u^\sim(t)u(t) = 0$ and $u^\dagger(t)u(t) = 1$, one obtains

$$\hat{a} = u^\dagger(t)\hat{\chi}(t) \quad \text{and} \quad \hat{a}^* = \hat{\chi}^\sim(t)u(t), \tag{7.40}$$

[4] $G^{(+)}(t,t')$ and $G^{(-)}(t,t')$ must not be confused with $G^+(t,t')$ and $G^-(t,t')$.

leading to the supercommutators:

$$[\hat{a},\hat{a}] = u^{\dagger}[\hat{\chi}(t),\hat{\chi}^{\sim}(t)]u^{*}(t) = u^{\dagger}\mathbf{1}_2 u^{*}(t) = 0 \tag{7.41}$$

$$[\hat{a}^{*},\hat{a}^{*}] = u^{\sim}[\hat{\chi}(t),\hat{\chi}^{\sim}(t)]u(t) = u^{\sim}\mathbf{1}_2 u(t) = 0 \tag{7.42}$$

$$[\hat{a},\hat{a}^{*}] = u^{\dagger}[\hat{\chi}(t),\hat{\chi}^{\sim}(t)]u(t) = u^{\dagger}\mathbf{1}_2 u(t) = 1 \tag{7.43}$$

These anticommutation relations are the same as the ones in the algebraic consideration of the Fermi oscillator. It follows immediately $\hat{a}^2 = 0$ and $\hat{a}^{*2} = 0$.
Now the Hamiltonian in terms of $\hat{a}$ and $\hat{a}^*$ can be calculated:

$$\begin{aligned}\hat{H} &= \hat{L}\frac{\overleftarrow{\partial}}{\partial\dot{\hat{\chi}}}\dot{\hat{\chi}} - \hat{L} = -\frac{\mathrm{i}}{2}\omega\hat{\chi}^{\sim}M\hat{\chi} = -\frac{\mathrm{i}}{2}\omega(\hat{\chi}_1\hat{\chi}_2 - \hat{\chi}_2\hat{\chi}_1)\\ &= -\mathrm{i}\omega\hat{\chi}_1\hat{\chi}_2 = -\frac{\mathrm{i}}{2}\omega(\hat{a}\mathrm{e}^{-i\omega t} + \hat{a}^{*}\mathrm{e}^{i\omega t})(\mathrm{i}\hat{a}\mathrm{e}^{-i\omega t} - \mathrm{i}\hat{a}^{*}\mathrm{e}^{i\omega t})\\ &= \frac{1}{2}\omega(\hat{a}^{*}\hat{a} - \hat{a}\hat{a}^{*}) = \omega\hat{a}^{*}\hat{a} - \frac{1}{2}\omega. \end{aligned}\tag{7.44}$$

Heisenberg's equation of motion is verified by:

$$-\mathrm{i}[\hat{\chi},\hat{H}] = -\frac{1}{2}\omega(\hat{\chi}(\hat{\chi}^{\sim}M\hat{\chi}) - (\hat{\chi}^{\sim}M\hat{\chi})\hat{\chi}) \tag{7.45}$$

$$= -\frac{1}{2}\omega\begin{pmatrix}\hat{\chi}_1(\hat{\chi}_1\hat{\chi}_2 - \hat{\chi}_2\hat{\chi}_1) - (\hat{\chi}_1\hat{\chi}_2 - \hat{\chi}_2\hat{\chi}_1)\hat{\chi}_1\\ \hat{\chi}_2(\hat{\chi}_1\hat{\chi}_2 - \hat{\chi}_2\hat{\chi}_1) - (\hat{\chi}_1\hat{\chi}_2 - \hat{\chi}_2\hat{\chi}_1)\hat{\chi}_2\end{pmatrix} \tag{7.46}$$

$$= -\frac{1}{2}\omega\begin{pmatrix}2\hat{\chi}_2\\ -2\hat{\chi}_1\end{pmatrix} \tag{7.47}$$

$$= -\omega M\hat{\chi} = \dot{\hat{\chi}}. \tag{7.48}$$

7.2.3 Energy Basis Supervectors

As the Hamiltonian is the energy operator, i.e. it is the operator of a physical observable, the term $\hat{a}^{*}\hat{a}$ has to have eigenvalues, the possible results of measurement according to the postulates of quantum mechanics.[5]

[5]The demand, that the Hamilton operator has at least one eigenvalue is necessarily included in the definition of harmonic oscillators in quantum mechanics, see section 7.1.1.

Let n be the eigenvalues of $\hat{a}^*\hat{a}$ with the corresponding normalized eigenvectors $|n\rangle$, $\langle n|n\rangle = 1$.

Suppose there is a nonvanishing n. Then it is $\langle n|\hat{a}^*\hat{a}|n\rangle = n$, so $\hat{a}|n\rangle$ is not the zero supervector as its norm is not zero. $\hat{a}|n\rangle$ is obviously the eigenvector for $\hat{a}^*\hat{a}$ with the eigenvalue 0 as $\hat{a}^*\hat{a}\hat{a}|n\rangle = 0$. Modulo a phase factor, it is $|0\rangle = \hat{a}|n\rangle$.

The supervector $\hat{a}^*|0\rangle$ is obviously not the zero supervector and normalized as $\langle 0|\hat{a}\hat{a}^*|0\rangle = \langle 0|1 - \hat{a}^*\hat{a}|0\rangle = \langle 0|0\rangle = 1$. Because of $(\hat{a}^*\hat{a})\hat{a}^*|0\rangle = \hat{a}^*[\hat{a}, \hat{a}^*]|0\rangle = \hat{a}^*|0\rangle$, it is the eigenvector with the eigenvalue 1, so modulo a phase factor it is $|1\rangle = \hat{a}^*|0\rangle$.

Suppose there was another, linearly independent eigenvector $|n'\rangle$ with $n \notin \{0, 1\}$, then $\hat{a}|n'\rangle$ is eigenvector with eigenvalue 0. Using the simplest representation of the operator superalgebra, it is $\hat{a}|n'\rangle = Z|0\rangle$ with $Z \in \Lambda_\infty$.

If n' had nonvanishing body, Z would have nonvanishing body because of $n' = \langle n'|\hat{a}^*\hat{a}|n'\rangle = Z^*\langle 0|0\rangle Z = Z^*Z$. As $|n'\rangle = \hat{a}^*\hat{a}|n'\rangle\frac{1}{n'} = \hat{a}^*|0\rangle\frac{Z}{n'} = |1\rangle\frac{Z}{n'}$, this is a contradiction to the assumption that $|n'\rangle$ is linearly independent of $|0\rangle$ and $|1\rangle$.

On the other hand, suppose n' has vanishing body. Then $\hat{a}^*|n'\rangle$ is not the zero supervector as $\langle n'|\hat{a}\hat{a}^*|n'\rangle = \langle n'|(1 - \hat{a}^*\hat{a})|n'\rangle = 1 - n'$. It follows $\hat{a}^*\hat{a}(\hat{a}^*|n'\rangle) = 1(\hat{a}^*|n'\rangle)$, so modulo a phase it is $|1\rangle = \hat{a}^*|n'\rangle 1/\sqrt{1 - n'}$ and $|0\rangle = |n'\rangle 1/\sqrt{1 - n'}$, a contradiction again.

Altogether for the basis consisting of energy eigenvectors, the following rules are obtained:

$$\begin{aligned} \hat{a}^*|0\rangle = |1\rangle \qquad & \hat{a}|1\rangle = |0\rangle \\ \hat{a}|0\rangle = 0 \qquad & \hat{a}^*|1\rangle = 0 \end{aligned} \tag{7.49}$$

Note that we had to assume the existence of a nonvanishing eigenvalue.

7.2.4 Eigenvectors of $\hat{\chi}_1$ and $\hat{\chi}_2$

Let $|i, \pm 1/\sqrt{2}, t\rangle$ be the eigenvectors of $\hat{\chi}_i$ with the eigenvalues $\pm 1/\sqrt{2}$. With

$$\hat{\chi}_1(t) = \frac{1}{\sqrt{2}}(\hat{a}\mathrm{e}^{-\mathrm{i}\omega t} + \hat{a}^*\mathrm{e}^{\mathrm{i}\omega t}) \quad \text{and} \quad \hat{\chi}_2(t) = \frac{\mathrm{i}}{\sqrt{2}}(\hat{a}\mathrm{e}^{-\mathrm{i}\omega t} - \hat{a}^*\mathrm{e}^{\mathrm{i}\omega t}) \tag{7.50}$$

it is easy to verify the relations

$$|1, \pm 1/\sqrt{2}, t\rangle = \frac{1}{\sqrt{2}}(|0\rangle \pm \mathrm{e}^{\mathrm{i}\omega t}|1\rangle) \text{ and} \tag{7.51}$$

$$|2, \pm 1/\sqrt{2}, t\rangle = \frac{1}{\sqrt{2}}(|0\rangle \mp \mathrm{i}\mathrm{e}^{\mathrm{i}\omega t}|1\rangle) \tag{7.52}$$

or inversely

$$\begin{aligned} |0\rangle &= \frac{1}{\sqrt{2}}(|1, 1/\sqrt{2}, t\rangle + |1, -1/\sqrt{2}, t\rangle) \\ &= \frac{1}{\sqrt{2}}(|2, 1/\sqrt{2}, t\rangle + |2, -1/\sqrt{2}, t\rangle) \text{ and} \qquad (7.53) \\ |1\rangle &= \frac{1}{\sqrt{2}}\mathrm{e}^{-\mathrm{i}\omega t}(|1, 1/\sqrt{2}, t\rangle - |1, -1/\sqrt{2}, t\rangle) \\ &= \frac{\mathrm{i}}{\sqrt{2}}\mathrm{e}^{-\mathrm{i}\omega t}(|2, 1/\sqrt{2}, t\rangle - |2, -1/\sqrt{2}, t\rangle). \qquad (7.54) \end{aligned}$$

The supervectors $|i, \pm 1/\sqrt{2}, t\rangle$ are necessarily impure, as otherwise the eigenvalue equations would be equalities between supervectors of opposite type, i.e. $\hat{\chi}_1(t)|1, \pm 1/\sqrt{2}, t\rangle = \pm\frac{1}{\sqrt{2}}|1, \pm 1/\sqrt{2}, t\rangle$.

The minimal super Hilbert space has total dimensions 2, and since a pair of orthonormal impure supervectors exist, its basis is necessarily $(1, 1)$, i.e. one a-type dimension and one c-type dimension.

Contrary to the eigenvectors of the $\hat{\chi}_i$, the energy basis eigenvectors $|0\rangle$ and $|1\rangle$ may be fixed to be pure, but they must be of opposite type, as the total dimension of the Hilbert space is $(1, 1)$ and since equations (7.49) imply a change of the parity from $|0\rangle$ to $|1\rangle$. From now on, we will regard $|0\rangle$ as a c-type and $|1\rangle$ as an a-type supervector.

As the operator algebra is invariant under exchange of $\hat{a}$ and $\hat{a}^*$, one may introduce

a theory of holes, which is sometimes more convenient. The energy spectrum always remains $\pm\frac{1}{2}\omega$.

7.2.5 Coherent States

Define the time-dependent operator

$$\hat{a}(t) := \hat{a}\mathrm{e}^{-\mathrm{i}\omega t} = \frac{1}{\sqrt{2}}[\hat{\chi}_1(t) - \mathrm{i}\hat{\chi}_2(t)] \tag{7.55}$$

$$\hat{a}^*(t) = \hat{a}^*\mathrm{e}^{\mathrm{i}\omega t} = \frac{1}{\sqrt{2}}[\hat{\chi}_1(t) + \mathrm{i}\hat{\chi}_2(t)] \tag{7.56}$$

Inversely it is

$$\hat{\chi}_1(t) = \frac{1}{\sqrt{2}}[\hat{a}(t) + \hat{a}^*(t)], \qquad \hat{\chi}_2(t) = \frac{\mathrm{i}}{\sqrt{2}}[\hat{a}(t) - \hat{a}^*(t)]$$

with

$$\hat{\chi}_1(t)|0\rangle = \frac{1}{\sqrt{2}}\mathrm{e}^{\mathrm{i}\omega t}|1\rangle \qquad \hat{\chi}_1(t)|1\rangle = \frac{1}{\sqrt{2}}\mathrm{e}^{-\mathrm{i}\omega t}|0\rangle \tag{7.57}$$

$$\hat{\chi}_2(t)|0\rangle = -\frac{\mathrm{i}}{\sqrt{2}}\mathrm{e}^{\mathrm{i}\omega t}|1\rangle \qquad \hat{\chi}_2(t)|1\rangle = \frac{\mathrm{i}}{\sqrt{2}}\mathrm{e}^{-\mathrm{i}\omega t}|0\rangle. \tag{7.58}$$

Let a' be an arbitrary complex a-number. Decompose a' by

$$a' = \frac{1}{\sqrt{2}}(\chi'_1 - \mathrm{i}\chi'_2) \tag{7.59}$$

where χ'_1 and χ'_2 are real a-numbers.

Define the coherent state c-type supervector by

$$|a', t\rangle := \mathrm{e}^{-\frac{1}{2}a'^*a'}\mathrm{e}^{\mathrm{i}\hat{H}t}(|0\rangle - a'|1\rangle) \tag{7.60}$$

$$= \left(1 - \frac{1}{2}a'^*a'\right)\sum_{n=0}^{\infty}\frac{(\mathrm{i}\omega\hat{a}^*\hat{a}t)^n}{n!}\left(\mathrm{e}^{-\frac{\mathrm{i}}{2}\omega t}|0\rangle - a'\mathrm{e}^{-\frac{\mathrm{i}}{2}\omega t}|1\rangle\right) \tag{7.61}$$

$$= \left(1 - \frac{1}{2}a'^*a'\right)\left(\mathrm{e}^{-\frac{\mathrm{i}}{2}\omega t}|0\rangle - a'\mathrm{e}^{\frac{\mathrm{i}}{2}\omega t}|1\rangle\right) \tag{7.62}$$

$$= \left(1 + \frac{\mathrm{i}}{2}\chi'_1\chi'_2\right)\left(\mathrm{e}^{-\frac{\mathrm{i}}{2}\omega t}|0\rangle - \frac{1}{\sqrt{2}}\mathrm{e}^{\frac{\mathrm{i}}{2}\omega t}(\chi'_1 - \mathrm{i}\chi'_2)|1\rangle\right). \tag{7.63}$$

From (7.60), it is easy to see, that $|a',t\rangle$ is a c-type supervector, as the a-type variable a' appears only together with the a-type variable a'^* or with the a-type supervector $|1\rangle$.

$|a',t\rangle$ is the eigenvector of $\hat{a}(t)$ with the eigenvalue a':

$$\hat{a}(t)|a',t\rangle = \left(1-\frac{1}{2}a'^*a'\right)a'\mathrm{e}^{\frac{\mathrm{i}}{2}\omega t}\mathrm{e}^{-\mathrm{i}\omega t}\hat{a}|1\rangle = a'|a',t\rangle \tag{7.64}$$

Because of parity conservation in the eigenvalue equation it is clear that the eigenvalues of $\hat{a}(t)$ have to be of a-type.

The dual of this coherent state right eigenvector is given by:

$$\langle a'^*,t|\hat{a}^*(t) = \langle a'^*,t|a'^* \tag{7.65}$$

where $\langle a'^*,t|$ would usually be denoted by $\langle a',t|$. The left eigenvector can be rewritten as

$$\begin{aligned}
\langle a'^*,t| &:= \mathrm{e}^{-\frac{1}{2}a'^*a'}(\langle 0| + a'^*\langle 1|)\mathrm{e}^{-\mathrm{i}\hat{H}t} && (7.66)\\
&= \left(1-\frac{1}{2}a'^*a'\right)\left(\mathrm{e}^{\frac{\mathrm{i}}{2}\omega t}\langle 0| + a'^*\mathrm{e}^{-\frac{\mathrm{i}}{2}\omega t}\langle 1|\right) && (7.67)\\
&= \left(1+\frac{\mathrm{i}}{2}\chi_1'\chi_2'\right)\left(\mathrm{e}^{\frac{\mathrm{i}}{2}\omega t}\langle 0| + \frac{1}{\sqrt{2}}(\chi_1'+\mathrm{i}\chi_2')\mathrm{e}^{-\frac{\mathrm{i}}{2}\omega t}\langle 1|\right), && (7.68)
\end{aligned}$$

so that the coherent state supervectors are normalized:

$$\begin{aligned}
\langle a'^*,t|a',t\rangle &= \left(1-\frac{1}{2}a'^*a'\right)^2\left(\mathrm{e}^{\frac{\mathrm{i}}{2}\omega t}\langle 0| + a'^*\mathrm{e}^{-\frac{\mathrm{i}}{2}\omega t}\langle 1|\right)\left(\mathrm{e}^{-\frac{\mathrm{i}}{2}\omega t}|0\rangle - a'\mathrm{e}^{\frac{\mathrm{i}}{2}\omega t}|1\rangle\right)\\
&= (1-a'^*a')(1+a'^*a') = 1.
\end{aligned}$$

Furthermore, they satisfy the differential equations:

$$\begin{aligned}
\left(\tfrac{1}{2}\chi_1' + \frac{\overrightarrow{\partial}}{\partial\chi_1'}\right)\langle a'^*,t| &= \langle a'^*,t|\chi_1(t)\\
\left(\tfrac{1}{2}\chi_2' + \frac{\overrightarrow{\partial}}{\partial\chi_2'}\right)\langle a'^*,t| &= \langle a'^*,t|\chi_2(t)
\end{aligned} \tag{7.69}$$

$$\begin{aligned}\left(\tfrac{1}{2}a'^{*}+\tfrac{\overrightarrow{\partial}}{\partial a'}\right)\langle a'^{*},t| &= \langle a'^{*},t|a^{*}(t)\\ \left(\tfrac{1}{2}a'+\tfrac{\overrightarrow{\partial}}{\partial a'^{*}}\right)\langle a'^{*},t| &= \langle a'^{*},t|a(t)\end{aligned}\quad . \tag{7.70}$$

In the bosonic case, the set of coherent states is overcomplete, e.g. every state vector of the infinite-dimensional bosonic Hilbert space can be constructed from the infinite set of coherent state vectors:

$$|a',t\rangle_{bosonic}\in\left\{\mathrm{e}^{-\frac{1}{2}a'^{*}a'}\sum_{n=0}^{\infty}\frac{a'^{n}}{\sqrt{n!}}\mathrm{e}^{\mathrm{i}(n+\frac{1}{2})\omega t}|n\rangle \mid a'\in\mathbb{X}\right\} \tag{7.71}$$

where usually $\mathbb{X}=\mathbb{R}$ and B.S. DeWitt would use $\mathbb{X}=\mathbb{R}^1_c\times\mathbb{R}^0_a$.

Though the Hilbert space in the fermionic case is just of dimension 2, and there seems to be an infinite number of coherent states, this set is *not* complete. Supernumbers are no field, especially they have only a multiplicative inverse, if their body[6] does not vanish. *The clear meaning of a linear combination as known from $\mathbb{R}^n$ is lost*, as i.e. $\chi'(\chi'|1\rangle+|0\rangle)=\chi'|0\rangle$, which could be misunderstood as a "linear dependence of $\chi'|1\rangle$ and $|0\rangle$". It is obvious, that from the coherent states (7.62) the pure a-type vector $|1\rangle$ cannot be constructed. In the bosonic case this is possible as the coefficients are c-type supernumbers, including all invertible supernumbers. Though the coherent state supervectors do not form a complete set, they satisfy an integral identity which allows a partition of unity:

$$\begin{aligned}&\frac{1}{2\pi\mathrm{i}}\int \mathrm{d}a'\mathrm{d}a'^{*}|a',t\rangle\langle a'^{*},t|\\ =\;&\frac{1}{2\pi}\int \mathrm{d}\chi'_2\mathrm{d}\chi'_1\left\{\mathrm{e}^{-\frac{\mathrm{i}}{2}\omega t}|0\rangle\left(+\frac{\mathrm{i}}{2}\chi'_1\chi'_2\right)+\frac{1}{\sqrt{2}}\mathrm{e}^{\frac{\mathrm{i}}{2}\omega t}|1\rangle(\chi'_1+\chi'_2)\right\}\\ &\times\left\{\left(1+\frac{\mathrm{i}}{2}\chi'_1\chi'_2\right)\mathrm{e}^{\frac{\mathrm{i}}{2}\omega t}\langle 0|+\frac{1}{\sqrt{2}}(\chi'_1+\mathrm{i}\chi'_2)\mathrm{e}^{-\frac{\mathrm{i}}{2}\omega t}\langle 1|\right\}\\ =\;&\frac{\mathrm{i}}{2\pi}\int \mathrm{d}\chi'_2\mathrm{d}\chi'_1(|0\rangle\langle 0|+|1\rangle\langle 1|)\chi'_1\chi'_2\\ =\;&|0\rangle\langle 0|+|1\rangle\langle 1|=1.\end{aligned} \tag{7.72}$$

[6] the pure complex part of a supernumber

(Note that when changing integration variables the super Jacobian has to be included and that $\int \mathrm{d}\chi\chi = (2\pi\mathrm{i})^{\frac{1}{2}}$, see chapter 2.)

7.2.6 Evaluation of the Functional Integral

The amplitude $\langle a''^*, t''|a', t'\rangle$[7] is given by:

$$\begin{aligned}\langle a''^*, t''|a', t'\rangle &= \mathrm{e}^{-\frac{1}{2}a''^*a''-\frac{1}{2}a'^*a'}(\mathrm{e}^{\frac{\mathrm{i}}{2}\omega t''}\langle 0| + \mathrm{e}^{-\frac{\mathrm{i}}{2}\omega t''}a''^*\langle 1|) \\ &\quad \times(\mathrm{e}^{-\frac{\mathrm{i}}{2}\omega t'}|0\rangle + \mathrm{e}^{\frac{\mathrm{i}}{2}\omega t'}|1\rangle a') \\ &= \mathrm{e}^{-\frac{1}{2}a''^*a''-\frac{1}{2}a'^*a'+\frac{\mathrm{i}}{2}\omega(t''-t')}[1 + a''^*\mathrm{e}^{-\mathrm{i}\omega(t''-t')}a'] \\ &= \exp\left[-\frac{1}{2}a''^*a'' + a''^*\mathrm{e}^{-\mathrm{i}\omega(t''-t')}a' - \frac{1}{2}a'^*a' + \frac{\mathrm{i}}{2}\omega(t''-t')\right].\end{aligned} \tag{7.73}$$

When $t''-t'$ is infinitesimal, the exponent in the final expression may be written as:

$$\begin{aligned}&\mathrm{i}\left[\frac{\mathrm{i}}{2}a''^*(a''-a') - \frac{\mathrm{i}}{2}(a''^*-a'^*)a' - \omega a''^*a'(t''-t') + \frac{1}{2}\omega(t''-t')\right] \\ =\ &\mathrm{i}\int_{a',t'}^{a''^*,t''}\mathrm{d}t\left[\frac{\mathrm{i}}{2}(a^*\dot{a} - \dot{a}^*a) - \omega a^*a + \frac{1}{2}\omega\right].\end{aligned} \tag{7.74}$$

The last expression was obtained by dividing by $t''-t'$ and assuming that $a(t)$ and $a^*(t)$ are a-number valued functions of t passing through the endpoints $a(t') = a'$ and $a(t'') = a''$ with $t''-t'$ (and so $a''-a'$) infinitesimal. The right hand side of (7.74) is the superclassical Lagrangian with the χ replaced by a and the vacuum energy $\frac{1}{2}\omega$ added.

Equation (7.74) defines the action integral as the limit of differences for infinitesimal $t''-t'$. B.S. DeWitt wants the superclassical action to be understood as defined by this limit.

[7] Bryce DeWitt does the calculation only for coherent states. It is shown above, that these states are not complete, but relation (7.72) (pseudo-completeness relation) can be used to insert a 1.

When $t'' - t'$ is finite, together with (7.72) the usual path integral description is:

$$\begin{aligned}\langle a''^{*},'' | a', t' \rangle &= \int \mathrm{d}a_N^* \mathrm{d}a_N \ldots \mathrm{d}a_1^* \mathrm{d}a_1 \langle a_{N+1}^*, t_{N+1} | a_N, t_N \rangle \times \\ &\quad \langle a_N^*, t_N | a_{N-1}, t_{N-1} \rangle \ldots \langle a_2^*, t_2 | a_1, t_1 \rangle \langle a_1^*, t_1 | a_0, t_0 \rangle \left(\frac{1}{2\pi \mathrm{i}} \right)^N \end{aligned} \quad (7.75)$$

where

$$\begin{aligned} a_{N+1}^* = a''^*, \; a_0 = a' \text{ and} \\ t'' = t_{N+1} > t_N > \ldots > t_2 > t_1 > t_0 = t'. \end{aligned} \quad (7.76)$$

After passing to the limit $N \to \infty$, $t_n - t_{n-1} \to 0$ and because of the dynamical equation (7.28), we get:

$$\langle a''^*, t'' | a', t' \rangle = Z' \int \mathrm{d}\chi \exp \left\{ \mathrm{i} \int_{a',t'}^{a''^*,t''} \mathrm{d}t \left[\frac{\mathrm{i}}{2} (a^* \dot{a} - \dot{a}^* a) - \omega a^* a + \frac{1}{2} \omega \right] \right\} \quad (7.77)$$

with

$$\begin{aligned} Z' &= \left(\frac{1}{2\pi} \right)^{(t''-t')/\mathrm{d}t - 1}, \\ \mathrm{d}\chi &= \prod_{t'<t<t''} \mathrm{d}\chi_1(t) \mathrm{d}\chi_2(t). \end{aligned} \quad (7.78)$$

(Note that $1/\sqrt{\mathrm{sdet} G^+}$ due to the change of variables is just a constant and has been absorbed in the factor Z'.)

To evaluate the Gaussian integral (7.77), Δt in the right hand side of (7.75) is considered as infinitesimal, so via (7.74) we can insert (7.73) leading to the differenced

form of the exponent in (7.77):

$$\begin{aligned}
& \mathrm{i}\int_{a',t'}^{a''^*,t''} \mathrm{d}t \left[\frac{\mathrm{i}}{2}(a^*\dot{a}-\dot{a}^*a)-\omega a^*a+\frac{1}{2}\omega\right] \\
= & -\frac{1}{2}a^*_{N+1}a_{N+1}+a^*_{N+1}\mathrm{e}^{\mathrm{i}\omega\Delta t}a_N-\frac{1}{2}a^*_N a_N+\frac{\mathrm{i}}{2}\omega\Delta t \\
= & -\frac{1}{2}a^*_N a_N+a^*_N\mathrm{e}^{\mathrm{i}\omega\Delta t}a_{N-1}-\frac{1}{2}a^*_{N-1}a_{N-1}+\frac{\mathrm{i}}{2}\omega\Delta t \\
& \vdots \\
= & -\frac{1}{2}a^*_2 a_2+a^*_2\mathrm{e}^{\mathrm{i}\omega\Delta t}a_1-\frac{1}{2}a^*_1 a_1+\frac{\mathrm{i}}{2}\omega\Delta t \\
= & -\frac{1}{2}a^*_1 a_1+a^*_1\mathrm{e}^{\mathrm{i}\omega\Delta t}a_0-\frac{1}{2}a^*_0 a_0+\frac{\mathrm{i}}{2}\omega\Delta t. \qquad (7.79)
\end{aligned}$$

where $\Delta t = \frac{t''-t'}{N+1}$. A differential equation for the stationary, superclassical trajectory is obtained by differentiating (7.79) with respect to the a_i and a_i^*:

$$\left.\begin{aligned}
a^*_{N+1}\mathrm{e}^{-\mathrm{i}\omega\Delta t}-a^*_N &= 0 \\
a^*_N\mathrm{e}^{-\mathrm{i}\omega\Delta t}-a^*_{N-1} &= 0 \\
&\vdots \\
a^*_2\mathrm{e}^{-\mathrm{i}\omega\Delta t}-a^*_1 &= 0
\end{aligned}\right\} \qquad (7.80)$$

$$\left.\begin{aligned}
-a_N+\mathrm{e}^{-\mathrm{i}\omega\Delta t}a_{N-1} &= 0 \\
&\vdots \\
-a_2+\mathrm{e}^{-\mathrm{i}\omega\Delta t}a_1 &= 0 \\
-a_1+\mathrm{e}^{-\mathrm{i}\omega\Delta t}a_0 &= 0
\end{aligned}\right\} \qquad (7.81)$$

(Note that a_0 and a^*_{N+1} are fixed by (7.76).)

The solutions are evidentely

$$\begin{aligned}
a^*_n = a^*_{N+1}\mathrm{e}^{-\mathrm{i}\omega(N+1-n)\Delta t}, \qquad & a_n = \mathrm{e}^{-\mathrm{i}\omega n\Delta t}a_0 \\
a^*_{st}(t) = \mathrm{e}^{-\mathrm{i}\omega(t''-t)}a''^*, \qquad & a_{st}(t) = \mathrm{e}^{-\mathrm{i}\omega(t-t')}a'. \qquad (7.82)
\end{aligned}$$

where 'st' denotes the stationary trajectory in the limit $N \to \infty$. Together with (7.57), it follows:

$$\chi_{st}(t) = \frac{1}{\sqrt{2}} \begin{pmatrix} \mathrm{e}^{-\mathrm{i}\omega(t-t')}a' + \mathrm{e}^{-\mathrm{i}\omega(t''-t)}a''^* \\ \\ \mathrm{i}\mathrm{e}^{-\mathrm{i}\omega(t-t')}a' - \mathrm{i}\mathrm{e}^{-\mathrm{i}\omega(t''-t)}a''^* \end{pmatrix}. \tag{7.83}$$

The functions $a_{st}(t)$ and a^*_{st} are only complex conjugates if $a' = a'' = 0$; so in general, $\chi_{st}(t)$ is not real. In this case, the stationary trajectory lies not in the configuration space $\mathbb{R}^0_c \times \mathbb{R}^2_a$ of the superclassical model, but in its complex extension $\mathbb{C}^0_c \times \mathbb{C}^2_a$. Calculating (7.79) for the stationary point, it is found that the $N+1$ terms $\frac{\mathrm{i}}{2}\omega\Delta t$ add up to $\frac{\mathrm{i}}{2}\omega(t''-t')$. From (7.82) it follows

$$a^*_{st}(t)a_{st}(t) = a''^*\mathrm{e}^{-\mathrm{i}\omega(t''-t')}a' \tag{7.84}$$

The N terms $-\frac{1}{2}a^*_n a_n$ add up to $-Na''^*\mathrm{e}^{-\mathrm{i}\omega(t'-t)}a'$.
The $N+1$ terms $a^*_n\mathrm{e}^{-\mathrm{i}\omega\Delta t}a_{n-1}$ add up to $(N+1)a''^*\mathrm{e}^{-\mathrm{i}\omega(t''-t')}a'$.
Together with (7.76) the sum of (7.79) is equal to the final exponent in (7.73), which itself follows from the functional integral by the normalization condition $\langle a'^*, t|a', t\rangle = 1$ of the coherent states.
a_{st} and a^*_{st} certainly follow the dynamical equation (7.28):

$$\dot{a}_{st}(t) = -\mathrm{i}\omega a_{st}(t) \qquad \dot{a}^*_{st}(t) = \mathrm{i}\omega a^*_{st}(t) \tag{7.85}$$

Naively inserting this expressions into the integral on the left hand side of (7.79) leads to a vanishing integrand (up to the zero point energy), a contradiction to the whole calculation:

$$\begin{aligned} & \mathrm{i}\int_{a',t'}^{a''^*,t''} \mathrm{d}t \left[\frac{\mathrm{i}}{2}(a^*_{st}\dot{a}_{st} - \dot{a}^*_{st}a_{st}) - \omega a^*_{st}a_{st} + \frac{1}{2}\omega\right] \\ = & \mathrm{i}\int_{a',t'}^{a''^*,t''} \mathrm{d}t \left[\frac{\mathrm{i}}{2}(a^*_{st}(-\mathrm{i}\omega a_{st}) - \mathrm{i}\omega a^*_{st}a_{st}) - \omega a^*_{st}a_{st} + \frac{1}{2}\omega\right] \\ = & \mathrm{i}\int_{a',t'}^{a''^*,t''} \mathrm{d}t \left[\frac{1}{2}\omega\right] \end{aligned} \tag{7.86}$$

This apparent contradiction is due to endpoint contributions, as the stationary trajectory does not join smoothly to its endpoint values. Equations (7.82) should be written as:

$$a^*_{st}(t) = \lim_{\epsilon\to 0}[\theta(t,t'+\epsilon)\mathrm{e}^{-\mathrm{i}\omega(t''-t)}a''^* + \theta(t'+\epsilon,t)a'^*] \tag{7.87}$$

$$a_{st}(t) = \lim_{\epsilon\to 0}[\theta(t''-\epsilon,t)\mathrm{e}^{-\mathrm{i}\omega(t-t')}a' + \theta(t,t''-\epsilon)a''] \tag{7.88}$$

The theta-function becomes a delta-function in the derivative which keeps the integral from vanishing.

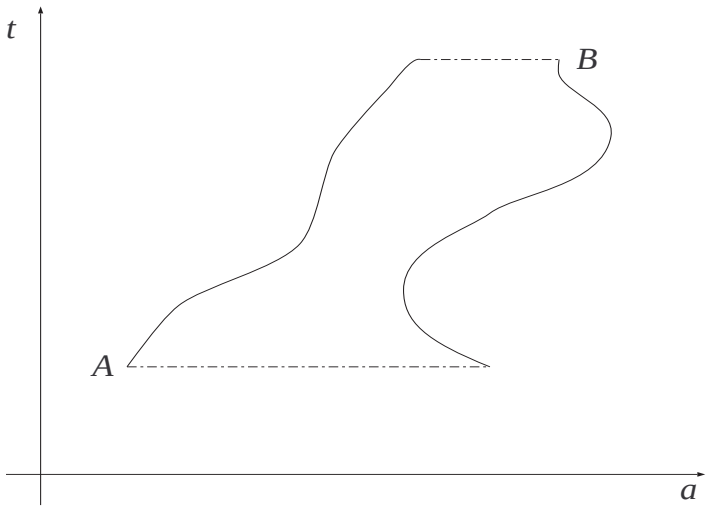

Figure 7.1: The stationary trajectory does not join smoothly to its endpoint values. The trajectory for $a_{st}(t)$ is the left one, the trajectory for $a^*_{st}(t)$ is on the right.

If a source term $\eta\chi$ is added to the integrand of the action (7.26), then the Schwinger variational principle yields:

$$\frac{\delta}{\delta\eta(t)}\langle a''^*,t''|a',t'\rangle = \langle a''^*,t''|\chi(t)|a',t'\rangle \tag{7.89}$$

On the other hand, differentiation of the functional integral (7.77) leads to $\chi_{st}(t)\langle a''^*,t''|a',t'\rangle$. These expressions have to be identical for a vanishing source,

which is shown by

$$\langle a''^*, t''|a(t)|a', t'\rangle = \langle a''^*, t''|\mathrm{e}^{-\mathrm{i}\omega(t-t')}a(t')|a', t'\rangle$$
$$= \mathrm{e}^{-\mathrm{i}\omega(t-t')}a'\langle a''^*, t''|a', t'\rangle = a_{st}(t)\langle a''^*, t''|a', t'\rangle \tag{7.90}$$
$$\langle a''^*, t''|a^*(t)|a', t'\rangle = \langle a''^*, t''|a^*(t'')\mathrm{e}^{-\mathrm{i}\omega(t-t')}|a', t'\rangle$$
$$= \mathrm{e}^{-\mathrm{i}\omega(t-t')}a''^*\langle a''^*, t''|a', t'\rangle = a^*_{st}(t)\langle a''^*, t''|a', t'\rangle \tag{7.91}$$

so we obtain after substituting a with χ:

$$\langle a''^*, t''|\chi(t)|a', t'\rangle = \chi_{st}\langle a''^*, t''|a', t'\rangle \qquad t' < t < t''. \tag{7.92}$$

7.2.7 The Feynman Propagator

The Feynman Propagator is the vacuum expectation value of a time ordered product. It is calculated from Wick's Theorem[8]:

$$T\{\chi(s)\chi^\sim(t)\} =: \chi(s)\chi^\sim(t) : +\langle 0|T\{\chi(s)\chi^\sim(t)\}|0\rangle \tag{7.93}$$

It is $\langle 0|T\{\chi(s)\chi^\sim(t)\}|0\rangle = -\mathrm{i}G(s,t)$ (B.S. DeWitt's differes from the usual one), and the expectation value $\langle a''^*, t''|T\{x(s)x^\sim(t)\}|a', t'\rangle$ leads to

$$G(s,t)\langle a''^*, t''|a', t'\rangle = \mathrm{i}\langle a''^*, t''|T\{\chi(s)\chi^\sim(t)\}|a', t'\rangle - \mathrm{i}\chi_{st}(s)\chi^\sim_{st}(t)\langle a''^*, t''|a', t'\rangle. \tag{7.94}$$

Together with

$$a(s)a(t) = 0 \quad a^*(s)a^*(t) = 0 \quad [a(s), a^*(t)] = \mathrm{e}^{-\mathrm{i}\omega(s-t)} \tag{7.95}$$

[8]It may also be obtained by differentiating $\ln\langle a''^*, t''|a', t'\rangle$ twice with respect to the source

it follows

$$\begin{aligned}
& \langle a''^*, t''|\chi(s)\chi^\sim(t)|a', t'\rangle \\
= & \frac{\mathrm{i}}{2}\langle a''^*, t''| \begin{pmatrix} a(s)+a^*(s) \\ \mathrm{i}a(s)-\mathrm{i}a^*(s) \end{pmatrix} (a(t)+a^*(t), \mathrm{i}a(t)-\mathrm{i}a^*(t))\,|a', t'\rangle \\
= & \frac{\mathrm{i}}{2}\Bigg[\begin{pmatrix} a^*_{st}(s)a_{st}(t)-a^*_{st}(t)a_{st}(s) & \mathrm{i}a^*_{st}(s)a_{st}(t)+\mathrm{i}a^*_{st}(t)a_{st}(s) \\ -\mathrm{i}a^*_{st}(s)a_{st}(t)-\mathrm{i}a^*_{st}(t)a_{st}(s) & a^*_{st}(s)a_{st}(t)-a^*_{st}(t)a_{st}(s) \end{pmatrix} \\
& + \begin{pmatrix} 1 & -\mathrm{i} \\ \mathrm{i} & 1 \end{pmatrix} \mathrm{e}^{-\mathrm{i}\omega(s-t)} \Bigg] \langle a''^*, t''|a', t'\rangle \\
= & \mathrm{i}[\chi_{st}(s)\chi^\sim_{st}(t)+u(s)u^\sim(t)]\langle a''^*, t''|a', t'\rangle. \qquad (7.96)
\end{aligned}$$

Putting this in (7.94) leads to the expression for the Feynman propagator:

$$\begin{aligned}
G(s,t) &= \mathrm{i}[\theta(s,t)u(s)u^\dagger(t)-\theta(t,s)u^*(s)u^\sim(t)] \\
&= -\theta(s,t)G^{(+)}(s,t)+\theta(t,s)G^{(-)}(s,t) \\
&= G^-(s,t)+G^{(-)}(s,t) = G^+(s,t)-G^{(+)}(s,t) \qquad (7.97)
\end{aligned}$$

$G(s,t)$ propagates positive frequencies to the future and negative frequencies to the past, i.e. for $s > t$ it behaves as $\mathrm{e}^{-\mathrm{i}\omega s}$ in s and as $\mathrm{e}^{\mathrm{i}\omega t}$ in t. As the stationary trajectory is fixed by the boundary conditions

$$\begin{aligned}
\chi_{st\ 1}(t')-\mathrm{i}\chi_{st\ 2}(t') &= \chi'_1-\mathrm{i}\chi'_2 \\
\chi_{st\ 1}(t')+\mathrm{i}\chi_{st\ 2}(t') &= \chi''_1+\mathrm{i}\chi''_2.
\end{aligned}$$

one obtains the corresponding conditions for the Feynman propagator:

$$(1,-\mathrm{i})G(t',s)=0, \qquad G(s,t')\begin{pmatrix} 1 \\ -\mathrm{i} \end{pmatrix}=0$$

$$(1,\mathrm{i})G(t'',s)=0, \qquad G(s,t'')\begin{pmatrix} 1 \\ \mathrm{i} \end{pmatrix}=0$$

7.3 The Fermi Oscillator in the New Representation

In this section, we develop a representation of the Fermi oscillator which is constructed in a straightforward way from the formalism worked out in the previous chapters. The well-known algebraic description of the Fermi oscillator demands anticommuting operators, which are obtained from quantizing Graßmann variables. As there are obviously no classical objects with anticommuting dynamical variables, we can only define a pseudo-classical model from which we will obtain the quantum system by canonical quantization.

7.3.1 Choosing a Pseudo-Classical Model

The mechanics with ordinary (commuting) dynamical variables is described on $S_M = TM \times_M T^*M$ (the Lagrangian part on the tangential, the Hamiltonian formulation on the cotangent bundle); an obvious choice for Graßmann dynamical variables is $\Sigma_M = \Pi TM \times_M \Pi T^*M$. Since we will constrain our considerations for the Fermi oscillator to the one-dimensional case, we are left with the coordinates x, ξ and π. The next step is to find a pseudo classical Hamiltonian. In analogy to the bosonic case, where H is a function of x and p, we look for a function of the two Graßmann variables ξ and π. The most general function reads:

$$H(\xi, \pi) = \alpha_1 + \alpha_2\xi + \alpha_3\pi + \alpha_4\pi\xi \tag{7.98}$$

As the Hamiltonian has to be invariant under a change of coordinates, we have to demand $\alpha_2 = \alpha_3 = 0$ as ξ and π are obviously dependent of the charts.
The term $\pi\xi$ seems familiar, and reminds of the $p_i\dot{x}^i$ from the Legendre transform, which enables us to change between Hamilton and Lagrange formalism. So one is easily lead to think that its appearance in the Hamiltonian is natural. To see that this is wrong, we need to clarify the picture and distinguish between the mathematical analogies of Graßmann variables with the ordinary ones due to their construction

and the physical analogy between bosonic and fermionic dynamical variables:

mathematical analogies:	$x \leftrightarrow$ (no correspondence)	$\dot{x} \leftrightarrow \xi$	$p \leftrightarrow \pi$
physical analogies:	$x \leftrightarrow \xi$	$\dot{x} \leftrightarrow \dot{\xi}$	$p \leftrightarrow \pi$

We also see, that it is necessary to introduce further Graßmann variables $\dot{\xi}$ and $\dot{\pi}$ to describe the dynamics of the system.

7.3.2 Canonical Quantization of Systems with Graßmann Variables

To get more hints on how to choose α_1 and α_4, we quantize the system by canonical quantization. For a bosonic system, this is done by defining

$$[\hat{u}, \hat{v}]_- = \mathrm{i}\hbar\widehat{\{u, v\}}_{p,q,Poisson} + O(\hbar^2) \tag{7.99}$$

where $\{u, v\}_{p,q,Poisson}$ denotes the Poisson brackets given by:[9].

$$\{u, v\}_{p,q,Poisson} = \left(\partial_{q^i} u\right)\left(\partial_{p_i} v\right) - \left(\partial_{p_i} u\right)\left(\partial_{q^i} v\right). \tag{7.100}$$

The obvious generalization of the Poisson brackets are the Schouten brackets:

$$\{u, v\}_{p,q,Schouten} = \{u, v\}_{p,q} = \left(\partial_{q^i} u\right)\left(\partial_{p_i} v\right) - (-1)^{\tilde{u}\tilde{v}}\left(\partial_{p_i} u\right)\left(\partial_{q^i} v\right). \tag{7.101}$$

Now we are ready to quantize our system by canonical quantization[10]:

$$[\hat{u}, \hat{v}]_+ = \mathrm{i}\hbar\widehat{\{u, v\}}_{\pi,\xi} + O(\hbar^2). \tag{7.102}$$

This leads to the following results:

$$\left[\hat{\xi}, \hat{\xi}\right]_+ = \mathrm{i}\hbar\left(\left(\partial_\xi \xi\right)\left(\partial_\pi \xi\right) - (-1)^{\tilde{\xi}\tilde{\xi}}\left(\partial_{\pi_i} \xi\right)\left(\partial_\xi \xi\right)\right) = \mathrm{i}\hbar(0+0) = 0 \tag{7.103}$$

$$[\hat{\pi}, \hat{\pi}]_+ = \mathrm{i}\hbar\left(\left(\partial_\xi \pi\right)\left(\partial_\pi \pi\right) - (-1)^{\tilde{\pi}\tilde{\pi}}\left(\partial_{\pi_i} \pi\right)\left(\partial_\xi \pi\right)\right) = \mathrm{i}\hbar(0+0) = 0 \tag{7.104}$$

$$\left[\hat{\xi}, \hat{\pi}\right]_+ = \mathrm{i}\hbar\left(\left(\partial_\xi \xi\right)\left(\partial_\pi \pi\right) - (-1)^{\tilde{\xi}\tilde{\pi}}\left(\partial_{\pi_i} \xi\right)\left(\partial_\xi \pi\right)\right) = \mathrm{i}\hbar(1+0) = \mathrm{i}\hbar, \tag{7.105}$$

[9] We use the abbreviation $\partial_x = \partial/\partial x$

[10] We will *not* use "God-given units" $c = \hbar = 1$ in this section, as $\hbar$ often gives insights in the meaning of terms, which are otherwise lost.

which seem quite similar to the Heisenberg commutation relations for position and momentum operators. If we now rescale our two operators by:

$$\hat{f} = \frac{\hat{\xi}}{\sqrt{\mathrm{i}\hbar}} \text{ and } \hat{f}^\dagger = \frac{\hat{\pi}}{\sqrt{\mathrm{i}\hbar}} \tag{7.106}$$

we get the canonical anticommutation relations for fermionic creation and annihilation operators: $[\hat{f}, \hat{f}]_+ = [\hat{f}^\dagger \hat{f}^\dagger]_+ = 0$, $[\hat{f}, \hat{f}^\dagger]_+ = 1$. One could argue that we should rather follow the same steps which lead from bosonic position and momentum operators to bosonic creation and annihilation operators, but this brings up completely different anticommutation relations, as we will show below. The only choice is which of the operators $\hat{\xi}$ and $\hat{\pi}$ becomes the creation and which becomes the annihilation operator. Our choice here will prove to be the most comfortable.

Comparing our general, now quantized, Hamilton operator

$$\hat{H} = \alpha_1 + \alpha_4 \hat{\pi}\hat{\xi} = \alpha_1 + \alpha_4(\hat{\pi}\hat{\xi}) = \alpha_1 + \alpha_4 \mathrm{i}\hbar \hat{f}^\dagger \hat{f} \tag{7.107}$$

with the Hamiltonian for the Bose oscillator $\hat{H}_{bos} = \hbar\omega(\hat{b}^\dagger\hat{b} - \frac{1}{2})$, $\hat{b}^\dagger$, $\hat{b}$ bosonic creation and annihilation operators, we easily see, that we should choose $\alpha_4 = \omega/\mathrm{i} = -\mathrm{i}\omega$.

Having in mind a supersymmetric quantum field theory, which is basically the sum of bosonic and fermionic oscillators for each point in space, we would like α_1 to cancel the infinite zero-point energy of the bosonic Hamiltonian. This infinity arises from the constant part in the Hamiltonian density which diverges, if it is integrated over the whole space: $\int H_{bos}\mathrm{d}x \geq \int \frac{\omega}{2}\,\mathrm{d}x \to \infty$.

Altogether, this fixes α_1 to $-\frac{\hbar\omega}{2}$ and we get the the Hamilton operator for the Fermi oscillator:

$$H = \hbar\omega\left(\hat{f}^\dagger\hat{f} - \frac{1}{2}\right) = \hbar\omega\left(\frac{\hat{\pi}}{\sqrt{\mathrm{i}\hbar}}\frac{\hat{\xi}}{\sqrt{\mathrm{i}\hbar}} - \frac{1}{2}\right) = \hbar\omega\left(\frac{\hat{\pi}\hat{\xi}}{\mathrm{i}\hbar} - \frac{1}{2}\right) \tag{7.108}$$

7.3.3 The Two Other Operators

In the case of the bosonic oscillator, the creation and annihilation operators are constructed from the position and momentum operators by

$$\hat{b} = \frac{1}{\sqrt{2\hbar\omega}}(\omega\hat{x} + \mathrm{i}\hat{p}) \text{ and } \hat{b}^\dagger = \frac{1}{\sqrt{2\hbar\omega}}(\omega\hat{x} - \mathrm{i}\hat{p}). \tag{7.109}$$

As we obtained our creation and annihilation operators by rescaling the position and momentum operators, it is interesting to ask, what kind of operators a similar construction yields in the fermionic case. As we do not know, if we should regard our operators rather as position/momentum or creation/annihilation operators, we have to examine the transform mentioned above as well as its inverse, so that we obtain two pairs of operators:

$$\begin{aligned}
\hat{\phi} &= \frac{1}{\sqrt{2\hbar\omega}}(\omega\hat{\xi} + \mathrm{i}\hat{\pi}) \quad , \quad \hat{\phi}^\dagger = \frac{1}{\sqrt{2\hbar\omega}}(\omega\hat{\xi} - \mathrm{i}\hat{\pi}). \\
\hat{\phi}_x &= \sqrt{\frac{\hbar}{2\omega}}(\hat{f} + \hat{f}^\dagger) \quad , \quad \hat{\phi}_p = -\mathrm{i}\sqrt{\frac{\hbar\omega}{2}}(\hat{f} - \hat{f}^\dagger).
\end{aligned}$$

which obey the anticommutation relations:

$$\left[\hat{\phi}, \hat{\phi}\right]_+ = \frac{1}{2\hbar}\left(\mathrm{i}[\hat{\xi}, \hat{\pi}]_+ + \mathrm{i}[\hat{\pi}, \hat{\xi}]_+\right) = -1 \tag{7.110}$$

$$\left[\hat{\phi}^\dagger, \hat{\phi}^\dagger\right]_+ = \frac{1}{2\hbar}\left(-\mathrm{i}[\hat{\xi}, \hat{\pi}]_+ - \mathrm{i}[\hat{\pi}, \hat{\xi}]_+\right) = 1 \tag{7.111}$$

$$\left[\hat{\phi}, \hat{\phi}^\dagger\right]_+ = \frac{1}{2\hbar}\left(-\mathrm{i}[\hat{\xi}, \hat{\pi}]_+ + \mathrm{i}[\hat{\pi}, \hat{\xi}]_+\right) = 0 \tag{7.112}$$

$$\left[\hat{\phi}_x, \hat{\phi}_x\right]_+ = \frac{\hbar}{2\omega}\left([\hat{f}, \hat{f}^\dagger]_+ + \mathrm{i}[\hat{f}^\dagger, \hat{f}]_+\right) = \frac{\hbar}{\omega} \tag{7.113}$$

$$\left[\hat{\phi}_p, \hat{\phi}_p\right]_+ = -\frac{\hbar}{2\omega}\left(-[\hat{f}, \hat{f}^\dagger]_+ - \mathrm{i}[\hat{f}^\dagger, \hat{f}]_+\right) = \hbar\omega \tag{7.114}$$

$$\left[\hat{\phi}_x, \hat{\phi}_p\right]_+ = -\mathrm{i}\frac{\hbar}{2}\left(-[\hat{f}, \hat{f}^\dagger]_+ + \mathrm{i}[\hat{f}^\dagger, \hat{f}]_+\right) = 0. \tag{7.115}$$

These relations remind strongly of the anticommutation relations of the operators in B.S. DeWitt's approach $[\hat{\chi}_1(t), \hat{\chi}_1(t)] = 1$, $[\hat{\chi}_2(t), \hat{\chi}_2(t)] = 1$ and $[\hat{\chi}_1(t), \hat{\chi}_2(t)] = 0$ and basically underline the fact, that his creation/annihilation operators correspond

to our position/momentum operators.

Because of the anticommutation relations, we see, that the variables ξ and π correspond by quantization and rescaling to the fermionic annihilation and creation operators, while $\hat{\phi}$ and $\hat{\phi}^\dagger$ rather correspond to the fermionic analogue of bosonic momentum and position operators.

Using

$$\hat{\xi} = \sqrt{\frac{\hbar}{2}}(\hat{\phi}^\dagger + \hat{\phi}) \text{ and } \hat{\pi} = \mathrm{i}\sqrt{\frac{\hbar}{2}}(\hat{\phi}^\dagger - \hat{\phi}), \tag{7.116}$$

we can rewrite the Hamiltonian for the Fermi oscillator in terms of $\hat{\phi}$ and $\hat{\phi}^\dagger$

$$\begin{aligned} \hat{H} &= \hbar\omega\left(\frac{\hat{\pi}\hat{\xi}}{\mathrm{i}\hbar} - \frac{1}{2}\right) = \hbar\omega\left(\frac{1}{\mathrm{i}\hbar}\mathrm{i}\frac{\hbar}{2}\left(\hat{\phi}^\dagger\hat{\phi}^\dagger - \hat{\phi}^\dagger\hat{\phi} - \hat{\phi}\hat{\phi}^\dagger + \hat{\phi}\hat{\phi}\right) - \frac{1}{2}\right) \\ &= \hbar\omega\left(\frac{1}{2}(2\hat{\phi}^\dagger\hat{\phi} + 1) - \frac{1}{2}\right) = \hbar\omega\hat{\phi}^\dagger\hat{\phi} \end{aligned} \tag{7.117}$$

Some textbooks on supersymmetry (as, for example. [18]) start with this quantized Hamiltonian and the anticommutation relations for $\hat{\phi}$ and $\hat{\phi}^\dagger$ and derive from them our Hamiltonian for the Fermi Oscillator $H = \hbar\omega\left(\hat{f}^\dagger\hat{f} - \frac{1}{2}\right)$. It seems, that you either have to do without distinguishing between momentum/position and creation/annihilation operators or without the simple canonical quantization rule.

7.3.4 The Heisenberg Equation of Motion

In quantum mechanics with quantized ordinary variables, the Heisenberg equation of motion is

$$\mathrm{i}\hbar\frac{\partial}{\partial t}\hat{A}_H = [\hat{A}_H, \hat{H}_H] \tag{7.118}$$

where A_H is an operator in the Heisenberg picture.

The generalization to systems with Graßmann variables is again straightforward, just replace the commutator with the supercommutator as it was already done in B.S. DeWitt's approach. Nevertheless, as the Hamilton operator has even parity

(c-type), the supercommutator becomes again the simple commutator:

$$\begin{aligned} \mathrm{i}\hbar\frac{\partial}{\partial t}\hat{\xi} &= [\hat{\xi},\hat{H}] = [\hat{\xi},\hat{H}]_- = \left[\hat{\xi},\hbar\omega\left(\frac{\hat{\pi}}{\sqrt{\mathrm{i}\hbar}}\frac{\hat{\xi}}{\sqrt{\mathrm{i}\hbar}}-\frac{1}{2}\right)\right]_- \\ &= \hbar\frac{\omega}{\mathrm{i}\hbar}\left(\hat{\xi}\hat{\pi}\hat{\xi}-\hat{\xi}\hat{\xi}\hat{\pi}\right) = \hbar\frac{\omega}{\mathrm{i}\hbar}\mathrm{i}\hbar\hat{\xi} = \hbar\omega\hat{\xi}. \end{aligned} \tag{7.119}$$

This result is up to a factor of $\mathrm{i}M^{-1}$ identical with the result in B. DeWitt's approach, as he calculates with $\hbar = 1$.

Similarly, we get the equation for $\hat{\pi}$:

$$\mathrm{i}\hbar\frac{\partial}{\partial t}\hat{\pi} = [\hat{\pi},H]_- = \hbar\frac{\omega}{\mathrm{i}\hbar}\left(\hat{\pi}\hat{\pi}\hat{\xi}-\hat{\pi}\hat{\xi}\hat{\pi}\right) = -\hbar\frac{\omega}{\mathrm{i}\hbar}\mathrm{i}\hbar\hat{\pi} = -\hbar\omega\hat{\pi}. \tag{7.120}$$

These dynamical equations suggest the solutions (operators in Heisenberg picture):

$$\begin{aligned} \hat{\xi}(t) &= \mathrm{e}^{-\mathrm{i}\omega t}\hat{\xi_0} \quad \text{and} \quad \hat{\pi}(t) = \mathrm{e}^{\mathrm{i}\omega t}\hat{\pi_0} \\ \hat{f}(t) &= \mathrm{e}^{-\mathrm{i}\omega t}\hat{f}_0 \quad \text{and} \quad \hat{f}^\dagger(t) = \mathrm{e}^{\mathrm{i}\omega t}\hat{f}\dagger_0 \end{aligned}$$

Comparing the last two equations with the ones for the Bose oscillator, we see that the analogy is preserved:

$$\hat{b}(t) = \mathrm{e}^{-\mathrm{i}\omega t}\hat{b}_0 \quad \text{and} \quad \hat{b}^\dagger(t) = \mathrm{e}^{\mathrm{i}\omega t}\hat{b}_0^\dagger$$

In terms of $\hat{\pi}$ and $\dot{\hat{\xi}}$, the Hamiltonian gets the simple form

$$\hat{H} = \hbar\omega\left(\frac{\hat{\pi}}{\sqrt{\mathrm{i}\hbar}}\frac{\hat{\xi}}{\sqrt{\mathrm{i}\hbar}}-\frac{1}{2}\right) = \hat{\pi}\dot{\hat{\xi}}-\frac{\hbar\omega}{2}. \tag{7.121}$$

This expression now includes the term $\hat{\pi}\dot{\hat{\xi}}$ which corresponds to $\hat{p}\dot{\hat{x}}$ and has unit of energy.

7.3.5 Discussion of Pseudo-Classical Dynamics

After quantization gave us all the remaining constraints for the Hamilton operator, we can rederive the classical Hamilton function

$$H(\pi,\xi) = \hbar\omega\left(\frac{\pi\xi}{\mathrm{i}\hbar}-\frac{1}{2}\right). \tag{7.122}$$

Using the physical analogies $\xi \leftrightarrow x$ and $\pi \leftrightarrow p$, we can rewrite the Hamilton equation of motion by:

$$\dot{q} = -\frac{\partial H(p,q)}{\partial p} \quad \rightarrow \quad \dot{\xi} = -\frac{\partial H(\xi,\pi)}{\partial \pi} = -\mathrm{i}\omega\xi \tag{7.123}$$

$$\dot{p} = \frac{\partial H(p,q)}{\partial q} \quad \rightarrow \quad \dot{\pi} = \frac{\partial H(\xi,\pi)}{\partial \xi} = \mathrm{i}\omega\pi \tag{7.124}$$

These results are obviously consistent with the Heisenberg equation of motion derived in the last section. But even more important is that our pseudo-classical system is indeed a Fermi oscillator satisfying the differential equation for a harmonic oscillator:

$$\ddot{\xi} + \omega^2\xi = -\mathrm{i}\omega\dot{\xi} + \omega^2\xi = -\omega^2\xi + \omega^2\xi = 0. \tag{7.125}$$

Since the choice of our Hamiltonian is fixed up to constant factors, we have shown that the only possible Fermi system is that of a Fermi oscillator.

Our next goal is to obtain a pseudo-classical Lagrange function $L(\xi,\dot{\xi})$. Naively using the inverse Legendre-transform, we obtain[11]:

$$L(\xi,\dot{\xi}) = \pi\dot{\xi} - H = \pi(-\mathrm{i}\omega\xi) - \hbar\omega\left(\frac{\pi\xi}{\mathrm{i}\hbar} - \frac{1}{2}\right) = \frac{\hbar\omega}{2} \tag{7.126}$$

Looking more closely at the basics of Legendre-transform, we notice, that it is actually ill-defined in our case. This operation would transform a function of variables ξ and π into a function of ξ and $\partial_\pi H(\pi(\partial_\pi H))$ which requires the existence of a *unique* inverse function $\pi(\partial_\pi H)$. As $\partial_\pi H$ is constant in π, this condition is not fulfilled.

Let us try another approach: Using the most general Lagrangian

$$L(\xi,\dot{\xi}) = \alpha_0 + \alpha_1\xi + \alpha_2\dot{\xi} + \alpha_3\xi\dot{\xi} \tag{7.127}$$

we could derive equations of motion by putting the variation of the action $S = \int L(\xi,\dot{\xi})\mathrm{d}t$ equal to zero:

$$\frac{\delta S}{\delta\xi(t)} = 0 \Rightarrow \frac{\partial L(\xi,\dot{\xi})}{\partial\xi} = 0 = \alpha_1 + \alpha_3\dot{\xi}. \tag{7.128}$$

[11]Though this calculation is ill defined, it might be interesting that this result appears in B.S. DeWitt's path integral if one does not consider endpoint contributions exactly where the Lagrangian should appear, see (7.86).

As the terms with α_1 and α_3 have opposite parity, they cannot cancel each other and have to vanish separately: $\alpha_1 = \alpha_3 = 0$. $\alpha_2\dot{\xi}$ is not invariant, so we demand $\alpha_2 = 0$. We are left with a constant part as in the case of the naive Legendre-transform: $L(\xi, \dot{\xi}) = \alpha_0$. Not only does varying the action not tell us anything about the system's dynamic, but it constrains our Lagrange function to an uninteresting expression.

Altogether we see that the Lagrange function for our Hamilton function cannot be derived from Legendre-transforms but even if it could in some other way, it would be constant and of no use for us. Especially path integrals cannot be done in the usual way.[12]

7.3.6 Eigenstates of the Hamiltonian

In the case of the Bose oscillator, the n-th energy eigenfunctions in position and momentum representation is basically the n-th Hermite polynomial. Similarly, eigenfunctions for the Fermi oscillator can be determined in both position and momentum representation (which corresponds to restricting the considerations from Σ to ΠTM and ΠT^*M resp.). The results from basic algebraic calculations with fermionic creation and annihilation operators in B.S. DeWitt's approach (section 7.2.3) can simply be taken over for this approach. But we will get the same results from choosing a special representation:

π-representation

Working on ΠT^*M, we can represent the operators (similarly to the position representation in ordinary quantum mechanics) by $\hat{\pi}_\pi = \pi$ and $\hat{\xi}_\pi = \mathrm{i}\hbar\partial_\pi$ which obey obviously the anticommutation relations of $\hat{\xi}$ and $\hat{\pi}$. We will call this representation the π-representation.

[12] Systems without Lagrangian (and thus without action) are not as uncommon as they appear to be. Path integrals can often be constructed by different approaches.

The Hamilton operator here has the form:

$$\hat{H} = \hbar\omega \left(\frac{\hat{\pi}}{\sqrt{\mathrm{i}\hbar}} \frac{\hat{\xi}}{\sqrt{\mathrm{i}\hbar}} - \frac{1}{2} \right) = \hbar\omega \left(\pi\partial_\pi - \frac{1}{2} \right). \tag{7.129}$$

An arbitrary function of π has the form $f(\pi) = c_1 + c_2\pi$ where c_1 and c_2 are complex numbers. We can represent f by a vector:

$$f(\pi) = c_1 + c_2\pi \simeq \begin{pmatrix} c_1 \\ c_2 \end{pmatrix}. \tag{7.130}$$

Using the vector notation, we can describe the action of the Hamilton operator on a function by a 2×2-matrix, where the energy eigenfunctions are given by the eigenvectors of this matrix:

$$\hat{H} f \simeq \hbar\omega \begin{pmatrix} -\frac{1}{2} & 0 \\ 0 & 1 - \frac{1}{2} = +\frac{1}{2} \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = E \begin{pmatrix} c_1 \\ c_2 \end{pmatrix}. \tag{7.131}$$

We immediately get the results:

$$\text{Eigenvalue: } -\frac{1}{2}\hbar\omega \qquad \text{Eigenfunction: } f(\pi) = \alpha \tag{7.132}$$

$$\text{Eigenvalue: } +\frac{1}{2}\hbar\omega \qquad \text{Eigenfunction: } f(\pi) = \alpha\pi \tag{7.133}$$

where α is in both cases an arbitrary complex number. The exact value of α will be constraint by the normalization of the states in the next section to $\alpha = 1$.
We notice, that the parity of the two eigenfunctions is opposite, so that the parity of the annihilation operator $\hat{f}|1\rangle = |0\rangle$ is obviously odd.

ξ-representation

The same results can be obtained on ΠTM, in the ξ-representation. In ordinary quantum mechanics, the momentum operator in this representation looks like $\hat{p}_x = -\mathrm{i}\hbar\partial_x$ where the minus sign is due to the antisymmetry of the commutator $[\hat{x}, \hat{p}] =$

$-[\hat{p}, \hat{x}]$. Since the anticommutator is symmetric, we get rid of this sign, so we will use the representation $\hat{\xi}_\xi = \xi$ and $\hat{\pi}_\xi = \mathrm{i}\hbar\partial_\xi$ which obey again obviously the anticommutation relations of $\hat{\xi}$ and $\hat{\pi}$.

The Hamilton operator now has the form:

$$\hat{H} = \hbar\omega \left(\frac{\hat{\pi}}{\sqrt{\mathrm{i}\hbar}} \frac{\hat{\xi}}{\sqrt{\mathrm{i}\hbar}} - \frac{1}{2} \right) = \hbar\omega \left(\partial_\xi \xi - \frac{1}{2} \right), \tag{7.134}$$

We will again represent an arbitrary function $f(\xi) = c_1 + c_2\xi$ by a vector $(c_1, c_2)^T$. The corresponding equation for the eigenvalues is here:

$$\hat{H} f \simeq \hbar\omega \begin{pmatrix} 1 - \frac{1}{2} = +\frac{1}{2} & 0 \\ 0 & -\frac{1}{2} \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = E \begin{pmatrix} c_1 \\ c_2 \end{pmatrix}. \tag{7.135}$$

The parity of the eigenfunctions is the opposite of (7.132) and (7.133):

$$\text{Eigenvalue: } -\frac{1}{2}\hbar\omega \qquad \text{Eigenfunction: } f(\xi) = \alpha\xi \tag{7.136}$$

$$\text{Eigenvalue: } +\frac{1}{2}\hbar\omega \qquad \text{Eigenfunction: } f(\xi) = \alpha 1 \tag{7.137}$$

These results are certainly consistent with the algebraic considerations. We see that the parities of the states $|0\rangle$ and $|1\rangle$ depend on the representation.

7.3.7 Normalization of Fermionic States

With the representation of different states by functions of supernumbers, the question of how to normalize these functions arises. Remembering the fact that quantum mechanical states are rays in a Hilbert space rather than vectors, we have to rescale an arbitrary function to unit length.

We certainly want the dual product to make use of Berezin integration. As the result should be a real number, the integral must not change parity and thus has to be either an integral over two Graßmann variables or include Graßmann variables

in the measure. This features strongly remind of the dual product in the holomorphic representation[13] and in a rough analogy we define the dual product in the ξ-representation:

$$\langle f|g\rangle := \int \frac{\mathrm{d}\xi\mathrm{d}\bar{\xi}\mathrm{e}^{-\xi\bar{\xi}}}{Z^2} f(\bar{\xi})g(\xi) =: \int \mathrm{d}\gamma(\xi,\bar{\xi})f(\bar{\xi})g(\xi). \tag{7.139}$$

where $\bar{\xi}$ does *not* denote the complex conjugate of ξ (remember that in B.S. DeWitt's approach the Graßmann variables are considered to be real) but another Graßmann variable.

$\mathrm{d}\gamma(\xi,\bar{\xi}) = \frac{\mathrm{d}\xi\mathrm{d}\bar{\xi}\mathrm{e}^{-\xi\bar{\xi}}}{Z^2}$ can be called a "measure" though Berezin integration does not allow the strict development of an measure as its done for real integration (see e.g. [3] and [5]).

For the π-representation we simply substitute ξ with π:

$$\langle f|g\rangle := \int \frac{\mathrm{d}\pi\mathrm{d}\bar{\pi}\mathrm{e}^{-\pi\bar{\pi}}}{Z^2} f(\bar{\pi})g(\pi) =: \int \mathrm{d}\gamma(\pi,\bar{\pi})f(\bar{\pi})g(\pi). \tag{7.140}$$

We can easily verify, that all the equations $\langle 0|0\rangle = \langle 1|1\rangle = 1$ and $\langle 1|0\rangle = \langle 0|1\rangle = 0$ can be obtained from this definition of the dual product, e.g.:

$$\langle 0|0\rangle = \int \frac{\mathrm{d}\xi\mathrm{d}\bar{\xi}\mathrm{e}^{-\xi\bar{\xi}}}{Z^2}\bar{\xi}\xi = \int \frac{\mathrm{d}\xi\mathrm{d}\bar{\xi}(1-\xi\bar{\xi})}{Z^2}\bar{\xi}\xi = 1$$
$$\langle 0|1\rangle = \int \frac{\mathrm{d}\xi\mathrm{d}\bar{\xi}\mathrm{e}^{-\xi\bar{\xi}}}{Z^2}\xi 1 = \int \frac{\mathrm{d}\xi\mathrm{d}\bar{\xi}(1-\xi\bar{\xi})}{Z^2}1\xi = 0.$$

It is obvious, that this dual product can easily be extended to a complete fermionic Fock space by integrating over a $\mathrm{d}\gamma^i(\xi^i,\bar{xi}^i)$ for each degree of freedom, where it

[13] The holomorphic representation combines the dynamical variables p and q to one complex variable with $z = p + \mathrm{i}q$ and $\bar{z} = p - \mathrm{i}q$ (up to factors) and the eigenvectors of the corresponding quantized operator $\hat{z} = \hat{b}$ are the coherent states. Quantum mechanical states are represented by holomorphic functions (polynomials) so that $\langle z|n\rangle = \frac{z^n}{\sqrt{n!}}$. The dual product in this representation is given by:

$$\langle f|g\rangle = \int \mathrm{d}z\mathrm{d}\bar{z}\mathrm{e}^{-z\bar{z}} f(\bar{z})g(z). \tag{7.138}$$

As our set of coherent state is undercomplete, we cannot use the direct analogy of the holomorphic representation in the fermionic case. (E.g., the formula $\langle \pi|n\rangle = \frac{\pi^n}{\sqrt{n!}}$ can not work in the case $n = 1$, as the dual product is performed by Berezin integration over π which yields always a real number.)

remains valid, i.e. it reproduces the same results as the algebraic calculations.

7.3.8 Coherent States

As already discussed in the algebraic considerations, $|0\rangle$ is an eigenstate of the annihilation operator with the eigenvalue 0. All eigenvalues α for different eigenstate have to be a-numbers and such a state has the form: $|0\rangle - \alpha|1\rangle$. We do not want to introduce further Graßmann variables in our model, but only allow the naturally appearing ones ξ, $\dot{\xi}$, π and $\dot{\pi}$. As the time derivatives of the dynamical variables are proportional to the variables, we do not have to consider them separately. Furthermore, ξ is only available in the ξ-representation on ΠTM and π only on ΠT^*M. As multiplication with these Graßmann variables is equivalent to acting with creation and annihilation operators on the equations, the new eigenvalues break down to zero:

$$\begin{aligned} \hat{f}_\pi \left(|0\rangle_\pi + \pi|1\rangle_\pi\right) &= \partial_\pi(1 + \pi \cdot \pi) = 0 \\ \hat{f}_\xi \left(|0\rangle_\xi + \xi|1\rangle_\xi\right) &= \partial_\xi(\xi - \xi \cdot 1) = 0 \end{aligned}$$

Altogether, we get neither a new eigenstate nor new eigenvalues (what would be a contradiction, as there can only be one eigenvalue for each eigenstate).

This is an interesting difference to B.S. DeWitt: While he gets a undercomplete set of coherent states, we get only one coherent state. Even with the undercomplete set of coherent states, B.S. DeWitt has a partition of unity enabling him to construct path integrals, which we do not have in our case.

As one could argue that our interpretation of $\hat{\xi}$ and $\hat{\pi}$ was wrong and we should rather consider $\hat{\phi}$ and $\hat{\phi}^\dagger$, we briefly calculate the eigenvalues of $\hat{\phi}$ as an example:

$$\hat{\phi}|\alpha\rangle = \frac{1}{\sqrt{2\hbar\omega}}(\omega\hat{\xi} + \mathrm{i}\hat{\pi})\left(\alpha_0|0\rangle + \alpha_1|1\rangle\right) = (-1)^{\tilde{\alpha_1}}\sqrt{\frac{\mathrm{i}\omega}{2}}\alpha_1|0\rangle - (-1)^{\tilde{\alpha_0}}\sqrt{\frac{1}{2\mathrm{i}\omega}}\alpha_0|1\rangle \tag{7.141}$$

We get two constraints from demanding, that $|\alpha\rangle$ be an eigenstate of $\hat{\phi}$ with the eigenvalue ϕ and one from the normalization:

$$\alpha_0 = \phi(-1)^{\tilde{\alpha_1}}\sqrt{\frac{\mathrm{i}\omega}{2}}\alpha_1\ ,\ \ \alpha_1 = \phi(-1)^{\tilde{\alpha_0}}\sqrt{\frac{1}{2\mathrm{i}\omega}}\alpha_0 \text{ and } |\alpha_0|^2 + |\alpha_1|^2 = 1 \tag{7.142}$$

Allowing only c-type values for α_0 and α_1, we get from these conditions $\alpha_1 = -1/2\,\phi^2\alpha_1$, so the only eigenvalues are $\phi_{1,2} = \pm\mathrm{i}\sqrt{2}$, inserting all equations in the normalization condition leads to

$$\left|-\sqrt{\frac{1}{2\mathrm{i}\omega}}\alpha_0(\pm\mathrm{i})\sqrt{2}\right|^2 + |\alpha_0|^2 = 1 \Rightarrow \alpha_0 = \frac{1}{1+\omega}\mathrm{e}^{\mathrm{i}\theta}. \tag{7.143}$$

where θ is an arbitrary phase. For α_1 we obtain

$$\alpha_1 = -\sqrt{\frac{1}{2\omega\mathrm{i}}}\frac{1}{1+\omega}\mathrm{e}^{\mathrm{i}\theta}(\pm\mathrm{i})\sqrt{2} = \frac{1}{\sqrt{\omega}(1+\omega)}\mathrm{e}^{\mathrm{i}(\theta+\frac{3}{4}\pi\pm\mathrm{i}\frac{1}{2}\pi)}. \tag{7.144}$$

As in the case of the coherent state, we do not get new eigenvalues by allowing the Graßmann variables contained in the representations. We rewrite the conditions (7.142) with a Graßmann variable ζ as eigenvalue ϕ:

$$\begin{aligned}
\alpha_0 = \zeta(-1)^{\tilde{\alpha_1}}\sqrt{\frac{\mathrm{i}\omega}{2}}\alpha_1 \quad &\text{and} \quad \alpha_1 = \zeta(-1)^{\tilde{\alpha_0}}\sqrt{\frac{1}{2\mathrm{i}\omega}}\alpha_0 \\
\Rightarrow \alpha_0 \quad &= \quad \zeta(-1)^{\tilde{\alpha_1}}\sqrt{\frac{\mathrm{i}\omega}{2}}\zeta(-1)^{\tilde{\alpha_0}}\sqrt{\frac{1}{2\mathrm{i}\omega}}\alpha_0 = 0 \\
\Rightarrow \alpha_1 \quad &= \quad 0.
\end{aligned}$$

So there is no eigenstate with Graßmann eigenvalues, as the conditions for such a state require the coefficients α_0 and α_1 to vanish.

7.3.9 Extension to the Supersymmetric Oscillator

We can now easily find a representation of the SUSY oscillator on our supermanifolds ΠTM and ΠT^*M. The fermionic component will be given by a function of a Graßmann variable, as discussed above. As the manifold M has only been used to determine the dimension of the fibers ΠTM and ΠT^*M, it can now become the

domain of functions representing states of the bosonic oscillator. The states of the SUSY oscillator thus are given by functions on the symmetric supermanifold ΠTM (x, ξ representation) or ΠT^*M (p, π representation). A SUSY-operator in a representation is simply the product of the separate bosonic and fermionic operators in this representation, e.g.:

$$[\hat{A}_{SUSY}]_{(x,\xi)} = [\hat{A}_{Bose}\hat{A}_{Fermi}]_{(x,\xi)} = [\hat{A}_{Bose}]_x[\hat{A}_{Fermi}]_\xi \tag{7.145}$$

where the brackets denote the enclosed operator in a special representation.

x, ξ representation

A supersymmetric state is given by:

$$|\alpha\rangle_{SUSY} = |\alpha\rangle_{Bose} \otimes |\alpha\rangle_{Fermi} = |\alpha\rangle_{Bose} \otimes (\alpha_0|0\rangle_{Fermi} + \alpha_1|1\rangle_{Fermi}) \tag{7.146}$$

where normalization demands $|\alpha_0|^2 + |\alpha_1|^2 = 1$. This state will be represented by a function of x and ξ:

$$|\alpha\rangle_{SUSY} \simeq F(x,\xi) = \alpha_0 f_0(x)\xi + \alpha_1 f_1(x) \tag{7.147}$$

(Remember that the parity of the vacuum state is odd in the ξ representation.)
The eigenfunctions of the Hamiltonian are given by the functions where either α_0 or α_1 vanishes and f_0 or f_1 are eigenfunctions of the Bose oscillator $g(x) \sim \mathrm{e}^{-\frac{z^2}{2}} H_n(z)$ where $H_n(z)$ is the n-th Hermite-polynomial in z. The SUSY operators can immediately be written down as simple products of the representations of the bosonic and fermionic creation and annihilation operators. The commutation relation of the fermionic and bosonic operators among themselves are clear from the separate considerations of the oscillators, but the mixed relations still have to be proven:

$$\begin{aligned}\left[\hat{b}, \hat{f}\right]_{x,\xi} &= \left[\frac{1}{\sqrt{2\hbar\omega}}\left(\omega x + \hbar\partial_x\right), \frac{\hat{\xi}}{\sqrt{\mathrm{i}\hbar}}\right]_- = 0 \\ \left[\hat{b}, \hat{f}^\dagger\right]_{x,\xi} &= \left[\frac{1}{\sqrt{2\hbar\omega}}\left(\omega x + \hbar\partial_x\right), \frac{\partial_\xi}{\sqrt{\mathrm{i}\hbar}}\right]_- = 0\end{aligned}$$

Altogether, this representation is consistent with the algebraic definitions.

p, π representation

This representation works exactly in the same way as the x, ξ representation, as we are working on ΠT^*M, states are represented by functions:

$$|\alpha\rangle_{SUSY} \simeq f(p,\pi) = \alpha_0 f_0(p) + \alpha_1 f_1(p)\pi \tag{7.148}$$

where the coordinate x on the manifold has been renamed to p to indicate the change to momentum representation. The mixed commutation relations are fulfilled again, so that this representation is also consistent:

$$\begin{aligned}\left[\hat{b}, \hat{f}\right]_{p,\pi} &= \left[\frac{1}{\sqrt{2\hbar\omega}}\left(\mathrm{i}\hbar\omega\partial_p + \mathrm{i}p\right), \sqrt{\mathrm{i}\hbar}\partial_\pi\right]_- = 0 \\ \left[\hat{b}, \hat{f}^\dagger\right]_{p,\pi} &= \left[\frac{1}{\sqrt{2\hbar\omega}}\left(\omega x + \mathrm{i}\partial_x\right), \frac{\partial_\xi}{\sqrt{\mathrm{i}\hbar}}\right]_- = 0\end{aligned}$$

7.3.10 Transition Amplitudes

The calculation of the transition amplitude and probability from an arbitrary state of the Fermi oscillator to another one (to 0th order) is easily done:
Given two arbitrary states

$$\begin{aligned}|\alpha, t\rangle &= \mathrm{e}^{-\frac{\mathrm{i}}{2\hbar}\omega t}\alpha_0|0\rangle + \mathrm{e}^{+\frac{\mathrm{i}}{2\hbar}\omega t}\alpha_1|1\rangle \\ \langle\alpha', t'| &= \langle 0|{\alpha_0'}^*\mathrm{e}^{+\frac{\mathrm{i}}{2\hbar}\omega t'} + \langle 0|{\alpha_1'}^*\mathrm{e}^{-\frac{\mathrm{i}}{2\hbar}\omega t'},\end{aligned}$$

we can easily calculate the overlap:

$$\langle\alpha', t'|\alpha, t\rangle = {\alpha_0'}^*\alpha_0\mathrm{e}^{+\frac{\mathrm{i}}{2\hbar}\omega(t'-t)} + {\alpha_1'}^*\alpha_1\mathrm{e}^{-\frac{\mathrm{i}}{2\hbar}\omega(t'-t)} \tag{7.149}$$

The probability for a transition from $|\alpha, t\rangle$ to $|\alpha', t'\rangle$ is then given by:

$$\begin{aligned}|\langle\alpha', t'|\alpha, t\rangle|^2 &= |{\alpha_0'}^*\alpha_0\mathrm{e}^{+\frac{\mathrm{i}}{2\hbar}\omega(t'-t)} + {\alpha_1'}^*\alpha_1\mathrm{e}^{-\frac{\mathrm{i}}{2\hbar}\omega(t'-t)}|^2 \\ &= {\alpha_0'}^*\alpha_0\alpha_0'{\alpha_0}^* + {\alpha_1'}^*\alpha_1\alpha_1'{\alpha_1}^* \\ &\quad + {\alpha_0'}^*\alpha_0\alpha_1'{\alpha_1}^*\mathrm{e}^{+\frac{\mathrm{i}}{\hbar}\omega(t'-t)} + \alpha_0'{\alpha_0}^*{\alpha_1'}^*\alpha_1\mathrm{e}^{-\frac{\mathrm{i}}{\hbar}\omega(t'-t)} \\ &= |\alpha_0'|^2|\alpha_0|^2 + |\alpha_1'|^2|\alpha_1|^2 + 2\mathrm{Re}\left({\alpha_0'}^*\alpha_0\alpha_1'{\alpha_1}^*\mathrm{e}^{+\frac{\mathrm{i}}{\hbar}\omega(t'-t)}\right).\end{aligned}$$

The transition amplitude for two SUSY-states is obviously given by:

$$\begin{aligned} T(\alpha,\alpha') &= \langle\alpha'|\alpha\rangle = \left(\langle\alpha|_{Fermi}\otimes\langle\alpha|_{Bose}\right)\left(|\alpha'\rangle_{Bose}\otimes|\alpha'\rangle_{Fermi}\right) \\ &= \langle\alpha|_{Fermi}|\alpha'\rangle_{Fermi}\langle\alpha|_{Bose}|\alpha'\rangle_{Bose} \end{aligned}$$

As the dual products for the Bose case is known, and the Fermi case given above, $T(\alpha,\alpha')$ can be easily calculated and the dynamics of the system is completely covered.

7.3.11 Discussion of Path Integrals

Path integrals are the most appealing method of quantization. Besides their conceptual beauty from which the Hamilton principle evolves immediately, they allow non-perturbative insights in theories and are in some cases the only way to quantize. A path integral gives the transition function $K(b,a)$ from a state a to a state b by:

$$K(b,a) = \sum_{\text{all paths}} const. \times \mathrm{e}^{\frac{\mathrm{i}}{\hbar}S(\text{path})} \tag{7.150}$$

where $S(\text{path}) = \int \mathrm{d}t L(q(t),\dot{q}(t),t)$ is the action of the path. The vague "sum over all paths" is performed by breaking the path into a number of points and then integrating over the position of these intermediate points. In the limit (number of points) $\to \infty$ we get the path integral

$$K(b,a) = const. \times \int \mathcal{D}x\, \mathrm{e}^{\frac{\mathrm{i}}{\hbar}S} \tag{7.151}$$

where $\int \mathcal{D}x = \int ... \int \mathrm{d}x_1 ... \mathrm{d}x_n$.

This first approach does obviously not work as we cannot define a Lagrangian for our system, as the Legendre-transformation is ill-defined. It is obvious as well that a constant Lagrangian, which would be required by the equations of motion, if it could be derived in another way, leads to a divergent path integral.

The second way path integrals are constructed is again by "time-slicing" the path

through the configuration space, but using the quantum mechanical time evolution operator $\mathrm{e}^{-\mathrm{i}/\hbar\,\hat{H}\Delta t}$ in the linearized version for short times $1-\mathrm{i}/\hbar\,\hat{H}\Delta t$ which holds true for infinite many intermediate points. The intermediate points can be inserted as there is a complete set of eigenstates (usually position eigenstates) allowing a partition of unity: $\hat{1}=\int \mathrm{d}q|q\rangle\langle q|$. Note that though the set of coherent states in B.S. DeWitt's approach are undercomplete, they still allowed the necessary partition of unity. In our representation, there is no complete set of eigenstates which would allow a partition of unity with integration. Nevertheless, we can use this time-slicing approach:

$$K(\alpha',\alpha)=\lim_{n\to\infty}\langle\alpha'|\left(\mathrm{e}^{-\frac{\mathrm{i}}{\hbar}H\Delta t}\left(|0\rangle\langle 0|+|1\rangle\langle 1|\right)\right)^n|\alpha\rangle \tag{7.152}$$

where $\Delta t=\frac{t}{n+1}$. First, we calculate the term in brackets:

$$\begin{aligned}\mathrm{e}^{-\frac{\mathrm{i}}{\hbar}H\Delta t}\left(|0\rangle\langle 0|+|1\rangle\langle 1|\right) &\overset{\Delta t\to 0}{=} \left(1-\mathrm{i}\omega\left(\hat{f}^\dagger\hat{f}-\frac{1}{2}\right)\Delta t\right)\left(|0\rangle\langle 0|+|1\rangle\langle 1|\right)\\ &= \left(1+\frac{\mathrm{i}\omega\Delta t}{2}\right)|0\rangle\langle 0|+\left(1-\frac{\mathrm{i}\omega\Delta t}{2}\right)|1\rangle\langle 1|\\ &= c^*|0\rangle\langle 0|+c|1\rangle\langle 1|\end{aligned}$$

Using our dual product in the ξ-representation, we can rewrite $K(\alpha',\alpha)$ as:

$$\begin{aligned}K(\alpha',\alpha)=\int\frac{\mathrm{d}\xi^{n+1}\mathrm{d}\bar{\xi}^{n+1}}{Z^2}...\frac{\mathrm{d}\xi^1\mathrm{d}\bar{\xi}^1}{Z^2}(1-\xi^{n+1}\bar{\xi}^{n+1})\alpha'(\bar{\xi}^{n+1})\times\\ (1-\xi^n\bar{\xi}^n)(c^*\xi^{n+1}\bar{\xi}^n+c)...(1-\xi^2\bar{\xi}^2)(c^*\xi^3\bar{\xi}^2+c)(1-\xi^1\bar{\xi}^1)(c^*\xi^2\bar{\xi}^1+c)\alpha(\xi^1).\end{aligned}$$

Note that all the elements on the right are c-type since a-type objects as Berezin integrals and Graßmann variables appear in pairs. The "measure" $\mathrm{d}\gamma(\xi,\bar{\xi})$ has been split for further calculations:

$$(1-\xi^i\bar{\xi}^i)(c^*\xi^{i+1}\bar{\xi}^i+c)=(c^*\xi^{i+1}\bar{\xi}^i+c-c\xi^i\bar{\xi}^i)\to(c^*\xi^{i+1}\bar{\xi}^i-c\xi^i\bar{\xi}^i)$$

The single c is cancelled by Berezin integration over $\bar{\xi}^i$, so we obtain inside the integrals:

$$\begin{aligned}\alpha'(\bar{\xi}^{n+1})(1+\bar{\xi}^{n+1}\xi^{n+1})(c^*\xi^{n+1}\bar{\xi}^n+c\bar{\xi}^n\xi^n)...(c^*\xi^3\bar{\xi}^2+c\bar{\xi}^2\xi^2)(c^*\xi^2\bar{\xi}^1+c\bar{\xi}^1\xi^1)\alpha(\xi^1)\\ \to\alpha'(\bar{\xi}^{n+1})\xi^{n+1}(c^*)^n\bar{\xi}^1\alpha(\xi^1)+\alpha'(\bar{\xi}^{n+1})\bar{\xi}^{n+1}\xi^{n+1}(c)^n\bar{\xi}^1\xi^1\alpha(\xi^1)\end{aligned}$$

In the last expression, all the other factors (mixed in c and c^*) are cancelled by Berezin integration since they do not come with pairs of $\bar{\xi}^i \xi^i$.
The limit of the last expression is obviously consistent with the results obtained for the transition amplitudes (section 7.3.10). This discussion becomes much more complicated, if we allow interaction terms in the Hamiltonian.

7.3.12 Proposition of an Extension to a Supersymmetric Fock Space

"Do you really believe, that there is an infinite number of points in the universe? Do you really believe that?"

Willy Fischler

A Fock space is completely defined by the algebra

$$\begin{aligned} \hat{a}_i|0\rangle &:= 0 \\ \left[\hat{a}_i, \hat{a}_j^\dagger\right]_\mp &= \hat{a}_i\hat{a}_j^\dagger \mp \hat{a}_j^\dagger\hat{a}_i := \delta_{ij} \end{aligned}$$

where i runs over the degrees of freedom and the upper sign refers to a bosonic Fock space, the lower sign to a fermionic one.
We will call a space which is the tensor product of a fermionic and a bosonic Fock space with equally many degrees of freedom and with the additional commutation rule

$$\begin{aligned} \left[\hat{b}_i, \hat{f}_j\right] &= \left[\hat{b}_i, \hat{f}_j\right]_- := 0 \\ \left[\hat{b}_i, \hat{f}_j^\dagger\right] &= \left[\hat{b}_i, \hat{f}_j^\dagger\right]_- := 0 \end{aligned}$$

a supersymmetric Fock space.
It is quite obvious, that a Fock space is constructed by defining an independent supersymmetric oscillator for each degree of freedom, i.e. points in position or momentum space[14]. Let us assume, we have N degrees of freedom. Then a state is

[14]Note that we assume that the continuous space-time is reduced to a lattice

given by:

$$|\alpha\rangle_{Fock} = \bigotimes_{i=1}^{N} |\alpha_i\rangle_{SUSY} = \bigotimes_{i=1}^{N} (|\alpha_i\rangle_{Bose} \otimes |\alpha_i\rangle_{Fermi}) \tag{7.153}$$

To represent this space, we will have to use a manifold of dimension N so that ΠTM and ΠT^*M provide us with N ordinary and N Graßmann variables: $(x^1, ..., x^N, \xi^1, ..., \xi^N)$ and $(x^1, ..., x^N, \pi^1, ..., \pi^N)$ resp.

Functions on ΠTM and ΠT^*M automatically provide the properties demanded: they are symmetric under exchange of ordinary variables associated to bosons and antisymmetric under exchange of Graßmann variables associated to fermions.

The usual operator algebra is also conserved in this representation. This is already known for one oscillator. Since the operators constructed out of different variables - so referring to different degrees of freedom in position or momentum space - do not interfere with each other, i.e. bosonic operators commute: $[x^1, \partial_{x^2}]_- = 0$, fermionic ones anticommute: $[\pi_2, \partial_{\pi_3}]_+ = 0$, there remains nothing to prove.

The dual product of two Fock states is given by our dual product for the Fermi oscillator, extended in a straightforward way. As the bosonic part is clear, we consider only the fermionic sector.

Given two states in the fermionic Fock space $|\alpha\rangle$ and $|\beta\rangle$. Then its overlap is given in the ξ-representation by:

$$\begin{aligned} \langle\alpha|\beta\rangle &= \bigotimes_{i=1}^{N} \langle\alpha_{N-i+1}| \bigotimes_{i=1}^{N} |\beta_i\rangle = \langle\alpha_N|\beta_N\rangle ... \langle\alpha_1|\beta_1\rangle \\ &= \left(\prod_{i=1}^{N} \int \mathrm{d}\xi^i \mathrm{d}\bar{\xi}^i \mathrm{e}^{-\xi^i\bar{\xi}^i}\right) \left(\prod_{i=1}^{N} \alpha_i(\bar{\xi}^i)\beta(\xi^i)\right) \end{aligned}$$

As the integral terms are c-type, it does not matter, in which order the separate tensor products are calculated. If order should matter due to odd parity in the second term, the integral vanishes anyway. So it is clear that the different Graßmann variables do not interfere, and we found a valid representation for the dual product of the supersymmetric Fock space.

Chapter 8

Summary and Future Directions

"We apologize for the inconvenience." (God's Final Message to His Creation)
from Douglas Adams' "So Long and Thanks for All the Fish"

There are basically two different approaches to supermathematics. The more popular one uses abstract Graßmann variables and simply extends the rules of ordinary calculus on them. The second one uses the Graßmann variables as generators for the algebra of supernumbers and substitutes the real numbers by them. In both cases, the most interesting results are that differentiation and integration are equivalent and that a change of coordinate leads to an inverse Jacobi determinant.

We can construct supervector spaces (graded modules over the ring of supernumbers) and supermanifolds. On the latter, we can define further structures which allow a reduction of arbitrary supermanifolds to Graßmannian manifolds, which shows the algebraic equivalence of B.S. DeWitt's supermanifolds with those of T. Voronov.

There is an isomorphism of superfunctions and the de Rham complexes of differential forms on supermanifolds, which can be used to introduce a new representation for supersymmetric Fock spaces.

We obtained the most interesting results when applying the tools of supermathemat-

ics to model the Fermi oscillator. We found that the only one-dimensional fermionic Hamiltonian that can be constructed invariantly leads to a Fermi oscillator. This picture can be easily extended to the SUSY oscillator and even to a supersymmetric Fock space. We can define an analogue to path integrals which comes with some interesting features, as, for example, the gaussian measure. In contrast to B.S. DeWitt, our Fermi oscillator has only one coherent state, while he gets an uncountable set.

The next steps are quite clear: The main differences between our and B.S. DeWitt's description of the Fermi oscillator have to be examined closer, particularly the number of coherent states and the somewhat odd stationary (pseudo-classical) trajectory of B.S. DeWitt. The question of such a pseudo classical trajectory could be solved by finding more arguments in favor or against the existence of coherent states of the Fermi oscillator with a-type eigenvalues. The definition and usefulness of our path integrals have to be reconsidered and formulated in a tensorial way, which is currently being worked out by P. Cartier.

For the new approach of P. Cartier to supermanifolds, the definitions have to be extended to include connections on more complicated manifolds. The big advantage of this formulation, the tensorial integration, has to be worked out completely in a dimension independent manner to allow the transition to infinitely many dimensions as needed for path integrals.

Appendix A

Notation and Used Symbols

"The book of nature is written with mathematical symbols."

Galileo Galilei

To allow a quick understanding of the text, most of the used symbols are repeated here: In general, objects with symmetric properties have been labeled by latin letters (e.g. the ordinary variables x^i and the Graßmannian forms o), objects with an antisymmetric property by greek letters (e.g. the Graßmannian variables ξ^i and the ordinary forms ω).
Operators as the creation and annihilation operators of second quantization are marked by a hat: $\hat{b}$, $\hat{f}^\dagger$ etc.

Expression	Meaning
∂_μ	partial derivative with respect to x^μ or ξ^μ
∂	boundary operator (def. 5.1.3)
$D^\bullet$	set of differential operators
δA	variation of A

Expression	Meaning
e_i/ϵ_i	basis vector for ordinary/Graßmannian dimension, i.e. $\partial/\partial x^i$, $\partial/\partial\xi^i$ resp.
${}_ie$, ${}^ie$, e_i, e^i	basis vectors for a supervector space
ξ^i	generator of the Graßmann algebra/Graßmann variable
$\mathcal{A}^N$	set of functions of N Graßmann variables (rem. 2.1.1)
$I(u)$	Berezin integral over u
Z	constant of integration $Z = (2\pi\mathrm{i})^{-\frac{1}{2}}$ (def. 2.1.3)
z_B, z_S	body and soul of a supernumber $z = z_B + z_S$
Λ_N, Λ_∞	set of supernumbers (def. 2.2.1)
$\tilde{z}$	parity of a superobject, 0: even, 1: odd.
$[A, B]$	supercommutator of A and B (def. 2.2.4)
$v^\sim$	transponse of the supervector v
$\mathbb{R}_c$, $\mathbb{R}_a$, $\mathbb{C}_c$, $\mathbb{C}_a$	set of real/complex c-type/a-type supernumbers
$\mathbb{R}^{(n\|\nu)}$	space represented by n real and ν Graßmann variables, see section 4.1
$\mathbb{R}_c^n \times \mathbb{R}_a^\nu$	space represented by n elements of $\mathbb{R}_c$ and ν elements of $\mathbb{R}_a$, see section 4.1
sdet	superdeterminant (def. 2.2.22)
str	supertrace (def. 2.2.21)
$\delta(x)$	Dirac's delta distribution
$\{x_i\}_i$ / $(x_i)_i$	the set/tupel consisting of $x_1, ..., x_n$
T_xM, T_x^*M	tangent/cotangent space of M at x (def. 3.1.2)
TM, T^*M	tangent/cotangent bundle of M at x
$\mathcal{T}_q^p(\mathbb{X})$	set of tensors (def. 3.2.1) of type (p,q) on the linear space $\mathbb{X}$
$\mathcal{X}(M)$	set of vector fields on a manifold M

Expression	Meaning
Ω^n/O^n	ordinary/Graßmannian forms of order n
$\Omega^\bullet$	direct sum of all forms: $\Omega^\bullet = \bigoplus \Omega^n$
d_r	exterior derivative of an r-form (def. 3.2.7)
e_f	acts on forms like $\mathrm{d}f\wedge$ (def. 3.5.1)
i_X	contraction of a form with a vector field X, interior product (def. 3.2.8)
$\mathcal{L}_X$	Lie derivative $\mathcal{L}_X = [\mathrm{i}_X, \mathrm{d}]$ (def. 3.2.9, 3.4.3)
ΠTM	parity-changed tangent bundle (def. 4.2.2)
ΠT^*M	parity-changed cotangent bundle (def. 4.2.2)
S_M	dualized manifold $S_M = TM \times_M T^*M$
Σ_M	dualized super manifold $\Sigma_M = \Pi TM \times_M \Pi T^*M$
$\mathfrak{C}_q$	set of smooth q-chains (def. 5.1.5)
M_f	multiplication with the function f
$L_2(G)$	set of functions f with Lebesgue-integral $\int_G \lvert f(x)\rvert^2 \mathrm{d}x < \infty$
$\mathbb{H}^n$	n-particle Hilbert space
$\mathbb{F}$	Fock space, $\mathbb{F} = \bigoplus \mathbb{H}^n$
$\hat{b}^\dagger$, $\hat{b}$, $\hat{f}^\dagger$, $\hat{f}$	bosonic and fermionic creation and annihilation operators
$\mathrm{d}\gamma(\xi, \bar{\xi})$	$\mathrm{d}\gamma(\xi, \bar{\xi}) = \frac{\mathrm{d}\xi \mathrm{d}\bar{\xi} \mathrm{e}^{-\xi\bar{\xi}}}{Z^2}$, measure for the dual product

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Vita

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In 1988 the whole family moved to Thessaloniki, Greece, and was allowed to experience Greece's cultural treasures and its incredible friendly people for four years. During this time, Christian Sämann attended the "Deutsche Schule Thessaloniki".

After returning to Germany in 1992, he also returned to the "Winfriedschule Fulda" and graduated from it in 1996 with the Abitur.

As his civil service, he worked with handicapped children at the "Arbeiterwohlfahrt" in Fulda for the following year.

In 1997, he started studying mathematics and physics at the Julius-Maximilian Universität at Würzburg in northern Bavaria and received his Vordiplom in both subjects in 1999.

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