The Golden Section

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There are many interesting cases, in which the *golden section* surprisingly arises. After establishing a connection between the *golden ratio* and an expression in terms of trigonometric functions, we deal with three of those cases: the Fibonacci sequence, a certain Berezinian and the pentagram.

1. Preliminary Lemma

For the successive sections we need the basic triangular relation

Lemma 1 *It is*

$$\cos 36^\circ - \sin 18^\circ = \frac{1}{2} \tag{1}$$

Proof: We consider a triangle with two equal sides of length 1 and the angles 36°, 72° and 72° (see figure 1.). This triangle can be divided in four equal triangles with corners A, B, C, D, F, and G. The perpendicular line to $\overline{AB}$ through C intersects $\overline{AB}$ in E, the parallel line to $\overline{BC}$ through E intersects $\overline{AC}$ in H and $\overline{FG}$ in J. The denoted angles follow immediately from the two angles $\alpha = 18^\circ$ at A. As $|AE| = \cos 36^\circ$ and $|AD| = \frac{1}{2}$, it remains to show that $|DE| = |FH| = \sin 18^\circ$. It is furthermore $|CG| = \sin 18^\circ$ and $|HC| = |HJ|$. As the triangles CGJ and FHJ are identical, we obtain $|FH| = |CG| = \sin 18^\circ$. ■

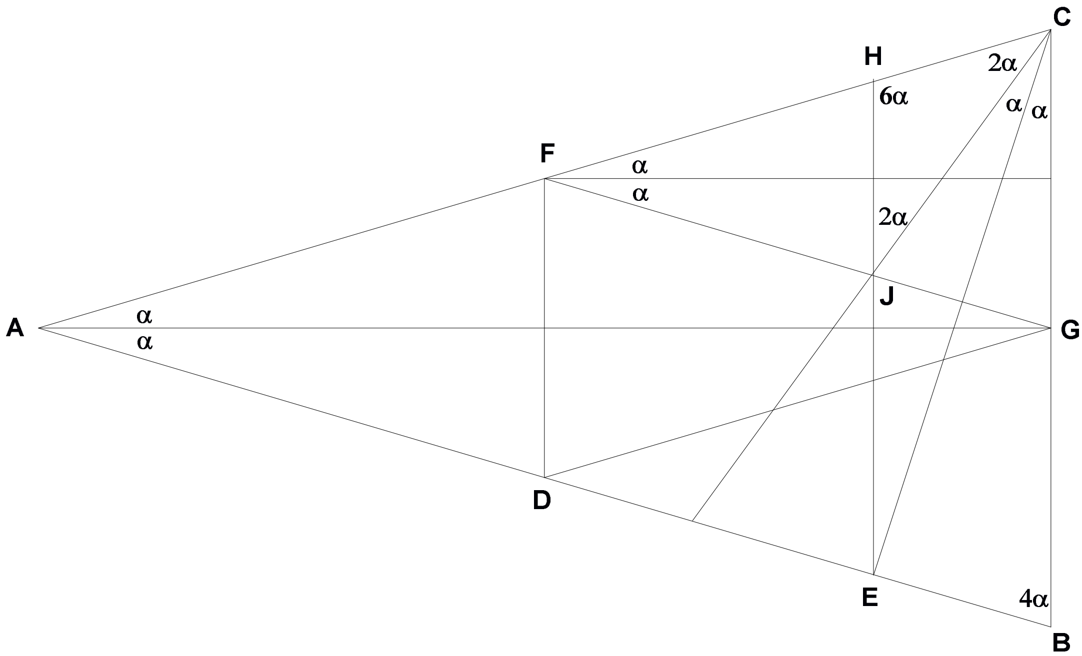

Figure 1: The proof of $\cos 36^\circ - \sin 18^\circ = \frac{1}{2}$.

2. The Golden Section

If a line L is split into two pieces A and B so that the ratios $|L|/|A|$ and $|A|/|B|$ are equal, this splitting is called the *golden section*. It was considered extremely aesthetical since the antiquity and can be found in many works of architects and painters.

In mathematics, the ratio τ given by the *golden section* appears in different contexts, so e.g. as the simplest continued fraction and as the ratio of two subsequent elements of the Fibonacci sequence.

Definition 1 *The* **golden ratio** *is the positive real number τ satisfying the equation*

$$\frac{1}{\tau} = \frac{\tau}{1-\tau} \tag{2}$$

Note that since this equation is equivalent to $\tau^2 + \tau = 1$, there are two solutions

$$-\frac{1}{2} \pm \frac{\sqrt{5}}{2}$$

where we discard the negative solution, so $\tau = \frac{1}{2}(\sqrt{5} - 1)$.

It is interesting that equation (2) is the simplest quadratic equation containing all possible monomials nontrivially with real solutions.

We introduce the notation $[a_0, a_1, a_2, ...]$ for continued fractions:

$$[a_0, a_1, a_2, ...] = \frac{1}{a_0 + \frac{1}{a_1 + \frac{1}{a_2 + \frac{1}{...}}}} \tag{3}$$

For a continued fraction we have the iteration formula

$$[a_0, a_1, ..., a_{n+1}] = \frac{1}{1 + [a_0, a_1, ..., a_n]}. \tag{4}$$

Now the simplest continued fraction is $[\bar{1}] = [1, 1, 1, ...]$. Treating the limits a little sloppily, the iteration formula directly yields

$$[\bar{1}] = \frac{1}{1 + [\bar{1}]} \Rightarrow [\bar{1}]^2 + [\bar{1}] = 1 \tag{5}$$

And with our results from above it follows immediately that $[\bar{1}] = \tau$.

Lemma 2 *The value of τ is* $2\sin 18°(\approx 0,618034)$.

Proof: We show that $2\sin 18°$ fulfills $1 = \tau^2 + \tau$:

$$\begin{aligned} 1 &= 4\sin^2 18° + 2\sin 18° \\ \Leftrightarrow \frac{1}{2} &= 2\left(\frac{1}{2} - \frac{1}{2}\cos 36°\right) + \sin 18° \\ \Leftrightarrow \frac{1}{2} &= \cos 36° - \sin 18° \end{aligned}$$

which holds because of lemma 1. ■

3. The Fibonacci sequence

Definition 2 *The* **Fibonacci**[1] **sequence** *is defined recursively by*

$$a_{n+1} = a_n + a_{n-1} \tag{6}$$

with the initial conditions $a_0 = a_1 = 1$*.*

Thus, the first elements of the Fibonacci sequence are easily calculated to be: $(1, 1, 2, 3, 5, 8, 13, ...)$. Looking at the ratio of two subsequent elements, we find the sequence $(1, 0.5, 0.\bar{6}, 0.6, 0.625, 0.\overline{615384}, ...)$. This sequence seems to converge and indeed we get an interesting result:

Theorem 1 *The limit of the ratio of two subsequent elements of the Fibonacci sequence is given by:*

$$\lim_{n\to\infty} \frac{a_n}{a_{n+1}} = 2\sin 18^\circ = \tau. \tag{7}$$

Proof: Let $Q = \lim_{n\to\infty} \frac{a_n}{a_{n+1}}$. It is

$$\frac{1}{Q} = \lim_{n\to\infty} \frac{a_n + 1}{a_n} = \lim_{n\to\infty} \frac{a_n + a_{n-1}}{a_n} = \lim_{n\to\infty} \left(1 + \frac{a_{n-1}}{a_n}\right) = 1 + Q. \tag{8}$$

This equation is obviously equivalent to equation (2) and with the restriction $Q > 0$ it follows immediately $Q = \tau$. ■

4. The Berezinian

The following section is based on a remark presented in [2]. The *Berezinian*[2] is the appropriate $\mathbb{Z}_2$-graded extension of the ordinary determinant and used in the field called *supermathematics* (see [3] for more details). For our purposes, the bare definition in a special case is sufficient.

Definition 3 *Given a block matrix of the form*

$$K = \begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix} , \tag{9}$$

its **Berezinian** *is given by*

$$\mathrm{Ber}(K) = \frac{\det(A)}{\det(B)} . \tag{10}$$

Furthermore, the **characteristic function** *of such a matrix* K *is defined by*

$$R_K(z) := \mathrm{Ber}(\mathbb{1} + Kz) . \tag{11}$$

Consider now the block diagonal matrix K with $A = 1$ and $B = \mathrm{diag}(-\frac{1}{2} + \frac{\sqrt{5}}{2}, -\frac{1}{2} - \frac{\sqrt{5}}{2})$, i.e. the latter contains the two solutions to equation (2). The characteristic function of this block matrix is then easily seen to be

$$R_K(z) = \frac{\det(1 + z)}{\det(\mathbb{1} + Bz)} = \frac{1 + z}{1 - z - z^2} . \tag{12}$$

[1]Leonardo of Pisa (Fibonacci), 1180-1250, Italian mathematician, introduced Indian and islamic mathematics in Europe.

[2]F. A. Berezin, Russian mathematician, founder of Graßmann calculus and supermathematics.

Lemma 3 *The coefficients in the power series expansion of $R_K(z)$ are just the terms of the Fibonacci sequence:*

$$R_K(z) = \sum_{n=0}^{\infty} a_{n+1} z^n \ . \tag{13}$$

To prove this, notice that we can rewrite $R_K(z)$ as a geometric series:

$$R_K(z) = (1+z)\sum_{n=0}^{\infty}(z+z^2)^n = \sum_{n=0}^{\infty}\sum_{i=0}^{\infty}\left(\begin{array}{c} n \\ i \end{array}\right)(z^{n-i}z^{2i} + zz^{n-i}z^{2i}) \ . \tag{14}$$

One can rearrange this sum into an extended Pascal's triangle which takes the form

$$\begin{array}{llll} 1(1+z) & & & \\ 1(z^1+z^2) & 1(z^2+z^3) & & \\ 1(z^2+z^3) & 2(z^3+z^4) & 1(z^4+z^5) & \\ 1(z^3+z^4) & 3(z^4+z^5) & 3(z^5+z^6) & 1(z^6+z^7) \\ 1(z^4+z^5) & 4(z^5+z^6) & 6(z^6+z^7) & \ldots \\ \ldots & & & \end{array} \tag{15}$$

The coefficient of z^n in the power series (13) is then determined by following a "stairway" through the above pattern and summing the encountered coefficients. For example, the coefficient of z^4 is $a_5 = 1+1+3+2+1 = 8$. On the other hand, one sees that such a coefficient can be derived from the sum of the coefficients obtained from the two stairways starting one line and two lines above the original one. This gives rise to the Fibonacci recursion relation

$$a_n = a_{n-1} + a_{n-2} \ , \tag{16}$$

and together with the observation that the above method yields $a_1 = 1$ and $a_2 = 2$, the induction is complete. ■

5. The Pentagram

Lemma 4 *The lines in a pentagram are split according to the golden section, i.e. $|AD|/|AC| = \tau$ (fig. 5.).*

Proof: The inner angles of a pentagon[3] are 108°. For the angles at A we have $\alpha + 2\beta = 108°$, considering the triangle ABC, we obtain $2\beta + 108° = 180°$. It follows $\alpha = \beta = 36°$ and thus, with the law of sines for the triangle BCD:

$$\frac{|AD|}{|DC|} = \frac{\sin 36°}{\sin 72°} = \frac{2\sin 18° \cos 18°}{\sin 72°} = 2\sin 18° = \tau \tag{17}$$

which was to be proven. ■

[3] The formula for a n-polygon is $\gamma = (n-2)\cdot 180°$, found by decomposing the polygon into n triangles and subtracting the angles of the triangles at the center $(2 \cdot 180°)$.

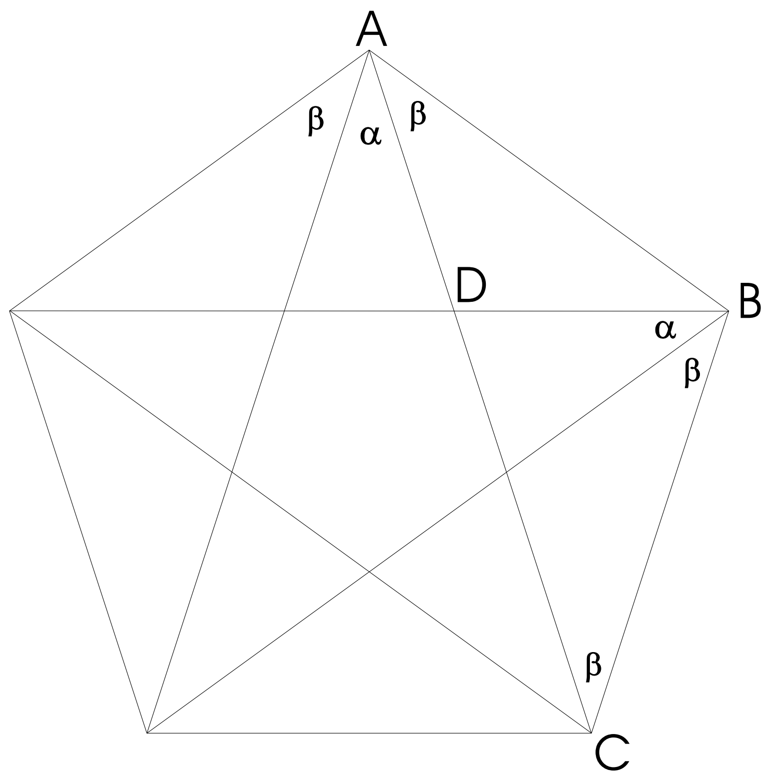

Figure 2: The pentagram

6. Concluding Remark

In many rather "esoteric" essays it is claimed that

$$\frac{6}{5} \cdot \left(\frac{1}{\tau}\right)^2 = \pi \tag{18}$$

which is approximately true only up to the 5th digit. As π is a transcendental[4] number while τ is only irrational, it is clear that an equality of this type can never be true.

References

[1] *Teubner-Taschenbuch der Mathematik I*, ed. by E. Zeidler, B.G. Teubner, 1996

[2] H. Khudaverdian, *Berezinians, recurrent sequences, Hamilton-Cayley identities*, talk presented at the conference "Supersymmetries and Quantum Symmetries" JINR, Dubna, Russia.

[3] P. Cartier, C. DeWitt-Morette, M. Ihl and C. Saemann, *Supermanifolds - Application to supersymmetry,* in "Multiple facets of quantization and supersymmetry: Michael Marinov memorial volume", Eds. M. Olshanetsky and A. Vainshtein, p.412, World Scientific, 2002 [math-ph/0202026].

[4] Transcendental numbers are never zeros of polynomials with rational coefficients.